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WEAK AND STRONG CONVERGENCE OF MANN'S-TYPE ITERATIONS FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

YISHENG SONG AND RUDONG CHEN

ABSTRACT. Let K be a nonempty closed convex subset of a Banach space E. Suppose $\{T_n\}$ $(n = 1, 2, \ldots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$, define $x_{n+1} = \lambda_{n+1}x_n + (1 - \lambda_{n+1})T_{n+1}x_n, n \geq 0$. If $\lambda_n \subset [0, 1]$ satisfies $\lim_{n\to\infty} \lambda_n = 0$, we proved that $\{x_n\}$ weakly converges to some $z \in F$ as $n \to \infty$ in the framework of reflexive Banach space E which satisfies the Opial's condition or has Fréchet differentiable norm or its dual E^* has the Kadec-Klee property. We also obtain that $\{x_n\}$ strongly converges to some $z \in F$ in Banach space E if K is a compact subset of E or there exists one map $T \in \{T_n; n = 1, 2, \ldots\}$ satisfy some compact conditions such as T is semicompact or satisfy Condition A or $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and so on.

1. Introduction

Let K be a nonempty closed convex subset of a Banach space E. A mapping $T : K \to K$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in K$. Mann [7] introduced the following iteration for T in a Hilbert space:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where $\{\lambda_n\}$ is a sequence in $[0, 1]$. Lately, Reich [9] studied this iteration in a uniformly convex Banach space with a Fréchet differentiable norm, and obtained that if T has a fixed point and $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$, then the sequence $\{x_n\}$ converges weakly to a fixed point of T. Shimizu and Takahashi [11] also introduced the following iteration procedure to approximate a common fixed points of finite family $\{T_n; n = 1, 2, \ldots, N\}$ of nonexpansive self-mappings: for any fixed $u, x_0 \in K$,

$$x_{n+1} = \lambda_{n+1}u + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0. \tag{1.2}$$

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Motivated by Shimizu and Takahashi [11], various iteration procedures for families of mappings have been studied by many authors. For instance, see [6, 8, 1, 5]. In particular, Jung [6] and O'Hara et al. [8] studied iteration scheme (1.2) for the family of nonexpansive self-mappings $\{T_n; n \in \mathbb{N}\}$ and proved several strong convergence theorems.

Motivated by Jung [6] and O'Hara et al. [8], we consider the following Mann's type iterative scheme: for a countable family of nonexpansive self-mappings $\{T_n; n \in \mathbb{N}\}$ and any fixed $x_0 \in K$,

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0. \tag{1.3}$$

In this paper, we prove several weak and strong convergence results by using a new conception of a uniformly asymptotically regular sequence $\{T_n\}$ of nonexpansive mappings. Our results is new also even if in a Hilbert space.

2. Preliminaries

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$ and J denotes the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\| \, \|f\|, \|x\| = \|f\|\}, \forall\, x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of a Banach space E is said *Fréchet differentiable* if, for any $x \in S(E)$, the unit sphere of E, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $y \in S(E)$. In this case, we have

$$\|x\|^2 + 2\langle h, J(x)\rangle \leq \|x + h\|^2 \leq \|x\|^2 + 2\langle h, J(x)\rangle + g(\|h\|)$$

for all $x, h \in E$, where $g(\cdot)$ is a function defined on $[0, \infty)$ such that $\lim_{t\to 0} \frac{g(t)}{t} = 0$ ([15]).

A Banach space E is said (i) *strictly convex* if $\|x\| = \|y\| = 1$, $x \neq y$ implies $\frac{\|x+y\|}{2} < 1$; (ii) *uniformly convex* if for all $\varepsilon \in [0, 2]$, $\exists \delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon \text{ whenever } \|x - y\| \geq \varepsilon.$$

It is well known that a uniformly convex Banach space E is reflexive and strictly convex [14, Theorem 4.1.6, 4.1.2].

In uniform convex Banach space, Reich [9] proved the following result which also is found in Tan and Xu [15, Lemma 4, Theorem 1].

Lemma 2.1 (Reich [9, Proposition]). *Let C be a closed convex subset of a uniform convex Banach space E, and let $\{T_n; n \geq 1\}$ be a sequence of nonexpansive self-mappings of C with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. If $x_1 \in C$ and $x_{n+1} = T_n x_n$ for $n \geq 1$, then for all $f_1, f_2 \in F$ and $t \in (0,1)$,*

(i) $\lim_{n\to\infty} \|tx_n - (1-t)f_1 - f_2\|$ *exists;*

(ii) *If the norm of E is also Fréchet differentiable, then* $\lim_{n\to\infty}\langle x_n, j(f_1 - f_2)\rangle$ *exists.*

The following Lemma can be found in [16, Theorem 2].

Lemma 2.2. *Let $q > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^q \leq \lambda\|x\|^q + (1-\lambda)\|y\|^q - \omega_q(\lambda)g(\|x-y\|)$$

for all $x, y \in B_r(0) = \{x \in E; \|x\| \leq r\}$ and $\lambda \in [0,1]$, where $\omega_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

Note that the inequality in Lemma 2.2 is known as *Xu's inequality.*

Now, we introduce the concept of asymptotically regular sequence of mappings and uniformly asymptotically regular sequence of mappings, respectively. Let C be a nonempty closed convex subset of a Banach space E, and $T_n : C \to C,\ \ n \geq 1$, then the mapping sequence $\{T_n\}$ is said *asymptotically regular* (in short, a.r.) if for all $m \geq 1$,

$$\lim_{n\to\infty} \|T_m(T_n x) - T_n x\| = 0, \quad \forall x \in C.$$

The mapping sequence $\{T_n\}$ is said *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $m \geq 1$,

$$\lim_{n\to\infty} \sup_{x\in C} \|T_m(T_n x) - T_n x\| = 0.$$

The following lemma was proved by Bruck in [3, 4].

Lemma 2.3 (Bruck [4]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \to C$ be nonexpansive. For each $x \in C$, if we define $T_n x = \frac{1}{n}\sum_{j=0}^{n-1} T^j x$, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|T_n x - T(T_n x)\| = 0.$$

Lemma 2.3 has been extended to a pair of mappings [11, Lemma 1].

Lemma 2.4 (Shimizu and Takahashi [11, Lemma 1]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $T, S : C \to C$ be two nonexpansive mappings such that $ST = TS$. For each $x \in C$, if we define*

$$T_n x = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j x,$$

then

$$\lim_{n\to\infty}\sup_{x\in C}\|T_nx-T(T_nx)\|=0 \text{ and } \lim_{n\to\infty}\sup_{x\in C}\|T_nx-S(T_nx)\|=0.$$

It is easily seen that the mapping sequence $\{T_n\}$ appeared in Lemma 2.4 and in Lemma 2.3 is u.a.r.. For more detail, see Refs. [12, 13, Examples].

Let K be a closed subset of a Banach space E. A mapping $T: K \to K$ is said *semicompact*, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ $(i \to \infty)$.

A Banach space E satisfies *Opial's condition* if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|, \forall y\in E \text{ with } x\neq y.$$

A Banach space E have the *Kadec-Klee property* if every sequence $\{x_n\}$ in E, as $n \to \infty$, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$. We know that dual of reflexive Banach spaces with Fréchet differentiable norms have the Kadec-Klee property (see [5]). But there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norms nor the Opial property but their dual have the Kadec-Klee property [5, Example 3.1].

In the sequel, we also need the following lemmas.

Lemma 2.5 (Browder [2]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Suppose $T: C \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, i.e.,*

$$x_n \rightharpoonup x, x_n - Tx_n \to 0 \text{ implies } x = Tx.$$

Lemma 2.6 ([5, Lemma 3.2]). *Let E be a uniformly convex Banach space such that its dual E^* has the Kadec-Klee property. Suppose $\{x_n\}$is a bounded sequence in E and $f_1, f_2 \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the weak limit set of $\{x_n\}$. If $\lim_{n\to\infty}\|tx_n + (1-t)f_1 - f_2\|$ exists for all $t \in [0,1]$, then $f_1 = f_2$.*

3. Main results

At first, we will show the approximating fixed point of a uniformly asymptotically regular sequence for nonexpansive self-mappings defined on a nonempty closed convex subset K of Banach space E.

Theorem 3.1. *Let K be a nonempty closed convex subset of Banach space E, and $\{T_n\}$ $(n = 1, 2, \ldots)$ is uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$, define*

$$x_{n+1} = \lambda_{n+1}x_n + (1-\lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

$\lambda_n \subset [0,1]$ and for any given $p \in F$, then

(i) *$\lim_{n\to\infty}\|x_n - p\|$ exists and $\{x_n\}$ is bounded;*

(ii) *If* $\lim_{n\to\infty}\lambda_n = 0$, *then for any fixed* $m \geq 1$,

$$\lim_{n\to\infty}\|x_n - T_m x_n\| = 0;$$

(iii) *If* E *is uniformly convex and* $\lambda_n \in [a, b]$ $(0 < a \leq b < 1)$, *then for any fixed* $m \geq 1$,

$$\lim_{n\to\infty}\|x_n - T_m x_n\| = 0.$$

Proof. (i) Take $p \in F$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\lambda_{n+1}(x_n - p) + (1 - \lambda_{n+1})(T_{n+1}x_n - p)\|\\ &\leq \lambda_{n+1}\|x_n - p\| + (1 - \lambda_{n+1})\|T_{n+1}x_n - p\|\\ &\leq \|x_n - p\|\\ &\vdots\\ &\leq \|x_0 - p\|.\end{aligned}$$

Therefore, $\{\|x_n - p\|\}$ is non-increasing and bounded below, and that (i) is proved.

(ii) Since $\{x_n\}$ is bounded by (i), then we get the boundedness of $\{T_{n+1}x_n\}$ from

$$\|T_{n+1}x_n\| \leq \|T_{n+1}x_n - p\| + \|p\| \leq \|x_n - p\| + \|p\|.$$

Using the condition $\lim_{n\to\infty}\lambda_n = 0$, we obtain that

$$\|x_{n+1} - T_{n+1}x_n\| = \lambda_{n+1}\|x_n - T_{n+1}x_n\| \to 0 \ (n \to \infty). \tag{3.1}$$

As $\{T_n\}$ $(n = 1, 2, \ldots)$ is uniformly asymptotically regular sequence of nonexpansive mapping, then for all $m \geq 1$,

$$\lim_{n\to\infty}\|T_m(T_{n+1}x_n) - T_{n+1}x_n\| \leq \lim_{n\to\infty}\sup_{x\in C}\|T_m(T_{n+1}x) - T_{n+1}x\| = 0, \tag{3.2}$$

where C is any bounded subset of K containing $\{x_n\}$. Thus,

$$\begin{aligned}\|x_{n+1} - T_m x_{n+1}\| &\leq \|x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_m(T_{n+1}x_n)\|\\ &\quad + \|T_m(T_{n+1}x_n) - T_m x_{n+1}\|\\ &\leq 2\|x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_m(T_{n+1}x_n)\|.\end{aligned}$$

By (3.1) and (3.2), we have

$$\lim_{n\to\infty}\|x_n - T_m x_n\| = 0.$$

(iii) As E is uniformly convex and $\{x_n\}$ is bounded, by Lemma 2.2, we take $q = 2$ and $r \geq \sup_{n\in\mathbb{N}}\|x_n\|$,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\lambda_{n+1}(x_n - p) + (1 - \lambda_{n+1})(T_{n+1}x_n - p)\|^2\\ &\leq \lambda_{n+1}\|x_n - p\|^2 + (1 - \lambda_{n+1})\|T_{n+1}x_n - p\|^2\\ &\quad - \lambda_{n+1}(1 - \lambda_{n+1})g(\|x_n - T_{n+1}x_n\|)\\ &\leq \|x_n - p\|^2 - \lambda_{n+1}(1 - \lambda_{n+1})g(\|x_n - T_{n+1}x_n\|).\end{aligned}$$

Hence, we get

$$\begin{aligned} a(1-b)g(\|x_{n+1}-T_{n+1}x_n\|) &\le \lambda_{n+1}(1-\lambda_{n+1})g(\|x_n-T_{n+1}x_n\|) \\ &\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2. \end{aligned}$$

By (i) $\lim_{n\to\infty}\|x_n-p\|$ exists, we have

$$a(1-b)g(\|x_n-T_{n+1}x_n\|)\to 0\ (n\to\infty).$$

Since $g:[0,+\infty)\to[0,+\infty)$ is a continuous strictly increasing convex function such that $g(0)=0$, then

$$\lim_{n\to\infty}\|x_n-T_{n+1}x_n\|=0. \tag{3.3}$$

Consequently, for all $m\ge 1$

$$\begin{aligned} \|x_n-T_mx_n\| &\le \|x_n-T_{n+1}x_n\| + \|T_{n+1}x_n - T_m(T_{n+1}x_n)\| \\ &\quad + \|T_m(T_{n+1}x_n)-T_mx_n\| \\ &\le 2\|x_n-T_{n+1}x_n\| + \|T_{n+1}x_n - T_m(T_{n+1}x_n)\|. \end{aligned}$$

Combining (3.3) and (3.2), we have

$$\lim_{n\to\infty}\|x_n-T_mx_n\|=0.$$

The proof is complete. □

Theorem 3.2. *Let E be a reflexive Banach space which satisfies Opial's condition, and K be a nonempty closed convex subset of E. Suppose $\{T_n\}$ ($n=1,2,\ldots$) is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F:=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. For $x_0\in K$ define*

$$x_{n+1}=\lambda_{n+1}x_n+(1-\lambda_{n+1})T_{n+1}x_n,\quad n\ge 0.$$

If $\lambda_n\subset[0,1]$ such that $\lim_{n\to\infty}\lambda_n=0$, then as $n\to\infty$, $\{x_n\}$ weakly converges to some common fixed point x^* of $\{T_n\}$.

Proof. By Theorem 3.1 (i) and (ii), we have $\{x_n\}$ is bounded and for any fixed $m\ge 1$,

$$\lim_{n\to\infty}\|x_n-T_mx_n\|=0.$$

We may assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ by the reflexivity of E and the boundedness of $\{x_n\}$. We claim that $x^*=T_mx^*$. Indeed, suppose $x^*\neq T_mx^*$, from E satisfying the Opial's condition, we obtain that

$$\begin{aligned} \limsup_{k\to\infty}\|x_{n_k}-x^*\| &< \limsup_{k\to\infty}\|x_{n_k}-T_mx^*\| \\ &\le \limsup_{k\to\infty}(\|x_{n_k}-T_mx_{n_k}\| + \|T_mx_{n_k}-T_mx^*\|) \\ &\le \limsup_{k\to\infty}\|x_{n_k}-x^*\|. \end{aligned}$$

This is a contradiction, therefore $x^*=T_mx^*$. Since $m\ge 1$ is arbitrary, then $x^*\in F$.

Now we prove $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_n\}$ doesn't converge weakly to x^*. Then there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which weakly converges to some $y \neq x^*$, $y \in K$. We also have $y \in F$. Because $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$ and E satisfies the Opial's condition, thus

$$\begin{aligned}\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{k\to\infty} \|x_{n_k} - x^*\| < \lim_{k\to\infty} \|x_{n_k} - y\| \\ &= \lim_{j\to\infty} \|x_{n_j} - y\| < \lim_{j\to\infty} \|x_{n_j} - x^*\| \\ &= \lim_{n\to\infty} \|x_n - x^*\|.\end{aligned}$$

Which is a contradiction, we must have $y = x^*$. Hence, $\{x_n\}$ converges weakly to $x^* \in F$. The proof is complete. □

Using the same methods as Theorem 3.2, we can easily obtain the following theorem.

Theorem 3.3. *Let E be a uniformly convex Banach space which satisfies the Opial's condition, and K be a nonempty closed convex subset of E. Suppose $\{T_n\}$ $(n = 1, 2, \dots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

If $\lambda_n \subset [0, 1]$ such that $\lim_{n\to\infty} \lambda_n = 0$ or $\lambda_n \in [a, b]$ $(0 < a \leq b < 1)$, then as $n \to \infty$, $\{x_n\}$ weakly converges to some common fixed point x^* of $\{T_n\}$.

Theorem 3.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, and K be a nonempty closed convex subset of E. Suppose $\{T_n\}$ $(n = 1, 2, \dots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

If $\lambda_n \subset [0, 1]$ such that $\lim_{n\to\infty} \lambda_n = 0$ or $\lambda_n \in [a, b]$ $(0 < a \leq b < 1)$, then as $n \to \infty$, $\{x_n\}$ weakly converges to some common fixed point x^* of $\{T_n\}$.

Proof. Theorem 3.1 guarantees $\{x_n\}$ is bounded and for any fixed $m \geq 1$,

$$\lim_{n\to\infty} \|x_n - T_m x_n\| = 0.$$

Since E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $x^* \in K$ and $\lim_{k\to\infty} \|x_{n_k} - T_m x_{n_k}\| = 0$. By Lemma 2.5, we have $x^* \in F(T_m)$. Since $m \geq 1$ is arbitrary, then $x^* \in F$.

Now we prove $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_n\}$ doesn't converge weakly to x^*. Then there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which weakly converges to some $y \in K$. We also have $y \in F$. Next we show $x^* = y$.

Set $A_n = \lambda_{n+1} I + (1 - \lambda_{n+1}) T_{n+1}, n \geq 0$, then it is clear that $\{A_n\}$ is a sequence of nonexpansive self-mappings of K with $F = \bigcap_{n=0}^{\infty} F(A_n) =$

$\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $x_{n+1} = A_n x_n$. Therefore, Lemma 2.1(ii) assures that $\lim_{n\to\infty}\langle x_n, j(x^* - y)\rangle$ exists. Hence, we have

$$\lim_{n\to\infty}\langle x_n, j(x^* - y)\rangle = \lim_{k\to\infty}\langle x_{n_k}, j(x^* - y)\rangle = \langle x^*, j(x^* - y)\rangle,$$

and

$$\lim_{n\to\infty}\langle x_n, j(x^* - y)\rangle = \lim_{l\to\infty}\langle x_{n_l}, j(x^* - y)\rangle = \langle y, j(x^* - y)\rangle.$$

Consequently,

$$\langle x^*, j(x^* - y)\rangle = \langle y, j(x^* - y)\rangle,$$

that is $\|x^* - y\| = 0$. We must have $y = x^*$. Thus $\{x_n\}$ converges weakly to $x^* \in F$. The proof is complete. □

Theorem 3.5. *Let E be a uniformly convex Banach space and its dual E^* have the Kadec-Klee property, and K be a nonempty closed convex subset of E. Suppose $\{T_n\}$ $(n = 1, 2, \dots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

If $\lambda_n \subset [0,1]$ such that $\lim_{n\to\infty}\lambda_n = 0$ or $\lambda_n \in [a,b]$ $(0 < a \leq b < 1)$, then as $n \to \infty$, $\{x_n\}$ weakly converges to some common fixed point x^* of $\{T_n\}$.

Proof. As in the proof of Theorem 3.3, we can reach the following objectives:

(1) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $x^* \in F$;

(2) the nonexpansive self-mappings sequence $\{A_n\}$ satisfies the conditions of Lemma 2.1.

Now we prove $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_n\}$ doesn't converge weakly to x^*. Then there exists another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ which weakly converges to some $y \in K$. We also have $y \in F$. Next we show $x^* = y$.

In fact, from Lemma 2.1(i), we have $\lim_{n\to\infty}\|tx_n - (1-t)x^* - y\|$ exists. Using Lemma 2.6 we obtain $y = x^*$. Thus $\{x_n\}$ converges weakly to $x^* \in F$. □

Corollary 3.6. *Let $\lambda_n \in [0,1]$ satisfy $\lim_{n\to\infty}\lambda_n = 0$ or $\lambda_n \in [a,b]$ $(0 < a \leq b < 1)$. Suppose K is a nonempty closed convex subset of a Banach space E, and let $S, T : K \to K$ be nonexpansive mappings with fixed points.*

(a) *Set $T_n x = \frac{1}{n}\sum_{j=0}^{n-1} T^j x$ and $x \in K$, for $x_0 \in K$ and $u \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

If E is uniformly convex Banach space which satisfies the Opial's condition or has Fréchet differentiable norm or its dual E^* have the Kadec-Klee property. Then as $n \to \infty$, $\{x_n\}$ weakly converges to some fixed point x^* of T.

(b) *Set $T_n x = \frac{2}{n(n+1)}\sum_{k=0}^{n-1}\sum_{i+j=k} S^i T^j x$ for $n \geq 1$ and $x \in K$. For $x_0 \in K$ and $u \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

Suppose that $ST = TS$ and $F(T) \bigcap F(S) \neq \emptyset$, and E a Hilbert space. Then as $n \to \infty$, $\{x_n\}$ weakly converges to some common fixed point x^* of T and S.

Proof. In case (a), take $w \in F(T)$ and define a subset D of K by

$$D = \{x \in K : \|x - w\| \leq r\},$$

where $r = \|w - x_0\|$. Then D is a nonempty closed bounded convex subset of K and $T(D) \subset D$ and $\{x_n\}, \{T_{n+1}x_n\} \subset D$. Also Lemma 2.4 implies

$$\lim_{n\to\infty} \sup_{x\in D} \|T_n x - T(T_n x)\| = 0, \tag{3.4}$$

and $\{T_n\}$ is an uniformly asymptotically regular sequence of nonexpansive mappings on D (see example in Preliminaries or Refs. [12, 13, Example]). It is clear that $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$ (using (3.4) and $T_n = \frac{1}{n}\sum_{j=0}^{n-1} T^j$). Consequently, using the same proof as Theorem 3.2, Theorem 3.3, and Theorem 3.4, we can obtain that $\{x_n\}$ weakly converges to $x^* \in P_{F_D(T)} \subset F(T)$, where $F_D(T) = \{x \in D : Tx = x\}$.

As for case (b). Let $w \in F(T) \bigcap F(S)$, using a similar argument to that of case (a) we find a nonempty closed bounded convex subset D of K and $T(D) \subset D$ and $S(D) \subset D$. Also Lemma 2.5 implies

$$\lim_{n\to\infty} \sup_{x\in D} \|T_n x - T(T_n x)\| = 0 \text{ and } \lim_{n\to\infty} \sup_{x\in D} \|T_n x - S(T_n x)\| = 0. \tag{3.5}$$

and $\{T_n\}$ is a uniformly asymptotically regular sequence of nonexpansive mappings on D (see example in Preliminaries or Refs. [12, 13, Example]). It is clear that $F(T) \bigcap F(S) = \bigcap_{n=0}^{\infty} F(T_n)$ (using (3.5) and $T_n = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j$). The reminder of the proof is the same as case (a), we can easily get the results. We omit it. □

Theorem 3.7. *Let K be a nonempty compact convex subset of Banach space E. Suppose $\{T_n\}$ $(n = 1, 2, \dots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$ define*

$$x_{n+1} = \lambda_{n+1}x_n + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0.$$

(i) *If $\lambda_n \subset [0, 1]$ such that $\lim_{n\to\infty} \lambda_n = 0$, then as $n \to \infty$, $\{x_n\}$ strongly converges to some common fixed point z of $\{T_n\}$.*

(ii) *If E is uniformly convex and $\lambda_n \in [a, b]$ $(0 < a \leq b < 1)$, then as $n \to \infty$, $\{x_n\}$ strongly converges to some common fixed point z of $\{T_n\}$.*

Proof. (i) By Theorem 3.1(i) and the compactness of K, we see that $\{x_n\}$ admits a strongly convergent subsequence $\{x_{n_k}\}$ whose limit we shall denote by z. Then, again by Theorem 3.1(ii), we have $z \in F(T_m)$ $(\forall m \in \mathbb{N})$. Since m is arbitrary, then $z \in F$. As $\forall p \in F$, $\lim_{n\to\infty} \|x_n - p\|$ exists by Theorem 3.1(i), z is actually the strong limit of the sequence $\{x_n\}$ itself.

(ii) Using the same method as (i), we can easily obtain the result, so we omit it. □

From the proof of Theorem 3.7, we can get the following corollary.

Corollary 3.8. *Let K be a nonempty closed convex subset of Banach space E. Suppose $\{T_n\}$ $(n = 1, 2, \ldots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For $x_0 \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

If λ_{n+1} is the same as Theorem 3.7, then as $n \to \infty$, $\{x_n\}$ strongly converges to some common fixed point z of $\{T_n\}$ if and only if there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to z \in F$ $(k \to \infty)$.

Theorem 3.9. *Let K be a nonempty closed convex subset of Banach space E. Suppose $\{T_n\}$ $(n = 1, 2, \ldots)$ is a uniformly asymptotically regular sequence of nonexpansive mappings from K to K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and there exists one map $T \in \{T_n; n = 1, 2, \ldots\}$ to be semicompact. For $x_0 \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

(i) *If $\lambda_n \subset [0, 1]$ such that $\lim_{n\to\infty} \lambda_n = 0$, then as $n \to \infty$, $\{x_n\}$ strongly converges to some common fixed point z of $\{T_n\}$.*

(ii) *If E is uniformly convex and $\lambda_n \in [a, b]$ $(0 < a \leq b < 1)$, then as $n \to \infty$, $\{x_n\}$ strongly converges to some common fixed point z of $\{T_n\}$.*

Proof. (i) By the hypotheses that there exists one map $T \in \{T_n; n = 1, 2, \ldots\}$ to be semicompact, we may assume that T_1 is semicompact without loss of generality. By Theorem 3.1 (i) and (ii), we see that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Using the definition of semicompact, then $\{x_n\}$ admits a strongly convergent subsequence $\{x_{n_k}\}$ whose limit we shall denote by z. It follows from Corollary 3.8 that the ultimateness are reached.

(ii) Using the same method as (i), we can easily obtain the result, we omit it. □

Corollary 3.10. *Let $\lambda_n \in [0, 1]$ satisfy $\lim_{n\to\infty} \lambda_n = 0$ or $\lambda_n \in [a, b]$ $(0 < a \leq b < 1)$. Suppose K is a nonempty closed convex subset of a Banach space E, and let $S, T : K \to K$ be semicompact nonexpansive mappings with fixed points.*

(a) *Set $T_n x = \frac{1}{n} \sum_{j=0}^{n-1} T^j x$ and $x \in K$, for $x_0 \in K$ and $u \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

If E is uniformly convex Banach space. Then as $n \to \infty$, $\{x_n\}$ strongly converges to some fixed point z of T.

(b) *Set $T_n x = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j x$ for $n \geq 1$ and $x \in K$. For $x_0 \in K$ and $u \in K$ define*

$$x_{n+1} = \lambda_{n+1} x_n + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0.$$

Suppose that $ST = TS$ and $F(T) \bigcap F(S) \neq \emptyset$, and E a Hilbert space. Then as $n \to \infty$, $\{x_n\}$ strongly converges to some common fixed point z of T and S.

Remark. The condition *semicompact* in Theorem 3.9 can be replaced by one of the following conditions, the result still holds.

(1) there exists one map $T \in \{T_n; n = 1, 2, \dots\}$ to satisfy Condition A ([15]), i.e., there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T))$$

for all $x \in K$, where $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

(2) there exists one map $T \in \{T_n; n = 1, 2, \dots\}$ such that $T(K)$ is contained in a compact subset of E.

(3) there exists one map $T \in \{T_n; n = 1, 2, \dots\}$ such that

$$\lim_{n \to \infty} d(x_n, F(T)) = 0.$$

References

[1] S. Atsushiba and W. Takahashi, *Strong convergence of Mann's-type iterations for nonexpansive semigroups in general Banach spaces*, Nonlinear Anal. **61** (2005), no. 6, 881–899.

[2] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[3] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), no. 2-3, 107–116.

[4] ———, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1981), no. 4, 304–314.

[5] J. G. Falset, W. Kaczor, T. Kuczumow, and S. Reich, *Weak convergence theorems for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal. **43** (2001), no. 3, Ser. A: Theory Methods, 377–401.

[6] J. S. Jung, *Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **302** (2005), no. 2, 509–520.

[7] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[8] J. G. O'Hara, P. Pillay, and H. K. Xu, *Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces*, Nonlinear Anal. **54** (2003), no. 8, 1417–1426.

[9] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), no. 2, 274–276.

[10] ———, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), no. 1, 287–292.

[11] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), no. 1, 71–83.

[12] Y. Song and R. Chen, *Iterative approximation to common fixed points of nonexpansive mapping sequences in reflexive Banach spaces*, Nonlinear Anal. **66** (2007), no. 3, 591–603.

[13] Y. Song, R. Chen, and H. Zhou, *Viscosity approximation methods for nonexpansive mapping sequences in Banach spaces*, Nonlinear Anal. **66** (2007), no. 5, 1016–1024.

[14] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[15] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), no. 2, 301–308.

[16] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), no. 12, 1127–1138.

Yisheng Song
College of Mathematics and Information Science
Henan Normal University
XinXiang, 453007 P. R. China
E-mail address: songyisheng123@yahoo.com.cn or songyisheng123@163.com

Rudong Chen
College of Mathematics and Information Science
Henan Normal University
XinXiang, 453007 P. R. China