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PROJECTIVE DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUPS I

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Abstract. Most of domains people have studied are convex bounded projective (or affine) domains. Edith Socié-Méthou [15] characterized ellipsoid in $\mathbb{R}^n$ by studying projective automorphism of convex body. In this paper, we showed convex and bounded projective domains can be identified from local data of their boundary points using scaling technique developed by several mathematicians. It can be found that how the scaling technique combined with properties of projective transformations is used to do that for a projective domain given local data around singular boundary point. Furthermore, we identify even unbounded or non-convex projective domains from its local data about a boundary point.

1. Introduction

In (G, X)-manifold theory, the developing map plays a crucial role since it is essentially a global chart map defined on the universal covering space of the manifold. Through the developing map, the covering action induces the holonomy group action on the developing image equivariantly and the holonomy group $H \subset G$ acts syndetically as an automorphism group when the (G, X)-manifold is compact. Therefore, the study of a syndetic domain which has a compact generating subset by automorphism group is directly related to the study of a compact (G, X)-manifold.

Kuiper [12] divided the convex subsets of S^n which is the double covering of the projective space $\mathbb{R}P^n$ into three types C^a, C^b and C^c-sets. In affine domains, Vey [16] showed that a convex saillant (which means that there is no affine full line) syndetic domain is a cone if the action is properly discontinuous. For projective domains, Benzécri [3] gave a deep and important description of convex bounded syndetic domains. He used the method which might be a source of inspiration of the "scaling method" so called by geometric analysts. In several complex variables, the scaling method was initiated by S. Pinchuk in the late 1970's. We can see the idea of the Pinchuk scaling technique in

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[14] and an example of the scaling method in [1], [2], [9] in several complex variables, when the automorphism group is non-compact.

The intrinsic metrics such as Hilbert metric, Kobayashi metric, and affine invariant metric defined by Vey, play a very important role in convex bounded or saillant projective domains as well as in several variable complex domains. As we see in papers like [3], [4], [11], [16], these metrics are crucial, for example, in the proof of Vey's theorems. Many results Benzécri proved in [3] can be drawn using the scaling method for the projective domains having non-compact automorphism groups without assuming that the domain is syndetic. Furthermore, we sometimes can handle even unbounded domains and non-convex domains which may contain an affine full line. This is one of the main theme in this paper.

The idea of the proof in Kim's book [10] of the Ball Characterization Theorem (Wong-Rosay Theorem) using the scaling method is that if one knows the local boundary equation(s) of a domain Ω, then Ω is determined by these equation(s) using the scaling method. We can apply the scaling method used in the study of complex domains to the projective domains and obtain some results about convex projective domains using affine scaling sequence together with Benzécri's theorem [3]. In the case of non-convex or unbounded, i.e., non-hyperbolic domains, the intrinsic metrics mentioned above cannot be used for these domains since they are not distance any more. Without using these intrinsic metrics, we'll scale the projective domains with a local description at a boundary point along with the simplex at another boundary point or interior point to find out that the whole original domain is really the domain described by the local equation(s). The following result proved using scaling technique may be proved in a different way.

Theorem. *Suppose that there is $q \in \Omega$ and $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$ such that*

$$\lim_{\nu \to \infty} \varphi_\nu(q) = p \in \partial\Omega.$$

If $\partial\Omega$ is C^2 in a neighborhood of p which is strictly convex, then Ω is projectively equivalent to a paraboloid, i.e., a ball.

The condition of this theorem is weaker than Edith Socié-Méthou's result [15] which assumes Hessian is non-degenerate at every boundary point. However, we cannot apply the scaling method directly to an unbounded domain but should observe how points converge to the accumulation point(s) carefully. Taking care of convergence, we can generalize the above theorem to the unbounded or non-convex domains:

Theorem. *Let $\{h(\Omega), G^h\}$ be an h-normalization of Ω of dimension n and $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$. Suppose that $\partial\Omega$ is C^2 in a neighborhood of $q \in \partial\Omega$ with $R(\varphi) = \{q\}$. If ${}^t M_{\varphi^h} H_p M_{\varphi^h} \neq 0$ on $\mathbb{R}^n$, where H_p is the Hessian matrix at $p = h_2(q) = \Pi(\mathbf{e}_{n+1})$, then Ω is projectively equivalent to one of quadratic domains $Q(b_2, \dots, b_n)$.*

As an application of this theorem, we get the following result for unbounded and non-convex domains.

Corollary. *Let $\Omega \subset \mathbb{R}P^n$ be a domain whose boundary is C^2 and not quadratic. If $\{\varphi_\nu\}$ is a sequence of automorphisms whose limit has a range of one point, then every point in an open dense subset converges to the boundary accumulation point q along degenerate directions at q.*

Section 2 contains basic concepts in projective geometry and basic properties of non-compact automorphism group. In Section 3, we introduce a kind of "normalization" of domain and its automorphism sequence to handle domain and its local equations in a convenient way and showed a few theorems which is used to prove main theorems. We proved the Ball Characterization Theorem for projective domains in Section 4. In Section 5, we applied the technique in Section 4 to domains with a local C^2 boundary equation at a boundary point without hyperbolic assumption.

2. Basic property of non-compact automorphism groups

We'll summarize some basic facts developed by Myrberg and then used by Kuiper and Benzécri in the study of projective domains and its automorphism groups. We will also discuss a basic theorems obtained as a consequence of non-compactness of the automorphism group.

A projective domain Ω of dimension n is an open subset of $\mathbb{R}P^n$. For a given projective domain Ω, $\mathrm{Aut}(\Omega)$ is the group of all projective transformation which preserves Ω, i.e., $\mathrm{Aut}(\Omega) = \{f \in PGL(n+1,\mathbb{R}) \mid f(\Omega) = \Omega\}$. Since $P\mathfrak{gl}(n+1,\mathbb{R})$ is a compactification of $PGL(n+1,\mathbb{R})$, there exists a convergent subsequence of $\{f_j\}$ for any sequence $\{f_j\} \subset PGL(n+1,\mathbb{R})$ whose limit is, in general, in $P\mathfrak{gl}(n+1,\mathbb{R})$.

Definition 1. We say that an element $f \in P\mathfrak{gl}(n+1,\mathbb{R})$ is *singular* if $f \in P\mathfrak{gl}(n+1,\mathbb{R}) - PGL(n+1,\mathbb{R})$.

The *kernel* $K(f)$ and the *range* $R(f)$ is the projectivization of the kernel and image of a representative matrix of a singular element f, respectively. Thus the domain of a singular element f is $\mathbb{R}P^n - K(f)$. Suppose that $f = \lim_{j\to\infty} f_j$ is singular. Choose a subsequence of $\{f_j^{-1}\}$ which converges to an element $g \in P\mathfrak{gl}(n+1,\mathbb{R})$. Then $R(f) \subset K(g)$ and $R(g) \subset K(f)$ (See [8]). The following proposition is not hard to show, so we'll not prove it.

Proposition 1. *Let $\gamma \in PGL(n+1,\mathbb{R})$ and $f \in P\mathfrak{gl}(n+1,\mathbb{R})$. Then*

(1) $K(f\gamma) = \gamma^{-1}K(f)$.
(2) $R(f) = R(f\gamma)$.
(3) $K(f) = K(\gamma f)$.
(4) $\gamma R(f) = R(\gamma f)$.

Remark. In (1), when f is a singular projective transformation, we can think of $f\gamma$ as a singular projective transformation given by the product of any representatives of f and γ.

General relations of range or kernel with a sequence of projective transformations were observed by Benzécri.

Theorem 1 (Benzécri). (1) *For any compact* $K \subset \mathbb{R}P^n - K(f)$, $f_j \mid_K$ *converges uniformly to* $f \mid_K$.

(2) *For any compact* $K \subset \mathbb{R}P^n - R(f)$, *and for any neighborhood* N *of* $K(f)$, *there exists* j_0 *such that* $f_j^{-1}(K) \subset N$ *for all* $j > j_0$.

(3) *For any open* U *with* $U \cap K(f) \neq \emptyset$, *there exists a compact subset* $B \subset U$ *such that* $\lim_{j\to\infty} f_j(B) = R(f)$.

(4) *Let* U *be an open set with* $U \cap R(f) \neq \emptyset$ *and* K *be a compact subset of* $f^{-1}(U \cap R(f))$ *and* $K \cap K(f) = \emptyset$. *Then there exists* j_0 *such that* $K \subset f_j^{-1}(U)$ *for all* $j > j_0$.

Proof. See Benzécri's paper [3]. □

Let Ω_1 and Ω_2 be open domains in $\mathbb{R}P^n$. We'll call (Ω_1, Ω_2) a *pair* if $\partial\Omega_1 = \partial\Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1^c = \overline{\Omega}_2$. Clearly, $\Omega_1 \cup \Omega_2 \cup \partial\Omega_1 = \mathbb{R}P^n$. Due to the above Theorem 1, we can get the following theorem. The essential content of the theorem must be well known but we state and prove in the useful form for our purpose.

Theorem 2. *Let* Ω *be a domain of a pair and its automorphism group* $\mathrm{Aut}(\Omega)$ *be non-compact. Then there exist* $q \in \Omega$, $p \in \partial\Omega$, *and a sequence* $\{f_j\} \subset \mathrm{Aut}(\Omega)$ *such that* $\lim_{j\to\infty} f_j(q) = p$ *with* $q \notin K(f)$, *where* $f = \lim_{j\to\infty} f_j$ *is singular.*

Remark. We will call such a point $p \in \partial\Omega$ a *boundary accumulation point.*

Proof. Being $\mathrm{Aut}(\Omega)$ non-compact, there is a sequence $\{f_j\}$ whose limit is singular. Let $g = \lim_{j\to\infty} f_j^{-1}$ by extracting a subsequence. Clearly, g is singular, if not, f is not singular. Let Ω, Ω' be the pair. We can consider three cases:

(i) $K(f) \subset \partial\Omega$,
(ii) $K(f) \cap \Omega \neq \emptyset$, and
(iii) $K(f) \cap \Omega' \neq \emptyset$.

(i) $K(f) \subset \partial\Omega$. Let $x \in K(f)$ and U be any neighborhood of x. Then there exists a compact subset $B \subset U$ such that $B \cap K(f) = \emptyset$, $B \cap \Omega \neq \emptyset$, $B \cap \Omega' \neq \emptyset$, and $\lim_{j\to\infty} f_j(B) = R(f)$. Now, we have two cases to be considered.

(a) $R(f) \cap \Omega = \emptyset$. Clearly, $R(f) \subset \overline{\Omega'}$. We see that $\lim_{j\to\infty} f_j(B \cap \Omega) \subset R(f) \subset \overline{\Omega'}$. Due to $\lim_{j\to\infty} f_j(B \cap \Omega) \subset \overline{\Omega}$, it follows that $\lim_{j\to\infty} f_j(B \cap \Omega) \subset \overline{\Omega} \cap \overline{\Omega'} = \partial\Omega$. Hence $f_j(q) \longrightarrow p$ for some $p \in \partial\Omega$ and $q \in B \cap \Omega \subset \Omega$ with $q \notin K(f)$.

(b) $R(f) \cap \Omega \neq \emptyset$. Let $y \in R(f) \cap \Omega \subset \Omega$. Since $R(f) \subset K(g)$, we have $y \in K(g)$. Take any neighborhood A of y contained in Ω. From theorem 1, there is a compact subset $B \subset A$ such that $B \cap K(g) = \emptyset$ and $\lim_{j\to\infty} f_j^{-1}(B) = R(g)$ which is contained in $K(f) \subset \partial\Omega$. Hence $f_j^{-1}(q) \to p$ for some $p \in \partial\Omega$ and $q \in B \subset \Omega$ with $q \notin K(g)$.

(ii) $K(f) \cap \Omega \neq \emptyset$. For any $y \in \Omega' - K(f)$, we see that $f_j(y) \to f(y) \in \overline{\Omega'} \cap R(f)$. This implies $R(f) - \Omega$ is nonempty. Take $x \in K(f) \cap \Omega$ and an open neighborhood U of x contained in Ω. There exists a compact subset $B \subset U$ so that $R(f) = \lim_{j\to\infty} f_j(B) \subset \overline{\Omega}$. Hence the result follows.

(iii) $K(f) \cap \Omega' \neq \emptyset$. Change the role of Ω and Ω' and use the same argument as the proof of (ii) □

3. h-normalization

We will define an "*h-normalization*" of a sequence $\{\varphi_\nu\}$ in $\mathrm{Aut}(\Omega)$, where $\mathrm{Aut}(\Omega) \subset PGL(n+1, \mathbb{R})$ is the automorphism group of a projective domain Ω in $\mathbb{R}P^n$. If φ is the limit of a subsequence of $\{\varphi_\nu\}$, an h-normalization makes a representative matrix of the limit φ diagonal and decomposes the kernel $K(\varphi)$ and range $R(\varphi)$ of the limit φ so that they are disjoint when φ is singular.

Let $A \in GL(n+1, \mathbb{R})$. It is a well-known fact that A has a polar decomposition BC, where B is a positive definite symmetric matrix and $C \in O(n+1, \mathbb{R})$. In addition, B is similar to a diagonal matrix by an orthogonal matrix $P \in O(n+1, \mathbb{R})$. Combining these, one has $A = PDQ$, where $P, Q \in O(n+1, \mathbb{R})$ and $D \in GL(n+1, \mathbb{R})$ is a diagonal matrix.

Suppose that $\Omega \subset \mathbb{R}P^n$ is a projective domain with its automorphism group $\mathrm{Aut}(\Omega) \subset PGL(n+1, \mathbb{R})$. If $\{\varphi_\nu | \nu = 1, 2, \ldots\}$ is a sequence in $\mathrm{Aut}(\Omega)$, then there exists a subsequential limit φ since the projectivization $P\mathfrak{gl}(n+1, \mathbb{R})$ is compact, where $\mathfrak{gl}(n+1, \mathbb{R})$ is the general linear group. We assume that $\varphi \in P\mathfrak{gl}(n+1, \mathbb{R}) - PGL(n+1, \mathbb{R})$, i.e., φ is singular. Especially, this is the case when $\mathrm{Aut}(\Omega)$ is non-compact. A representative matrix $A \in \mathfrak{gl}(n+1, \mathbb{R})$ of φ is a nonzero singular matrix so that the kernel and range of A are nontrivial. Denote $K(\varphi)$, $R(\varphi)$ the *projectivizations* of the kernel and range of A, respectively. Note that $\dim K(\varphi) + \dim R(\varphi) = n - 1$. Let $A_\nu = [a^\nu_{ij}] \in GL(n+1, \mathbb{R})$ be a representative matrix of φ_ν for each ν. From the previous paragraph, we have $A_\nu = P_\nu D_\nu Q_\nu$, where $P_\nu, Q_\nu \in O(n+1, \mathbb{R})$ and $D_\nu \in GL(n+1, \mathbb{R})$ is a diagonal matrix. Since $O(n+1, \mathbb{R})$ is compact, by extracting a subsequence, if necessary, there exist limits $P, Q \in O(n+1, \mathbb{R})$ such that $P = \lim_{\nu\to\infty} P_\nu$ and $Q = \lim_{\nu\to\infty} Q_\nu$. Then denoting $A = \lim_{\nu\to\infty} A_\nu$, $D = P^{-1} A Q^{-1}$ is a diagonal matrix because

$$D = \lim_{\nu\to\infty} D_\nu = \lim_{\nu\to\infty} P_\nu^{-1} A_\nu Q_\nu^{-1}.$$

(We can take an L_2-norm for the convergence.) Let h_1, h_2 be the projective transformations corresponding to Q, P^{-1}, respectively. One gets a singular map φ^h corresponding to $D \in \mathfrak{gl}(n+1, \mathbb{R})$. Thus, $\{\tilde{\varphi}_\nu = h_2 \varphi_\nu h_1^{-1}\}$ has a (subsequential) limit $\varphi^h = h_2 \varphi h_1^{-1}$. Let $d = (n+1) - (1 + \dim R(\varphi))$. We see that $P^{-1} A Q^{-1}$ is of the form, up to an order of the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$,

$$D(\gamma_{d+1},\dots,\gamma_n,1)=\left\{\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underbrace{\begin{matrix}\gamma_{d+1} & & & 0\\ & \gamma_{d+2} & & \\ & & \ddots & \\ 0 & & & 1\end{matrix}}_{1+\dim R(\varphi)}\end{array}\right\}\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}1+\dim R(\varphi),$$

where $0<|\gamma_i|\leq 1$ for $d+1\leq i\leq n$. Consequently, we have that $\Pi^{-1}(K(\varphi^h))\oplus\Pi^{-1}(R(\varphi^h))=\mathbb{R}^{n+1}$, where $\Pi:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}P^n$ is the canonical projection map. We obtained new domains $h_1(\Omega),h_2(\Omega)$ and a subset $G^h=\{f\in PGL(n+1,\mathbb{R})|f:h_1(\Omega)\to h_2(\Omega)\}$ of projective transformations such that both of $h_1(\Omega)$ and $h_2(\Omega)$ are projectively equivalent to the original domain Ω and G^h is homeomorphic to the automorphism group $\mathrm{Aut}(\Omega)$. The homeomorphism $\iota_h:\mathrm{Aut}(\Omega)\to G^h$ is given by $\iota_h(f)=h_2fh_1^{-1}$ for $f\in\mathrm{Aut}(\Omega)$. This is a kind of "normalization" of the pair $(\Omega,\{\varphi_\nu\})$. Note that if $h_1=h_2$, then G^h is a group isomorphic to $\mathrm{Aut}(\Omega)$ by ι_h. We can summarize as the following.

Proposition 2. *Let Ω be a projective domain and $\{\varphi_\nu\}\subset\mathrm{Aut}(\Omega)$ with a subsequential limit φ. Then there exists a pair of projective transformations h_1,h_2 such that a subsequential limit $h_2\varphi h_1^{-1}$ of the sequence $\{h_2\varphi_\nu h_1^{-1}\}$ has a diagonal matrix as its representative and satisfies*

$$(*)\qquad \Pi^{-1}(K(h_2\varphi h_1^{-1}))\oplus\Pi^{-1}(R(h_2\varphi h_1^{-1}))=\mathbb{R}^{n+1}.$$

Remark. Note that $R(h_2\varphi h_1^{-1})=h_2R(\varphi h_1^{-1})=h_2R(\varphi)$ and $K(h_2\varphi h_1^{-1})=K(\varphi h_1^{-1})=h_1K(\varphi)$.

Definition 2. For a given projective domain Ω and a sequence $\{\varphi_\nu\}\subset\mathrm{Aut}(\Omega)$, we define $\{h(\Omega),G^h\}$ to be the triple $\{h_1(\Omega),h_2(\Omega),G^h\}$ and call it an h-normalization of the pair Ω and $\{\varphi_\nu\}$. In this case, we say $\{\varphi_\nu\}$ is h-normalized. Denote a subsequential limit of an h-normalized sequence $\{\varphi_\nu\}$ by φ^h and $h=(h_1,h_2)$.

Remark. One can easily show that the diagonal matrix of an h-normalization for a given sequence $\{\varphi_\nu\}\subset\mathrm{Aut}(\Omega)$, hence the limit φ^h is independent of change of orthogonal basis of $\mathbb{R}^{n+1}$.

From now on, we always assume that φ^h has its representative matrix of the form $D(\gamma_{d+1},\dots,\gamma_n,1)$ in Fig.1. This can be done by the previous remark and the representative matrix is defined up to nonzero constant. After an h-normalization, one sees that $a^\nu_{ij}\to 0$ for $1\leq i\leq d,1\leq j\leq n+1$ and $a^\nu_{ij}\to\delta_{ij}\gamma_i$ for $d+1\leq i\leq n,1\leq j\leq n+1$. For each $i\in\{1,\dots,d\}$, by extracting a subsequence if necessary, there is a sequence $\{a^\nu_{is(i)}\}$ such that $\lim_{\nu\to\infty}\left|\frac{a^\nu_{ij}}{a^\nu_{is(i)}}\right|\leq 1$ for $1\leq j\leq n+1$, where $s(i)\in\{1,\dots,n+1\}$, i.e., the sequence $\{|a^{\nu}_{is(i)}|\}$ decreases at the slowest rate among $\{a^{\nu}_{ij}\}$ for $1 \leq j \leq n+1$. Note that, by a subsequence, if necessary, $a^{\nu}_{is(i)} \neq 0$ for all ν and $1 \leq i \leq d$ since $A_{\nu} \in GL(n+1, \mathbb{R})$. Let $\{a^{\nu}\} \in \{\{a_{is(i)}\} | 1 \leq i \leq d\}$ be a sequence, by extracting a subsequence, if necessary, such that $\lim_{\nu \to \infty} \left| \frac{a^{\nu}_{is(i)}}{a^{\nu}} \right| \leq 1$. We shall call the sequence $\{a^{\nu}\}$ *a primary sequence* of φ^h. Let $I = \{(i,j) | 1 \leq i \leq d, 1 \leq j \leq n+1\}$. For $1 \leq i \leq n$, $1 \leq j \leq n+1$, define

$$\xi_{ij} = \begin{cases} \lim_{\nu \to \infty} \dfrac{a^{\nu}_{ij}}{a^{\nu}} & \text{for } (i,j) \in I, \\ 0 & \text{for } (i,j) \notin I. \end{cases}$$

Let $p = [0 : \cdots : 0 : 1] \in \mathbb{R}P^n$. The affine coordinate $\psi : U \longrightarrow \mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$ defined by $\psi([x_1 : \cdots : x_{n+1}]) = \left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, 1\right)$ is called the *φ^h-related affine coordinate.* For simplicity, we write $\psi([x_1 : \cdots : x_{n+1}]) = \left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right)$ when the meaning is clear. So, $\psi(p) = (0,1)$ or $\psi(p) = 0$. We will handle projective domains through this affine coordinate with the reference point p. Let ξ^p_j be the vector $(\xi_{j1}, \dots, \xi_{jn+1})$. Let M_{ν} be an $n \times (n+1)$ matrix defined by

$$(i,j)\text{-th component of } M_{\nu} = \begin{cases} a^{\nu}_{ij} & \text{if } (i,j) \in I, \\ 0 & \text{if } (i,j) \notin I. \end{cases}$$

Define M_{φ^h} to be the $n \times (n+1)$ matrix $[\xi^p_{ij}]$ $1 \leq i \leq n$, $1 \leq j \leq n+1$. This is equivalent to

$$M_{\varphi^h} = \lim_{\nu \to \infty} \frac{1}{a^{\nu}} M_{\nu}.$$

Now, we will state several useful facts which are used throughout this paper.

Proposition 3. *Suppose that γ_k is a sequence of projective transformations which converges to a singular projective transformation γ. Let $x \in K(\gamma)$ and $\lim_{k \to \infty} \gamma_k(x) = y$. If σ is a line segment transverse to $K(\gamma)$ through x, then there is a subsequence $\{\gamma_m\}$ and a full affine line τ through y such that $\lim_{m \to \infty} \gamma_m(\sigma) = \tau$.*

Proof. This was observed by Benzécri [3]. See [13] for the proof. □

4. Ball Characterization Theorem

We will reprove the Wong-Rosay Theorem or, Ball Characterization Theorem using an affine scaling along simplex since its proof contains the main ideas of this paper. Let Ω be a domain in $\mathbb{R}P^n$ equipped with the metric induced from the standard Euclidean metric. Suppose that there exist a neighborhood N of $x \in \partial\Omega$ and a real-valued function ρ defined on $N \cap \Omega$ satisfying (1) $N \cap \Omega$ is defined by $\rho(x) < 0$ (2) The gradient $\nabla\rho = \left(\frac{\partial\rho}{\partial x_1}, \dots, \frac{\partial\rho}{\partial x_n}\right)$ is never zero at any point of $N \cap \Omega$. In this case, ρ is called a defining function of Ω at x. For $x \in \mathbb{R}P^n$, an *n-simplex at x* to be a convex n-simplex with a vertex x. The vertices other than x of the n-simplex at x are called *the end points of the n-simplex at x* and a line segment connecting x and an end point of the n-simplex at x is called an *axis of the n-simplex at x*. We call the simplex with vertices $(0,\dots,0),(1,0,\dots,0),\dots,(0,\dots,0,1)$ *the standard unit simplex* at 0 in an affine coordinate. Now, we will present one of main theorems which is the first application with our scaling method using simplex.

Theorem 3. *Suppose that there is $q \in \Omega$ and $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$ such that $\lim_{\nu\to\infty} \varphi_\nu(q) = p \in \partial\Omega$. If $\partial\Omega$ is C^2 in a neighborhood of p which is strictly convex, then Ω is projectively equivalent to a paraboloid, i.e., a ball.*

Remark. (1) $\partial\Omega$ is strictly convex in a neighborhood V of $p \in \partial\Omega$ if Hessian at p is positive definite in $V \cap \partial\Omega$.
(2) Theorem 3 is a "projective" version of Ball Characterization Theorem.
(3) We can find two different approaches for this theorem in Edith Socié-Méthou and B. Colbois and P. Verovic's papers ([4], [15]). The scaling method lets us relax the assumptions in those papers.

Proof. First, by the Proposition 3, one can see that $q \notin K(\varphi)$ for a subsequential limit φ of $\{\varphi_\nu\}$ since p is a strictly convex point. If one takes an open neighborhood $N \subset \Omega$ of q such that $N \cap K(\varphi) = \emptyset$, then the limit $\lim_{\nu\to\infty} \varphi_\nu(N)$ is open in $R(\varphi)$ containing p because the limit map φ is open. Thus $\varphi(N)$ must be tangential to $\partial\Omega$ at p. It follows that $\dim R(\varphi) \cap \partial\Omega = \dim R(\varphi)$. For the proof of this, see "A rigidity result for domains with a locally strictly convex point" by Kyeonghee Jo. By strictly convexity, one obtains $R(\varphi) = \{p\}$. Take a simplex E_0 at $p_0 \in \partial\Omega$ such that $\overline{E_0} \cap K(\varphi) = \emptyset$ and $n-1$ axes are tangent to $\partial\Omega$ at p_0. Then the first axis is transversal to $T_{p_0}\partial\Omega$. Take an affine coordinate $\phi : U_0 \longrightarrow \mathbb{R}^n \times \{1\}$ with $\phi(p) = 0$ and $p_0 \in U_0$. Denote $\varphi_\nu(p_0) = (c_1^\nu,\dots,c_n^\nu) = c^\nu$ and let $\{e_1^\circ,\dots,e_n^\circ\}$ be the end points of the axes of E_0, where the axis having end point e_1°, is not tangential to $\partial\Omega$. We know that $\varphi_\nu(\overline{E_0}) \to \varphi(\overline{E_0}) = p$ uniformly. Hence $\varphi_\nu(p_0) \to p = 0$ and $\varphi_\nu(e_1^\circ) \to 0$ uniformly. From the Taylor expansion of second order and an orthogonal basis change of $\mathbb{R}^{n+1}$, if necessary, there exists a neighborhood U of p such that $U \cap \Omega$ is defined by

$$\rho(x) = -x_1 + \delta_2 x_2^2 + \cdots + \delta_n x_n^2 + o(|'x_1|^2) < 0,$$

where ρ is the defining function and all $\delta_\ell \in \mathbb{R}$ are positive. Note that for all $\ell \in \{2,\dots,n\}$, one has $\delta_\ell \neq 0$ since p is a strictly convex point. For brevity, we write $'x_1 = (x_2,\dots,x_n)$. Now, we define affine coordinate changes as follows.
(I) For each ν,

$$\Phi_\nu = \begin{cases} y_1 = x_1 - c_1^\nu - \displaystyle\sum_{\ell=2}^{n} a_\ell^\nu (x_\ell - c_\ell^\nu) & \\ y_\ell = x_\ell - c_\ell^\nu & \ell = 2,\dots,n, \end{cases}$$

where a_ℓ^ν's are determined so that, in the coordinate $(y_1,\dots,y_n)$, (i) $\Phi_\nu(c^\nu)$ is the point $(0,\dots,0)\in\partial\Omega$ (ii) The tangent space to $\partial\Omega$ at $(0,\dots,0)$ is given by $\{y \mid y_1=0\}$. Let $d^\nu=(d_1^\nu,\dots,d_n^\nu)=\varphi_\nu(e_1^\circ)$. If $\Phi_\nu(d^\nu)=(\beta_1^\nu,\dots,\beta_n^\nu)$, $\beta_1^\nu=d_1^\nu-c_1^\nu-\sum_{\ell=2}^n a_\ell^\nu(d_\ell^\nu-c_\ell^\nu)$. If one consider an n-frame at p_0, there are 2^n simplexes at p_0 satisfying the same conditions as E_0. Thus, one can take the simplex E_0 at p_0 so that $\beta_1^\nu>0$ for all ν. For all sufficiently large ν, $\frac{\partial\rho}{\partial x_1}(c^\nu)\neq 0$ since $c^\nu\to 0$ as $\nu\to\infty$. So, determine a_ℓ^ν for $\ell=2,\dots,n$, so that

$$0 = \frac{\partial\rho_\nu}{\partial y_2}(0)=\sum_k^n\frac{\partial\rho}{\partial x_k}(c^\nu)\frac{\partial\Phi_k^{-1}}{\partial y_2}(0)=a_2^\nu\frac{\partial\rho}{\partial x_1}(c^\nu)+\frac{\partial\rho}{\partial x_2}(c^\nu)$$

$$\vdots$$

$$0 = \frac{\partial\rho_\nu}{\partial y_n}(0)=\sum_k^n\frac{\partial\rho}{\partial x_k}(c^\nu)\frac{\partial\Phi_k^{-1}}{\partial y_n}(0)=a_n^\nu\frac{\partial\rho}{\partial x_1}(c^\nu)+\frac{\partial\rho}{\partial x_n}(c^\nu).$$

It follows that $0\neq\frac{\partial\rho_\nu}{\partial y_1}(0)=\sum_k^n\frac{\partial\rho}{\partial x_k}(c^\nu)\frac{\partial\Phi_k^{-1}}{\partial y_1}(0)=\frac{\partial\rho}{\partial x_1}(c^\nu)$. Since all c_ℓ^ν's, d_ℓ^ν's, a_ℓ^ν's approach 0 as $\nu\to\infty$, we have $\beta_1^\nu\to 0$ as $\nu\to\infty$. The defining function $\rho_\nu(y)$ is, in this coordinate,

$$\rho_\nu(y)=-y_1+\sum_{\sigma,\tau=2}^n A_{\sigma\tau}^\nu y_\sigma y_\tau+(\delta_2y_2^2+\cdots+\delta_ny_n^2)+o(|'y_1|^2)<0,$$

where $A_{\sigma\tau}^\nu$ is a function of c_ℓ^ν's $(\ell=2,\dots,n)$ satisfying $A_{\sigma\tau}^\nu\to 0$ as $\nu\to\infty$. If we write

$$(EQ)\qquad \rho(x)-(-x_1+\delta_2x_2^2+\cdots+\delta_nx_n^2)=h(x_2,\dots,x_n),$$

then h is of class C^2 and $h(x_2,\dots,x_n)\in o(|'x_1|^2)$.

Lemma 1. *By the affine transformation Φ_ν, the defining function can be written as*

$$\rho_\nu(y)=-y_1+\sum_{\sigma,\tau=2}^n A_{\sigma\tau}^\nu y_\sigma y_\tau+(\delta_2y_2^2+\cdots+\delta_ny_n^2)+o(|'y_1|^2)<0,$$

where $A_{\sigma\tau}^\nu\to 0$ as $\nu\to\infty$ for $2\le\sigma,\tau\le n$.

Proof. By the second order Taylor expansion of h at $(c_2^\nu,\dots,c_n^\nu)$,

$$\begin{aligned}h(x_2,\dots,x_n) &= h(y_2+c_2^\nu,\dots,y_n+c_n^\nu)\\ &= h(c_2^\nu,\dots,c_n^\nu)+\sum_{\ell=2}^n\frac{\partial h}{\partial x_\ell}(c_2^\nu,\dots,c_n^\nu)y_\ell\\ &\quad+\sum_{j,k=2}^n\frac{\partial^2h}{\partial x_j\partial x_k}(c_2^\nu,\dots,c_n^\nu)y_jy_k+o(|'y_1|^2).\end{aligned}$$

By differentiating both sides of (EQ) for $2 \le \ell, j, k \le n$,

$$\begin{aligned} \frac{\partial\rho}{\partial x_\ell}(c^\nu) &= 2\delta_\ell c_\ell^\nu + \frac{\partial h}{\partial x_\ell}(c^\nu), \\ \frac{\partial^2\rho}{\partial x_j\,\partial x_k}(c^\nu) &= 2\delta_j\delta_{jk} + \frac{\partial^2 h}{\partial x_j\,\partial x_k}(c^\nu), \text{ where } \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases} \end{aligned}$$

Because ρ is of class C^2, every partial derivative is continuous so that $\frac{\partial h}{\partial x_\ell}(c^\nu) \to 0$, $\frac{\partial^2 h}{\partial x_j \partial x_k}(c^\nu) \to 0$. Note that $0 = \rho(c^\nu) = -c_1^\nu + (\delta_2(c_2^\nu)^2 + \cdots + \delta_n(c_n^\nu)^2) + h(c_2^\nu, \ldots, c_n^\nu)$ and $a_\ell^\nu = \frac{\partial\rho}{\partial x_\ell}(c^\nu)$. From (EQ), we have

$$\begin{aligned} \rho_\nu(y) = &- y_1 + c_1^\nu + \sum_\ell^n a_\ell^\nu y_\ell + \delta_2(y_2 + c_2^\nu)^2 + \cdots + \delta_n(y_n + c_n^\nu)^2 \\ &+ h(y_2 + c_2^\nu, \ldots, y_n + c_n^\nu) \\ = &- y_1 + \sum_\ell^n \delta_\ell y_\ell^2 + c_1^\nu + \sum_{\ell=2}^n \delta_\ell (c_\ell^\nu)^2 + 2\sum_\ell^n \delta_\ell c_\ell^\nu y_\ell + \sum_\ell^n a_\ell^\nu y_\ell \\ &+ h(y_2 + c_2^\nu, \ldots, y_n + c_n^\nu) \\ = &- y_1 + \sum_\ell^n \delta_\ell y_\ell^2 + c_1^\nu + \sum_{\ell=2}^n \delta_\ell (c_\ell^\nu)^2 + 2\sum_\ell^n \delta_\ell c_\ell^\nu y_\ell + \sum_\ell^n a_\ell^\nu y_\ell \\ &+ h(c_2^\nu, \ldots, c_n^\nu) + \sum_{\ell=2}^n \frac{\partial h}{\partial x_\ell}(c_2^\nu, \ldots, c_n^\nu) y_\ell + \sum_{j,k=2}^n \frac{\partial^2 h}{\partial x_j \partial x_k}(c_2^\nu, \ldots, c_n^\nu) y_j y_k \\ &+ o(|'y_1|^2) \\ = &- y_1 + \sum_\ell^n \delta_\ell y_\ell^2 + \sum_{\sigma,\tau=2}^n A_{\sigma\tau}^\nu y_j y_k + o(|'y_1|^2), \end{aligned}$$

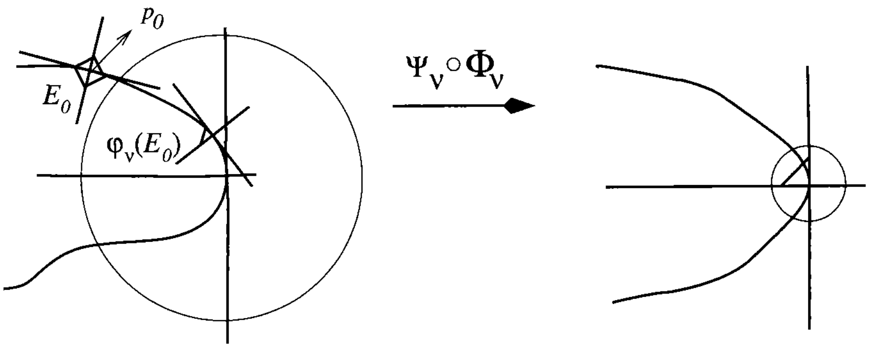
where $A_{\sigma\tau}^\nu = \frac{\partial^2 h}{\partial x_\sigma \partial x_\tau}(c_2^\nu, \ldots, c_n^\nu)$. ☐

The $n-1$ axes of $E_\nu = (\Phi_\nu \circ \varphi_\nu)(E_0)$ lie in the tangent space $T_{(\Phi_\nu \circ \varphi_\nu)(p_0)}\partial\Omega$ and the first axis is transversal to this tangent space by definition of Φ_ν and $\varphi_\nu \in \mathrm{Aut}(\Omega)$. So, β_1^ν cannot be 0 for all ν. Define $\epsilon_\nu > 0$ to be β_1^ν. Note that $\epsilon_\nu \to 0$ as $\nu \to \infty$ since $\varphi_\nu(E_0) \to p$ by $R(\varphi) = p$.

(II) By the affine coordinate change Φ_ν, one can see that $(\Phi_\nu \circ \varphi_\nu)(e_\ell^\circ)$ has the first coordinate 0 for $\ell \neq 1$. For each ν, define

$$\Psi_\nu = \begin{cases} u_1 = y_1 \\ u_\ell = -\dfrac{\beta_\ell^\nu}{\beta_1^\nu} y_1 + y_\ell \quad \ell = 2, \ldots, n. \end{cases}$$

This linear transformation sends the axis of the simplex E_ν which is not tangential at 0 to the normal direction at 0, i.e., on the x_1-axis and all the other axes are invariant, i.e., belong to the tangent space of the boundary $\partial\Phi_\nu(\Omega)$ at 0. Denote $B_\ell^\nu = |(\Phi_\nu \circ \varphi_\nu)(e_\ell^\circ)|$ and $(0, r_{2\ell}^\nu, \ldots, r_{n\ell}^\nu) = (\Psi_\nu \circ \Phi_\nu \circ \varphi_\nu)(e_\ell^\circ)$ for $\ell = 2, \ldots, n$.



(III) We have $(\Psi_\nu \circ \Phi_\nu \circ \varphi_\nu)(e_1^\circ) = (\beta_1^\nu, 0, \ldots, 0)$ with $\beta_1^\nu > 0$. Define R_ν to be the affine transformation defined by the matrix $\left(\frac{r_{ab}^\nu}{B_b^\nu}\right)^{-1}$ on $T_0(\partial\Lambda_\nu)$, where $\Lambda_\nu = (\Psi_\nu \circ \Phi_\nu \circ \varphi_\nu)(\Omega)$, so that $R_\nu((\Psi_\nu \circ \Phi_\nu \circ \varphi_\nu)(e_\ell^\circ)) \in u_\ell$-axis.
(IV) Define an affine coordinate change Γ_ν as follows.

$$\Gamma_\nu = \begin{cases} \tilde{x}_1 = w_1 \\ \tilde{x}_\ell = \dfrac{\sqrt{\epsilon_\nu}}{B_\ell^\nu} w_\ell \quad \ell = 2, \ldots, n. \end{cases}$$

Now, all axes of the simplex $(\Gamma_\nu \circ R_\nu \circ \Psi_\nu \circ \Phi_\nu \circ \varphi_\nu)(E_0)$ which are tangential to $(\Gamma_\nu \circ R_\nu \circ \Psi_\nu \circ \Phi_\nu \circ \varphi_\nu)(\Omega)$ at 0 are of length $\sqrt{\epsilon_\nu}$. The defining function in this $\tilde{x}$-coordinate is

$$\begin{aligned}\rho_\nu(\tilde{x}) = -\tilde{x}_1 &+ \sum_{\sigma,\tau=2}^{n} A_{\sigma\tau}^\nu \left(\frac{\beta_\sigma^\nu}{\beta_1^\nu}\tilde{x}_1 + \sum_{k=2}^{n} \frac{r_{\sigma k}^\nu}{\sqrt{\epsilon_\nu}}\tilde{x}_k\right)\left(\frac{\beta_\tau^\nu}{\beta_1^\nu}\tilde{x}_1 + \sum_{k=2}^{n} \frac{r_{\tau k}^\nu}{\sqrt{\epsilon_\nu}}\tilde{x}_k\right) \\ &+ \sum_{\ell=2}^{n} \delta_\ell \left(\frac{\beta_\ell^\nu}{\beta_1^\nu}\tilde{x}_1 + \sum_{k=2}^{n} \frac{r_{\ell k}^\nu}{\sqrt{\epsilon_\nu}}\tilde{x}_k\right)^2 \\ &+ o\left(|(\frac{\beta_2^\nu}{\beta_1^\nu}\tilde{x}_1 + \sum_{k=2}^{n} r_{2k}^\nu \frac{\tilde{x}_k}{\sqrt{\epsilon_\nu}}, \ldots, \frac{\beta_n^\nu}{\beta_1^\nu}\tilde{x}_1 + \sum_{k=2}^{n} r_{nk}^\nu \frac{\tilde{x}_k}{\sqrt{\epsilon_\nu}})|^2\right) < 0.\end{aligned}$$

Let D_ν be the intersection of a small neighborhood U of $p = 0$ and the domain defined by $\rho_\nu(\tilde{x}) > 0$. Now, we introduce a sequence L_ν to blow up D_ν defined by

$$L_\nu = \begin{cases} \hat{x}_1 = \tilde{x}_1/\epsilon_\nu \\ \hat{x}_\ell = \tilde{x}_\ell/\sqrt{\epsilon_\nu}\,. \end{cases}$$

Recall that $\epsilon_\nu = \beta_1^\nu$. This sequence preserves the tangent space and acts as a homothety on the tangent space at 0 and each transformation sends the last simplex to the standard unit simplex at 0. Then the defining function $\rho_\nu(\hat{x})$ of the domain $L_\nu(D_\nu)$ can be written as

$$\rho_\nu(\hat{x}) = -\epsilon_\nu \hat{x}_1 + \sum_{\sigma,\tau=2}^{n} A^\nu_{\sigma\tau}\left(\beta^\nu_\sigma \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{\sigma k}\hat{x}_k\right)\left(\beta^\nu_\tau \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{\tau k}\hat{x}_k\right)$$
$$+\sum_{\ell=2}^{n}\delta_\ell\left(\beta^\nu_\ell \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{\ell k}\hat{x}_k\right)^2$$
$$+\, o\left(|(\beta^\nu_2 \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{2k}\hat{x}_k, \ldots, \beta^\nu_n \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{nk}\hat{x}_k)|^2\right) < 0.$$

By our construction, we see that all $\sqrt{\epsilon_\nu}$, β^ν_ℓ's and r^ν_{ab}'s approach 0 as $\nu \to \infty$. Compare the decreasing rates of these values and divide the above equation by $\{|\alpha_\nu|^2\}$, where $\{\alpha_\nu\} \in \{\{\sqrt{\epsilon_\nu}\}, \{\beta^\nu_\ell\}, \{r^\nu_{ab}\} \mid \ell, a, b = 2, \ldots, n\}$ has the slowest decreasing rate by extracting a subsequence if necessary.

$$\rho_\nu(\hat{x}) = -\frac{\epsilon_\nu}{|\alpha_\nu|^2}\hat{x}_1 + \sum_{\sigma,\tau=2}^{n} A^\nu_{\sigma\tau}\left(\frac{\beta^\nu_\sigma}{|\alpha_\nu|}\hat{x}_1 + \sum_{k=2}^{n}\frac{r^\nu_{\sigma k}}{|\alpha_\nu|}\hat{x}_k\right)\left(\frac{\beta^\nu_\tau}{|\alpha_\nu|}\hat{x}_1 + \sum_{k=2}^{n}\frac{r^\nu_{\tau k}}{|\alpha_\nu|}\hat{x}_k\right)$$
$$+\sum_{\ell=2}^{n}\delta_\ell\left(\frac{\beta^\nu_\ell}{|\alpha_\nu|}\hat{x}_1 + \sum_{k=2}^{n}\frac{r^\nu_{\ell k}}{|\alpha_\nu|}\hat{x}_k\right)^2$$
$$+\frac{1}{|\alpha_\nu|^2}\, o\left(|(\beta^\nu_2 \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{2k}\hat{x}_k, \ldots, \beta^\nu_n \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{nk}\hat{x}_k)|^2\right) < 0.$$

Then all limits satisfy $\lim_{\nu\to\infty}\left|\frac{\sqrt{\epsilon_\nu}}{\alpha_\nu}\right| \le 1$, $\lim_{\nu\to\infty}\left|\frac{\beta^\nu_\ell}{\alpha_\nu}\right| \le 1$, and $\lim_{\nu\to\infty}\left|\frac{r^\nu_{ab}}{\alpha_\nu}\right| \le 1$ for $\ell, a, b = 2, \ldots, n$. Also, note that $c^\nu_\ell \to 0$, $A^\nu_{\sigma\tau} \to 0$ and

$$\frac{1}{|\alpha_\nu|^2}\, o\left(|(\beta^\nu_2 \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{2k}\hat{x}_k, \ldots, \beta^\nu_n \hat{x}_1 + \sum_{k=2}^{n} r^\nu_{nk}\hat{x}_k)|^2\right) \to 0$$

as $\nu \to \infty$. Combining all the coordinate changes above, we have a sequence of projective transformation

$$G_\nu := L_\nu \circ \Gamma_\nu \circ R_\nu \circ \Psi_\nu \circ \Phi_\nu \circ \varphi_\nu.$$

A subsequential limit yields a mapping $G : \Omega \longrightarrow G(\Omega) \in P\mathfrak{gl}(n+1, \mathbb{R})$.

Lemma 2. *$G : \Omega \longrightarrow G(\Omega)$ is a projective transformation.*

Proof. By our construction of Φ_ν, Ψ_ν, R_ν, Γ_ν, and L_ν, $G_\nu(E_0)$ is the standard unit simplex at 0 for all ν. So, $\overline{E}_0 \cap K(G) = \emptyset$ if G were singular. Otherwise, there must exist an affine half line contained in $\lim_{\nu\to\infty} G_\nu(\overline{E}_0)$ which is the closed standard unit simplex at 0. Since $G_\nu \mid_{\overline{E}_0} \to G \mid_{\overline{E}_0}$ uniformly, G cannot be singular. □

Since at least one of coefficients $\lim_{\nu\to\infty}\left|\frac{\sqrt{\epsilon_\nu}}{\alpha_\nu}\right|$, $\lim_{\nu\to\infty}\left|\frac{\beta^\nu_\ell}{\alpha_\nu}\right|$, and $\lim_{\nu\to\infty}\left|\frac{r^\nu_{ab}}{\alpha_\nu}\right|$ for $\ell, a, b = 2, \ldots, n$ is not be zero, the limit equation is not trivial. From

Proposition 2, if $K = int(E_0) \cap \Omega$, we see $G_\nu(K) \longrightarrow G(K)$ uniformly and the limit open set $G(K)$ should be inside the set defined by the limit equation of $\rho_\nu(\hat{x})$ above so that the limit set cannot be contained in a lower dimensional subset. Furthermore, the domain $\hat{\Omega}$ defined by the limit equation should be projectively equivalent to Ω since G is a projective transformation. This implies that the limit $\frac{\epsilon_\nu}{|\alpha_\nu|^2}$ as $\nu \to \infty$ cannot be zero. Observe that all the coefficients of the limit defining function cannot be 0 since δ_ℓ's have the same sign, i.e., $\delta_\ell > 0$ for all $\ell \in \{2, \ldots, n\}$. By a linear change of coordinate if necessary, the limit equation must be

$$(*) \qquad \hat{\rho}(x) = -x_1 + \sum_{\ell=2}^{n} g_\ell x_\ell^2 > 0 \quad \text{with } g_\ell \neq 0 \quad \forall\, \ell \in \{2, \ldots, n\}.$$

The domain $L_\nu(D_\nu)$ converges to the domain $\hat{\Omega}$ defined by $(*)$. Note that $\hat{\Omega} = Q(b_2, \ldots, b_n)$ such that all b_ℓ's are positive, which is projectively equivalent to an n-ball. □

Remark. In fact, Proposition 3 says that $\Omega \cap K(\varphi) = \emptyset$. If $\Omega \cap K(\varphi) \neq \emptyset$, the limit $\lim_{\nu\to\infty} \varphi_\nu(\ell)$ will be an affine full line in the closure $\overline{\Omega}$ for any small line segment $\ell \subset \Omega$ transversal to $K(\varphi)$. Since Ω is a ball, this is impossible.

Corollary 1. *Suppose that Ω is a syndetic domain. If $\partial\Omega$ is C^2 in a neighborhood of p which is strictly convex, then Ω is projectively equivalent to a paraboloid, i.e., a ball.*

Proof. Syndetic action implies that there is a compact subset $K \subset \Omega$ such that $\bigcup_{\varphi \in \mathrm{Aut}(\Omega)} \varphi(K) = \Omega$. If $\{U_n |\ n = 1, 2, \ldots\}$ is a family of open neighborhoods of p with $U_{n+1} \subset U_n$ and $\bigcap_n U_n = p \in \partial\Omega$, we can find $q_n \in K$ such that $\varphi_n(q_n) \in U_n$. Let $q \in K$ be a subsequential limit $\lim_{n\to\infty} q_n$. Then $\lim_{n\to\infty} \varphi_n(q) = p$. Thus we have an interior point $q \in K \subset \Omega$ and a sequence $\{\varphi_\nu\}$ of automorphism group of Ω such that $\lim_{\nu\to\infty}\varphi_\nu(q) = p$. Apply Theorem 3. □

5. Unbounded domains : quadratic domains

We can extend the idea of the previous section to unbounded projective domains with C^2 boundary point. In this section, we will prove: if a domain has a C^2 defining function at a boundary accumulation point, then it is one of the quadratic domains defined by

$$Q(a_2, \ldots, a_n) = \{[x_0 : \cdots : x_n] \in \mathbb{R}P^n | x_0 x_1 + a_2 x_2^2 + \cdots + a_n x_n^2 > 0, a_i \in \mathbb{R}\}.$$

Proofs depend on the dimension of the range through the boundary point. Unlike ellipsoid, we need an additional assumption which is analyzed in 5.1.

5.1. A geometric condition in scaling for quadratic domains

Let Ω be an n-dimensional projective domain with $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$. Suppose that $N_\varphi \cap \partial\Omega$ is C^2 for a tubular neighborhood N_φ of $R(\varphi)$. Let $\{h(\Omega), G^h\}$ be an h-normalization of Ω and $\{\varphi_\nu\}$. Suppose that $\lim_{\nu\to\infty} A_\nu = D(\gamma_{d+1}, \ldots, \gamma_n, 1)$ by a subsequence, where A_ν is a representative matrix of an h-normalized φ_ν and $D(\gamma_{d+1}, \ldots, \gamma_n, 1)$ is a diagonal matrix with diagonal elements $0, \ldots, 0, \gamma_{d+1}, \ldots, \gamma_n, 1$. If $v \in \mathbb{R}^n$, we can think of v as an $(n+1)$-vector by the affine coordinate $(v_1, \ldots, v_n, 1) \longrightarrow (v_1, \ldots, v_n)$, and vice versa. By definition of M_{φ^h}, $\lim_{\nu\to\infty} \frac{1}{a^\nu} M_\nu = M_{\varphi^h}$, where M_ν is an $n \times (n+1)$ matrix. Suppose that there exists a point $\mathbf{b} = (b_1, \ldots, b_n, 1) \in \mathbb{R}^n \times \{1\}$ such that ${}^t(M_{\varphi^h}(\mathbf{b}))H_p M_{\varphi^h}(\mathbf{b}) \neq 0$. This is equivalent to $\langle M_{\varphi^h}(\mathbf{b}), M_{\varphi^h}(\mathbf{b})\rangle_{H_p} \neq 0$, where $\langle \cdot, \cdot \rangle_{H_p}$ is the symmetric bilinear form on $\mathbb{R}^n$ defined by the Hessian H_p at $p = 0$. Since $\{\varphi_\nu\}$ is normalized, $\psi^{-1}(\mathbf{b}) \notin K(\varphi^h)$. In fact, $\psi^{-1}(w) \notin K(\varphi^h)$ for all $w \in \mathbb{R}^n \times \{1\}$. Since $|a^\nu| \neq 0$ for all ν and $\lim_{\nu\to\infty} a^\nu = 0$ for a primary sequence $\{a^\nu\}$, $M_{\varphi^h}(\mathbf{b}) \neq 0$ implies, by the definition of M_{φ^h}, that $M_\nu(\mathbf{b}) \neq 0$ for all ν in the φ^h-related affine coordinate ψ so that $|M_\nu(\mathbf{b})| \neq 0$ for all ν and $\lim_{\nu\to\infty}(\psi \circ M_\nu)(\mathbf{b}) = 0 = \psi(p)$, where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^n$. Recall that $\{a^\nu\} \in \{\{a^\nu_{ij}\} | 1 \leq i \leq d, 1 \leq j \leq n+1\}$ satisfies $\lim_{\nu\to\infty} \left|\frac{a^\nu_{ij}}{a^\nu}\right| \leq C$, $1 \leq i \leq d, 1 \leq j \leq n+1$, for some constant C. If $x = \psi^{-1}(\mathbf{b}) \in \mathbb{R}P^n - K(\varphi^h)$, $\lim_{\nu\to\infty}\varphi_\nu(x) = p$. Observe that

$$\frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|} = \frac{\left(\displaystyle\sum_{j=1}^{n+1} a^\nu_{1j} b_j, \ldots, \sum_{j=1}^{n+1} a^\nu_{nj} b_j\right)}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{n+1} a^\nu_{ij} b_j\right)^2}} = \frac{\left(\displaystyle\sum_{j=1}^{n+1} \frac{a^\nu_{1j}}{|a^\nu|} b_j, \ldots, \sum_{j=1}^{n+1} \frac{a^\nu_{nj}}{|a^\nu|} b_j\right)}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{n+1} \frac{a^\nu_{ij}}{a^\nu} b_j\right)^2}}.$$

This implies that

$$(*) \qquad \lim_{\nu\to\infty} \left\langle \frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|}, \frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|} \right\rangle_{H_p} = \left\langle \frac{M_{\varphi^h}(\mathbf{b})}{|M_{\varphi^h}(\mathbf{b})|}, \frac{M_{\varphi^h}(\mathbf{b})}{|M_{\varphi^h}(\mathbf{b})|} \right\rangle_{H_p} \neq 0.$$

Since H_p is a real symmetric matrix, the eigenvalues are real. If s, t are the number of positive and negative eigenvalues of H_p, respectively, we say that H_p is of signature (s, t). Define $Z(H_p) = \{x \in \mathbb{R}^n | \langle x, x \rangle_{H_p} = 0\}$. Let S^{n-1}_p be the unit sphere at the origin $\psi(p) = 0$ in the φ^h-related affine coordinate ψ, $(*)$ implies that the direction of convergence of x by $\{\varphi_\nu\}$, i.e., the limit of unit vectors $\left\{\frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|}\right\}$ is away from $S^{n-1}_p \cap Z(H_p)$.

Definition 3. Suppose $(\Omega, \{\varphi_\nu\})$ is normalized by $h = (h_1, h_2)$ and $p = h_2(q)$. We say that $x \in \mathbb{R}P^n - K(\varphi)$ converges to $R(\varphi)$ along a nondegenerate direction at q by $\{\varphi_\nu\}$ if $M_{\varphi^h}(\mathbf{b}) \neq 0$ and the limit of unit vectors $\left\{\frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|}\right\}$ is away from $S^{n-1}_q \cap Z_q$, where $Z_q = h_2^{-1}(Z(H_p))$ and $\psi(p) = (0, 1)$, and $\mathbf{b} = \psi(x) \in \mathbb{R}^n \times \{1\}$.

It's not hard to state similar definition when $\dim R(\varphi) \geq 1$ using projection.

Theorem 4. *Let Ω, $\{\varphi_\nu\}$ be a projective domain and a sequence of automorphisms with an h-normalization by $h = (h_1, h_2)$. Assume that there exists a neighborhood N_q of $R(\varphi) = \{q\} \subset \partial\Omega$ such that $\partial\Omega \cap N_q$ is C^2. Then the followings are equivalent.*

(1) ${}^tM_{\varphi^h} H_p M_{\varphi^h} \neq 0$ *on $\mathbb{R}^n$, where $p = h_2(q)$.*
(2) *There exists an open dense subset $\mathcal{U}$ such that each $u \in \mathcal{U}$ converges to $R(\varphi)$ along a nondegenerate direction at q.*

Proof. We may assume that $\{\varphi_\nu\}$ is h-normalized. Let $B = {}^tM_{\varphi^h} H_p M_{\varphi^h}$, where H_p is the $n \times n$ Hessian matrix of the defining function at p. B is symmetric $n \times n$ so that $V = \{\mathbf{v} \in \mathbb{R}^n | {}^t\mathbf{v}B\mathbf{v} = 0\}$ is of dimension $\leq n-1$. Letting $\mathcal{U} = \mathbb{R}P^n - K(\varphi^h) \cup V$ gives (2). Conversely, if (2) holds, take any point $x \in \mathcal{U}$ converging to $R(\varphi)$ along a nondegenerate direction at q. Then $M_{\varphi^h}(\mathbf{b}) \neq 0$ and $M_{\varphi^h}(\mathbf{b}) H_p M_{\varphi^h}(\mathbf{b}) \neq 0$, where $\psi(x) = \mathbf{b} \in \mathbb{R}^n \times \{1\}$ since

$$0 \neq \lim_{\nu\to\infty} |M_{\varphi^h}(\mathbf{b})|^2 \left\langle \frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|}, \frac{M_\nu(\mathbf{b})}{|M_\nu(\mathbf{b})|} \right\rangle_{H_p} = \left\langle M_{\varphi^h}(\mathbf{b}), M_{\varphi^h}(\mathbf{b}) \right\rangle_{H_p}.$$

□

5.2. Quadratic domains

We can handle quadratic domains which are neither convex nor bounded when there is a sequence of points converging in a non-degenerate direction.

Theorem 5. *Let $\{h(\Omega), G^h\}$ be an h-normalization of Ω of dimension n and $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$. Suppose that $\partial\Omega$ is C^2 in a neighborhood of $q \in \partial\Omega$ with $R(\varphi) = \{q\}$. If ${}^tM_{\varphi^h} H_p M_{\varphi^h} \neq 0$ on $\mathbb{R}^n$, where H_p is the Hessian matrix at $p = h_2(q) = \Pi(\mathbf{e}_{n+1})$, then Ω is projectively equivalent to one of $Q(b_2, \ldots, b_n)$.*

Remark. If we define $H_p = [\delta_{\sigma\tau}]$ and $\xi = [\xi^p_{ik}]$, then

$${}^tM_{\varphi^h} H_p M_{\varphi^h} = {}^t[\xi^p_{ik}][\delta_{\sigma\tau}][\xi^p_{ik}] = {}^t\xi[\delta_{\sigma\tau}]\xi.$$

Proof. We shall take the same notation $\{\varphi_\nu\}$ for an h-normalized sequence of $\{\varphi_\nu\}$. Let $[x_1 : \cdots : x_{n+1}]$ be a homogeneous coordinate of $x \in \mathbb{R}P^n$. Take a φ^h-related affine coordinate ψ defined by

$$\psi([x_1 : \cdots : x_{n+1}]) = \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1\right)$$

so that $\psi(p) = 0$ and from the Taylor series expansion of second order at 0, $h_2(\Omega)$ is expressed by

$$\rho(x_1, \ldots, x_n) = x_\lambda + \sum_{\sigma,\tau} \delta_{\sigma\tau} x_\sigma x_\tau + o(|'x_\lambda|^2) > 0$$

in a neighborhood of p, where $'x_\lambda = (x_1, \ldots, \widehat{x_\lambda}, \ldots, x_n)$. Here, $[\delta_{\sigma\tau}]$ is the $n \times n$ Hessian matrix at p. Since $R(\varphi^h) = p$, one point, every point except $K(\varphi^h)$ converges to p. Let $\pi_0 = T_{p_0} h_1(\Omega)$ be the tangent space at $p_0 =$

$h_1(q)$. Choose an n-simplex E_0 at p_0 with vertices $\{e_1^\circ, \ldots, e_n^\circ\}$ satisfying (a) $e_\lambda^\circ \notin \pi_0$. (b) $e_\ell^\circ \in \pi_0$ for $\ell \neq \lambda$. (c) $\overline{E}_0 \cap K(\varphi^h) = \emptyset$. Since $K(\varphi^h) \cap \overline{E}_0 = \emptyset$, we learn that $\lim_{\nu \to \infty} \varphi_\nu(\overline{E}_0) \subset R(\varphi^h) = p$ uniformly. Without loss of generality, we may assume that $\lambda = 1$. Let $\varphi_\nu(p_0) = c^\nu = (c_1^\nu, \ldots, c_n^\nu)$. Now, define affine coordinate changes $\Phi_\nu, \Psi_\nu, R_\nu, \Gamma_\nu, L_\nu$ and follow the proof of Theorem 3. To analyze the limit equation after applying the affine coordinate changes above, consider the coefficients of quadratic terms $\hat{x}_k\hat{x}_m$'s with $1 \leq k, m \leq n$. For notation, put $(r_{21}^\nu, \ldots, r_{n1}^\nu) = (\beta_2^\nu, \ldots, \beta_n^\nu)$. They are of the form $\sum_{\sigma,\tau=2} \delta_{\sigma\tau} \frac{r_{\sigma k}^\nu r_{\tau m}^\nu}{\alpha_\nu^2}$. Recall the definition of $(r_{2\ell}^\nu, \ldots, r_{n\ell}^\nu) = (\Phi_\nu \varphi_\nu)(e_\ell^\circ)$. In our φ^h-related affine coordinate ψ, this can be written as, with $v_{e_\ell^\circ} = (q_1, \ldots, q_n, 1)$ and $v_{p_0} = (p_1, \ldots, p_n, 1)$,

$$\begin{aligned} r_{i\ell}^\nu &= a_i^\nu \cdot v_{e_\ell^\circ} / a_{n+1}^\nu \cdot v_{e_\ell^\circ} - a_i^\nu \cdot v_{p_0} / a_{n+1}^\nu \cdot v_{p_0} \\ &= \frac{a^\nu \sum_{u=1}^{n+1} \frac{a_{iu}^\nu}{a^\nu} q_u}{\sum_{u=1}^{n+1} a_{n+1u}^\nu q_u} - \frac{a^\nu \sum_{u=1}^{n+1} \frac{a_{iu}^\nu}{a^\nu} p_u}{\sum_{u=1}^{n+1} a_{n+1u}^\nu p_u} \qquad (2 \leq i \leq n). \end{aligned}$$

Since $\varphi_\nu \to \varphi^h$ as $\nu \to \infty$ for a normalized $\{\varphi_\nu\}$, $a_{n+1}^\nu \to (0, \ldots, 0, 1)$. If $\{\alpha_\nu\} = \{r_{j\mu}^\nu\}$,

$$\begin{aligned} &\frac{r_{\sigma k}^\nu r_{\tau m}^\nu}{\alpha_\nu^2} = \frac{r_{\sigma k}^\nu}{r_{j\mu}^\nu} \cdot \frac{r_{\tau m}^\nu}{r_{j\mu}^\nu} \to \\ &\lim_{\nu \to \infty} \frac{a^\nu \sum_{u=1}^{n+1} \frac{a_{\sigma u}^\nu}{a^\nu}(v_u - p_u)}{a^\nu \sum_{u=1}^{n+1} \frac{a_{ju}^\nu}{a^\nu}(\tilde{v}_u - p_u)} \cdot \lim_{\nu \to \infty} \frac{a^\nu \sum_{u=1}^{n+1} \frac{a_{\tau u}^\nu}{a^\nu}(v'_u - p_u)}{a^\nu \sum_{u=1}^{n+1} \frac{a_{ju}^\nu}{a^\nu}(\tilde{v}_u - p_u)} \\ &= \lim_{\nu \to \infty} \frac{\sum_{u=1}^{n+1} \frac{a_{\sigma u}^\nu}{a^\nu}(v_u - p_u)}{\sum_{u=1}^{n+1} \frac{a_{ju}^\nu}{a^\nu}(\tilde{v}_u - p_u)} \cdot \lim_{\nu \to \infty} \frac{\sum_{u=1}^{n+1} \frac{a_{\tau u}^\nu}{a^\nu}(v'_u - p_u)}{\sum_{u=1}^{n+1} \frac{a_{ju}^\nu}{a^\nu}(\tilde{v}_u - p_u)} \\ &= \frac{\sum_{u=1}^{n} \xi_{\sigma u}(v_u - p_u)}{\sum_{u=1}^{n} \xi_{ju}(\tilde{v}_u - p_u)} \cdot \frac{\sum_{u=1}^{n} \xi_{\tau u}(v'_u - p_u)}{\sum_{u=1}^{n} \xi_{ju}(\tilde{v}_u - p_u)}, \end{aligned}$$

where $v_{e_\sigma^\circ} = (v_1, \ldots, v_n, 1)$, $v_{e_\tau^\circ} = (v'_1, \ldots, v'_n, 1)$, and $v_{e_\mu^\circ} = (\tilde{v}_1, \ldots, \tilde{v}_n, 1)$ are representative vectors of e_σ°, e_τ°, and e_μ°, respectively. From our choice of $\{\alpha^\nu\}$, we have

$$\left| \frac{\sum_{u=1}^{n} \xi_{\sigma u}(v_u - p_u)}{\sum_{u=1}^{n} \xi_{ju}(\tilde{v}_u - p_u)} \cdot \frac{\sum_{u=1}^{n} \xi_{\tau u}(v'_u - p_u)}{\sum_{u=1}^{n} \xi_{ju}(\tilde{v}_u - p_u)} \right| < \infty.$$

Notice that $(v_1-p_1,\dots,v_n-p_n)$, $(v'_1-p_1,\dots,v'_n-p_n)$, and $(\tilde{v}_1-p_1,\dots,\tilde{v}_n-p_n)$ are vectors corresponding to the edges of the simplex E_0 at p_0 in affine space, i.e., $\{e_1^\circ - p_0, e_2^\circ - p_0, \dots, e_n^\circ - p_0\}$, which is linearly independent in $\mathbb{R}^n$. Hence this is a basis of $\mathbb{R}^n$. Write the basis as $\{w_1,\dots,w_n\}$. Consequently, one gets

$$\begin{aligned} \sum_{\sigma,\tau=2} \delta_{\sigma\tau} \frac{r^\nu_{\sigma k} r^\nu_{\tau m}}{\alpha_\nu^2} &\to \frac{1}{(\xi^p_j \cdot w_\mu)^2} \sum_{\sigma,\tau} \delta_{\sigma\tau} (\xi^p_\sigma \cdot w_k)(\xi^p_\tau \cdot w_m) \\ &= \frac{1}{(\xi^p_j \cdot w_\mu)^2} \sum_{\sigma,\tau} \delta_{\sigma\tau} (\xi^p_\sigma \cdot w_k)(\xi^p_\tau \cdot w_m) \\ &= \frac{1}{(\xi^p_j \cdot w_\mu)^2} B(w_k, w_m)\ , \end{aligned}$$

where $B(\cdot\,,\cdot)$ is a symmetric bilinear form defined by the $(n+1)\times(n+1)$ matrix $B = {}^tM_{\varphi^h} H_p M_{\varphi^h} = {}^t[\xi^p_{ij}][\delta_{\sigma\tau}][\xi^p_{ij}]$. Since a bilinear form is uniquely determined by $B(w_k, w_m)$'s, where $\{w_1,\dots,w_n\}$ is a basis, we have $(\xi^p_j \cdot w_\mu) \neq 0$ and there must exist a coefficient $\sum_{\sigma,\tau=2} \delta_{\sigma\tau} \frac{r^\nu_{\sigma k} r^\nu_{\tau m}}{\alpha^2_\nu} \neq 0$ by the assumption ${}^tM_{\varphi^h} H_p M_{\varphi^h} \neq 0$ on $\mathbb{R}^n$ and $\delta_{1\tau} = 0 = \delta_{\sigma 1}$ for $1 \le \sigma, \tau \le n$. □

Remark. Let Ω be an affine domain defined by $\{(x,y) \in \mathbb{R}^2 \mid y > x^3\}$. Then

$$\left\{ \begin{bmatrix} \sqrt[3]{\lambda} & 0 \\ 0 & \lambda \end{bmatrix} \mid \lambda < 1 \right\}$$

has a subsequence whose limit φ satisfies $R(\varphi) = \{0\}$. As we can see in this example, which was advised me by K. H. Jo, φ and $Hess_\rho(p)$ does not satisfy the condition that ${}^tM_{\varphi^h} H_p M_{\varphi^h} \neq 0$ on $\mathbb{R}^2$ in Theorem 5.

Let us define a conical limit point p of the boundary of a domain as follows. There is a point $q \in \mathbb{R}P^n$ and an h-normalized sequence of automorphism group of the given domain such that $\lim_{\nu\to\infty} \varphi_\nu(q) = \varphi^h(q) = p$ and for all sufficiently large ν, $\varphi_\nu(q)$ is contained in a cone C_p which doesn't intersect the degenerate directions at p. The following corollary is an easy application of Theorem 5.

Corollary 2. *Let $\{h(\Omega), G^h\}$ be an h-normalization of Ω of dimension n and $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$. Suppose that $\partial\Omega$ is C^2 in a neighborhood of $q \in \partial\Omega$ with $R(\varphi) = \{q\}$. If $p = h_2(q) = \Pi(\mathbf{e}_{n+1})$ is a conical point, then Ω is projectively equivalent to one of $Q(b_2,\dots,b_n)$.*

In general, we do not know if the range of singular map consists of exactly one point when there is a boundary accumulation point. One can give similar theorems and proofs even for domains and sequences of automorphisms having dimension of range ≥ 1. There is a special C^2 boundary, flat boundary, in which ${}^tM_{\varphi^h}H_pM_{\varphi^h} = 0$. So, we prove that case separately.

Theorem 6. *Let Ω be an n-dimensional projective domain with $\{\varphi_\nu\} \subset \mathrm{Aut}(\Omega)$. If $\partial\Omega$ is flat in a tubular neighborhood of $R(\varphi)$ in an affine chart, then Ω is projectively equivalent to $Q_\lambda(0,\ldots,0)$.*

Proof. If $\partial\Omega \subset K(\varphi)$, then $\dim K(\varphi) = n-1 = \dim\partial\Omega$ and $\partial\Omega = K(\varphi)$, which clearly implies Ω is projectively equivalent to $Q_\lambda(0,\ldots,0)$. Suppose that $\partial\Omega - K(\varphi) \neq \emptyset$. If $p_0 \in \partial\Omega - K(\varphi) \neq \emptyset$, $p = \lim_{\nu\to\infty}\varphi_\nu(p_0) = \varphi(p_0) \in R(\varphi) \cap \partial\Omega$. From the assumption, Ω is defined by, without loss of generality, with $\lambda = 1$,

$$\rho(x) = x_1 > 0$$

in the tubular neighborhood of $R(\varphi)$. Define Φ_ν as before. Let $B_1^\nu = |\varphi_\nu(e_1^\circ)| = \sqrt{\beta_1^{\nu 2} + \cdots + \beta_n^{\nu 2}}$ and for each ν, define

$$\Psi_\nu = \begin{cases} u_1 = \dfrac{B_1^\nu}{\beta_1^\nu} y_1 \\ u_\ell = -\dfrac{\beta_\ell^\nu}{\beta_1^\nu} y_1 + y_\ell \quad \ell = 2,\ldots,n. \end{cases}$$

Define $\epsilon_\nu = |\alpha_\nu| > 0$ and R_ν, Γ_ν, L_ν as Theorem 3 to complete the proof. □

Remark. (i) If one assumes, in theorem 6, that there exists $q \in \Omega - K(\varphi)$ satisfying $\lim\limits_{\nu\to\infty} \varphi_\nu(q) = p \in \partial\Omega$, then $R(\varphi)$ cannot be transversal to $\partial\Omega$, for if one takes an open neighborhood $N_q \subset \Omega$ of q which does not meet the kernel $K(\varphi)$, then $\lim_{\nu\to\infty}\varphi_\nu(N_q) = \varphi(N_q)$ is an open subset of the range $R(\varphi)$ containing q as an interior point.

(ii) If Ω is hyperbolic and $\mathrm{Aut}(\Omega)$ contains a subgroup G which acts syndetically on Ω, then, for any boundary point $p \in \partial\Omega$, there exist $q \in \Omega$ and a sequence $\{\varphi_\nu\} \subset G$ such that $\lim\limits_{\nu\to\infty} \varphi_\nu(q) = p$ and $K(\varphi)\cap\Omega = \emptyset$. Therefore, $R(\varphi)$ cannot be transversal to $\partial\Omega$ at p for such domain Ω and G.

(iii) Consider the following affine domain Ω defined by $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid z > 0\}$ and

$$S = \left\{ \varphi_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{n} \end{bmatrix} \mid n = 1,2,\ldots \right\} \subset \mathrm{Aut}(\Omega).$$

Clearly, for $\varphi = \lim\limits_{\nu\to\infty}\varphi_\nu$, one has $R(\varphi) = \{(x,y,0) \in \mathbb{R}^3 \mid x,y \in \mathbb{R}\}$. Let $D \subset R(\varphi)$ be any 2-dimensional bounded domain containing 0. Then $D\times(0,\infty)$ is preserved by S but $D \times (0,\infty)$ is not projectively equivalent to $Q(0,\ldots,0)$.

This example shows that one needs to take a tubular neighborhood of $R(\varphi)$ in Theorem 6.

5.3. Cayley surface

This example shows why the direction of convergence is important in quadratic domain while it's not necessary for ellipsoid. Let Ω be the affine domain defined by $\{(x,y,z) \in \mathbb{R}^3 \mid z \geq xy - \frac{1}{3}x^3\}$. Ω is homogeneous by $\mathrm{Aut}(\Omega)$. A sequence of automorphisms

$$\varphi_\nu = \begin{bmatrix} \frac{1}{\nu} & 0 & 0 \\ 0 & \frac{1}{\nu^2} & 0 \\ 0 & 0 & \frac{1}{\nu^3} \end{bmatrix}$$

whose limit $\varphi = \lim_{\nu\to\infty}\varphi_\nu$ satisfies $R(\varphi) = \{0\}$. Ω and $\{\varphi_\nu\}$ are h-normalized with $h = (id, id)$. By a simple computation,

$$H_p = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{\varphi^h} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $p = 0$. So, ${}^tM_{\varphi^h} H_p M_{\varphi^h} = 0$ which shows that the hypothesis of Theorem 5 does not hold for Ω, $\{\varphi_\nu\}$. As a quadratic form on the tangent space $\mathbb{R}^3$ at p, H_p has a degenerate set consisting of two circles which is $Z(H_p) \cap S^{n-1}$. Every point except infinite points converges to the boundary accumulation point along these degenerate directions by the sequence $\{\varphi_\nu\}$. The affine automorphism group $\mathrm{Aut}(\Omega)$ consists of

$$\begin{pmatrix} c & 0 & 0 & a \\ ac & c^2 & 0 & b \\ bc & ac^2 & c^3 & ab - \frac{1}{3}a^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}, c > 0$. Since $\mathrm{Aut}(\Omega)$ acts transitively, we should find a sequences $\{p_\nu\}$ of points in Ω converging to p away from these degenerate directions by a sequence of automorphisms $\{\varphi_\nu\}$ before h-normalization. Theorem 5 says that after h-normalizing $(\Omega, \{\varphi_\nu\})$, the corresponding sequence $\{\tilde{p}_\nu\}$ should converge along its degenerate directions of the corresponding Hessian at p. If we choose a sequence of automorphism $A_n(a_n, b_n, c_n) \in \mathrm{Aut}(\Omega)$ with $a_n, b_n, c_n \to 0$ for $n = 1, 2, 3, \ldots$. $A_n = S_n^{-1} D_n S_n = P_n \tilde{D}_n Q_n$, where S_n's column vectors are eigenvectors of A_n, $P_n, Q_n \in O(4)$ and $D_n, \tilde{D}_n$ are diagonal matrices. Note that $S_n \to Id$ so we see rank of $\lim_n D_n$=rank of $\lim_n \tilde{D}_n$. Furthermore,

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This implies that the condition of Theorem 5 is not satisfied for the sequence A_n. We get the following statement obtained from this arguments.

Corollary 3. *Let $\Omega \subset \mathbb{R}P^n$ be a domain whose boundary is C^2 and not quadratic. If $\{\varphi_\nu\}$ is a sequence of automorphisms whose limit has a range of one point, then every point in an open dense subset converges to the boundary accumulation point q along degenerate directions at q.*

Finally, we can apply scaling method to get typical bounded domains even if we do not know whether they are bounded or not since scaling method enables us identify domains from local data around boundary points [18].

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References

[1] E. Bedford and S. Pinchuk, *Domains in C^2 with noncompact groups of holomorphic automorphisms*, Mat. Sb. (N.S.) **135(177)** (1988), no. 2, 147–157, 271; translation in Math. USSR-Sb. **63** (1989), no. 1, 141–151.

[2] ______, *Domains in C^{n+1} with noncompact automorphism group*, J. Geom. Anal. **1** (1991), no. 3, 165–191.

[3] J. P. Benzécri, *Sur les variétés localement affines et localement projectives*, Bull. Soc. Math. France **88** (1960), 229–332.

[4] B. Colbois and P. Verovic, *Rigidity of Hilbert metrics*, Bull. Austral. Math. Soc. **65** (2002), no. 1, 23–34.

[5] W. M. Goldman, *Two examples of affine manifolds*, Pacific J. Math. **94** (1981), no. 2, 327–330.

[6] ______, *Geometric structures on manifolds and varieties of representations*, Geometry of group representations (Boulder, CO, 1987), 169–198, Contemp. Math., 74, Amer. Math. Soc., Providence, RI, 1988.

[7] ______, *Convex real projective structures on compact surfaces*, J. Differential Geom. **31** (1990), no. 3, 791–845.

[8] H. Kim, *Actions of infinite discrete groups of projective transformation*, Proc. Second GARC Symposium on Pure and Applied Math. Part **III**, 1–9.

[9] K. T. Kim, *Complete localization of domains with noncompact automorphism groups*, Trans. Amer. Math. Soc. **319** (1990), no. 1, 139–153.

[10] ______, *Geometry of bounded domains and the scaling techniques in several complex variables*, Lecture Notes Series **13**, Research Institutes of Mathematics Global Analysis Research Center, Seoul National University.

[11] S. Kobayashi, *Projectively invariant distances for affine and projective structures*, Differential geometry (Warsaw, 1979), 127–152, Banach Center Publ., 12, PWN, Warsaw, 1984.

[12] N. H. Kuiper, *On convex locally-projective spaces*, Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 200–213. Edizioni Cremonese, Roma, 1954.

[13] K. S. Park, *Some results on the geometry and topology of affine flat manifolds*, Ph. D. thesis, Seoul National University, Aug, 1997.

[14] S. Pinchuk, *Holomorphic inequivalences of some classes of domains in $\mathbb{C}^n$*, Mat. USSR Sbornik **39** (1981), 61–86.

[15] E. Socié-Méthou, *Caracterisation des ellipsoides par leurs groupes d'automorphismes*, Ann. Sci. Ecole Norm. Sup. (4) **35** (2002), no. 4, 537–548.

[16] J. Vey, *Sur les automorphismes affines des ouverts convexes saillants*, Ann. Scuola Norm. Sup. Pisa (3) **24** (1970), 641–665.

[17] E. B. Vinberg and V. G. Kac, *Quasi-homogeneous cones*, Math. Notes **1** (1967), 231–235, (translated from Mathematicheskie Zametki, Vol. **1** (1967), no. 3, 347–354.

[18] C. W. Yi, *Projective domains with non-compact automorphism groups II*, preprint.

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