A Generic Craig Form for the Two-Dimensional Gaussian Q-Function

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ABSTRACT—In this letter we present a generic Craig form for the two-dimensional (2-D) Gaussian Q-function. The presented Craig form provides an alternative solution to the problems of computing probabilities involving a form of the 2-D Gaussian Q-function.

Keywords—Two-dimensional Gaussian Q-function, Craig form.

I. Introduction

There has been interest in computing probabilities involving a form of the two-dimensional (2-D) Gaussian *Q*-function defined by the double integral in [1] as equation (26.3.3):

$$Q(x,y;\rho)=\int_y^\infty\int_x^\infty \frac{1}{2\pi\sqrt{1-\rho}}\exp\left[-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right]dudv. \quad (1)$$

The 2-D Gaussian *Q*-function in (1) has been used in evaluating the performance of digital communication systems over fading channels [2]-[3]. Recently, a Craig form has been presented as an alternative form of the Gaussian *Q*-function and has been employed for the calculation of the average error probabilities over fading channels [4]-[5]. In [6], the 2-D Gaussian *Q*-function in (1) has been alternatively expressed in a Craig form as

$$Q(x,y;\rho)=\frac{1}{2\pi}\int_0^{\tan^{-1}[(\sqrt{1-\rho^2}x/y)/(1-\rho x/y)]}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta + \frac{1}{2\pi}\int_0^{\tan^{-1}[(\sqrt{1-\rho^2}y/x)/(1-\rho y/x)]}\exp\left(-\frac{y^2}{2\sin^2\theta}\right)d\theta,$$
$$\text{for} \quad x\ge 0 \quad \text{and} \quad y\ge 0. \quad (2)$$

Note that in (2), the $\tan^{-1}(\cdot)$ in the upper limit of the integral can be calculated by the following identity:

$$\tan^{-1}\left(\frac{x}{y}\right)=\frac{\pi}{2}\left[1-\text{sgn}(y)\right]+\text{sgn}(y)\tan^{-1}\left(\frac{x}{|y|}\right), \quad (3)$$

where $\text{sgn}\,x=+1$ if $x\ge 0$ and $\text{sgn}\,x=-1$ if $x<0$.

Here, we observe that $\tan^{-1}(\cdot)$ in (3) is a user-defined function, and (2) needs a redundant sorting to compute $Q(x,y:\rho)$ for the calculation of the integration area in the second, third, and fourth quadrants as was done in [2]. However, with regard to numerical computation, it is important to obtain a generic expression for the 2-D Gaussian *Q*-function. To solve these problems, we present a generic Craig form for the 2-D Gaussian *Q*-function. The presented Craig form can be useful to compute the 2-D Gaussian *Q*-function without user-defined function and redundant sorting.

II. Derivation

Our starting point is to make use of the identity of the tangent inverse function given in [7] as

$$\tan^{-1}u=\cos^{-1}\frac{1}{\sqrt{1+u^2}}, \quad \text{for} \quad u>0. \quad (4)$$

Applying (4) to the $\tan^{-1}(\cdot)$ in the upper limit of first term in the right hand side of (2) and using the trigonometry for $\sin^{-1}u+\cos^{-1}u=\pi/2$, we can rewrite the upper limit of (2) as

$$\tan^{-1}\frac{\sqrt{1-\rho^2}x/y}{1-\rho x/y}=\frac{\pi}{2}+\sin^{-1}\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}},$$
$$\text{for} \quad x>0 \text{ and } y>0. \quad (5)$$

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Substituting (5) to (2) yields

$$Q(x,y;\rho)=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{x^2}{2\sin^2\Phi}\right)d\Phi$$
$$+\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho y-x}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{y^2}{2\sin^2\Phi}\right)d\Phi,$$
$$\text{for } x>0 \text{ and } y>0. \quad (6)$$

For the second, third, and fourth quadrants of the integral regions, the 2-D Gaussian Q-function $Q(x, y:\rho)$ can be obtained with the help of equations (26.3.8) and (26.3.9) in [1] as follows:

$$Q(x,y;\rho)=-\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{x^2}{2\sin^2\Phi}\right)d\Phi$$
$$+\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho y-x}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{y^2}{2\sin^2\Phi}\right)d\Phi,$$
$$\text{for } x<0 \text{ and } y>0, \quad (7)$$

$$Q(x,y;\rho)=1-\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{x^2}{2\sin^2\Phi}\right)d\Phi$$
$$-\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho y-x}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{y^2}{2\sin^2\Phi}\right)d\Phi,$$
$$\text{for } x<0 \text{ and } y<0, \quad (8)$$

and

$$Q(x,y;\rho)=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{x^2}{2\sin^2\Phi}\right)d\Phi$$
$$-\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho y-x}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{y^2}{2\sin^2\Phi}\right)d\Phi,$$
$$\text{for } x>0 \text{ and } y<0. \quad (9)$$

Finally, we obtain a Craig form for the 2-D Gaussian Q-function in the form of a generic expression as in (10).

$$Q(x,y;\rho)=\frac{1}{2}\mathrm{sgn}[\mathrm{sgn}(x)+\mathrm{sgn}(xy)]-\frac{1}{2}\mathrm{sgn}(x)$$
$$+\mathrm{sgn}(x)\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho x-y}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{x^2}{2\sin^2\Phi}\right)d\Phi$$
$$+\mathrm{sgn}(y)\frac{1}{2\pi}\int_0^{\frac{\pi}{2}+\sin^{-1}\left(\frac{\rho y-x}{\sqrt{x^2-2\rho xy+y^2}}\right)}\exp\left(-\frac{y^2}{2\sin^2\Phi}\right)d\Phi,$$
$$\text{for } -\infty<x,y<\infty \text{ and } x,y\neq 0. \quad (10)$$

III. Conclusion

In this letter, we presented a generic Craig form for the 2-D Gaussian Q-function. The result of this work provides an alternative solution to the problems of computing probabilities involving a form of the 2-D Gaussian Q-function.

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