Prediction of Gain Expansion and Intermodulation Performance of Nonlinear Amplifiers

Muhammad Taher Abuelma'atti

A mathematical model for the input-output characteristic of an amplifier exhibiting gain expansion and weak and strong nonlinearities is presented. The model, basically a Fourier-series function, can yield closed-form series expressions for the amplitudes of the output components resulting from multisinusoidal input signals to the amplifier. The special case of an equal-amplitude two-tone input signal is considered in detail. The results show that unless the input signal can drive the amplifier into its nonlinear region, no gain expansion or minimum intermodulation performance can be achieved. For sufficiently large input amplitudes that can drive the amplifier into its nonlinear region, gain expansion and minimum intermodulation performance can be achieved. The input amplitudes at which these phenomena are observed are strongly dependent on the amplifier characteristics.

Keywords: Nonlinear amplifiers, intermodulation.

Manuscript received Apr. 20, 2006; revised Nov. 30, 2006.

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I. Introduction

Prediction of the intermodulation performance of nonlinear power amplifiers has attracted the attention of many researchers over the years. Of particular interest here is the phenomenon described as sweet points or gain expansion. *Gain expansion* refers to the fact that the input-output characteristic of the amplifier first follows a linear relationship; then, it experiences a faster rate of change; finally, it follows a slower rate ending in saturation (see Fig. 1(a)). In fact, the characteristic illustrated in Fig. 1(a) is a typical example of an amplifier exhibiting both weak and strong nonlinearities. On the other hand, the term *sweet points* refers to the fact that because of the gain expansion of the amplifier, the intermodulation distortion attains a minimum at a certain value of the input. This phenomenon has been observed and explicitly reported by many authors; see for example [1]-[11] and the references cited therein. Other authors have observed and reported the achievement of a minimum intermodulation distortion but without explicitly mentioning gain expansion or sweet points; see for example [12]-[14] and the references cited therein. However, despite the extensive description of this phenomenon, both implicitly and explicitly, only few attempts have been made to model it mathematically using a Taylor series expansion [1],[2],[5],[9],[14] ; a model comprising linear, sinusoidal, and hyperbolic terms [13]; or a Volterra series and describing function [6]. While the use of the Taylor series is mathematically simple and straightforward, it suffers from the difficulties faced in extracting its parameters and the need to use a large number of terms in order to obtain an accurate fit for the nonlinear characteristics of the amplifier exhibiting gain expansion. In fact, increasing the number of Taylor series terms usually allows the obtained model more flexibility in order to come closer to the input data. Thus, while the fit at the input data improves by increasing the number of terms, the overall fit deteriorates because of the oscillation between input data points. Moreover, the use of Taylor series expansion requires curve-fitting techniques that invariably demand extensive computing facilities and well-developed software. On the other hand, the use of models involving hyperbolic tangent functions cannot yield closed-form expressions for the intermodulation performance of the amplifier. Usually, such models are intended to model the device characteristic used in a simulation program like SPICE. The same argument applies to models based on the use of the Volterra series and the describing function. Moreover, the Volterra series is feasible only for small signal regimes where the input cannot travel deeply into the saturation region of the nonlinear amplifier. Therefore, prediction of the large signal performance, where the signal drives the amplifier deeply into saturation, requires the use of the describing function. This led to the simultaneous use of a Volterra series associated with a describing function approach [6]. By virtue of its derivation, this approach cannot lead to closed-form expressions to predict the gain expansion and intermodulation performance of nonlinear amplifiers working under large signal conditions driving the amplifier deeply into its saturation region.

It appears, therefore, that there is a need for a simple mathematical model to describe the nonlinear characteristics of amplifiers exhibiting gain expansion. Such a model, if available, would be very helpful for designers to obtain a quick understanding of the phenomenon and to predict, at least roughly and before using extensive software simulation packages, the input signal levels at which sweet points may occur. The main intention of this paper is to develop such a mathematical model and to illustrate its use in predicting the gain expansion and intermodulation performance of nonlinear amplifiers.

II. Proposed Model

Here we propose representing the normalized nonlinear input-output characteristic of an amplifier exhibiting gain expansion by the well known Fourier series model

$$y(x) = a_o + \sum_{m=1}^{M}\left(a_m \cos(\frac{m\pi}{T}x) + b_m \sin(\frac{m\pi}{T}x)\right), \qquad (1)$$

where x and y are the normalized input and output, respectively. The parameters a_o, a_m, b_m, and T are fitting parameters selected to provide the best fit between (1) and the input-output characteristic of the nonlinear amplifier. These parameters can be obtained using standard least-square approximations, Fourier transform, or the procedure described in [15] and [16]. This procedure is simple and does not require extensive computing facilities or well developed software. For convenience, a brief description of this procedure is given here. First, remove the offset at x_1=0, if any second, mirror image the nonlinear input-output characteristic of the amplifier to obtain a complete period (2T). Third, approximate the resulting characteristic by N–1 straight line segments joined end-to-end. The x-value of the segment joins is termed a knot. The number of knots and their positions must generally be chosen so that closer knots are placed in regions where the function $y(x)$ is changing rapidly. The knots are not necessarily equally-spaced. By denoting the slope of each segment as α_n it is easy to show that the a_o, a_m, and b_m can be expressed as [15], [16]

$$a_o = \frac{1}{2T}\left[\frac{1}{2}x_2 y_2 + \frac{1}{2}(x_N - x_{N-1})y_{N-1} + \sum_{s=2}^{N-2}(x_{s+1}-x_s)y_{s+1} - \frac{1}{2}(x_{s+1}-x_s)(y_{s+1}-y_s)\right], \qquad (2)$$

$$a_m = \frac{-T}{(m\pi)^2}\left[\alpha_1 - \alpha_{N-1} + \sum_{s=1}^{N-2}(\alpha_{s+1}-\alpha_s)\cos(\frac{m\pi}{T}x_{s+1})\right], \qquad (3)$$

and

$$b_m = \frac{-T}{(m\pi)^2}\left[\sum_{s=1}^{N-2}(\alpha_{s+1}-\alpha_s)\sin(\frac{m\pi}{T}x_{s+1})\right]. \qquad (4)$$

Using this procedure, the parameters a_o, a_m, b_m, and T of any nonlinear amplifier can be easily calculated.

If the normalized input x is formed of a multisinusoidal signal of the form

$$x = \sum_{k=1}^{K} X_k \sin\omega_k t, \qquad (5)$$

where $\sum_{k=1}^{K}|X_k| \le 1.0$, then combining (1) and (5) and using the trigonometric identities, we have

$$\sin(\vartheta \sin\omega t) = 2\sum_{l=0}^{\infty} J_{2l+1}(\vartheta)\sin(2l+1)\omega t$$

and

$$\cos(\vartheta \sin\omega t) = J_o(\vartheta) + 2\sum_{l=1}^{\infty} J_{2l}(\vartheta)\cos 2l\omega t\,,$$

where $J_l(\vartheta)$ is the ordinary Bessel function of order l. After simple mathematical manipulations, it is easy to show that the amplitude of the normalized odd-order output component of frequency $\sum_{k=1}^{K}\alpha_k\omega_k$ and order $\sum_{k=1}^{K}|\alpha_k|$, where α_k is a positive or negative integer or zero, will be given by

$$Y_{\alpha_1,\alpha_2,\cdots,\alpha_K} = 2\sum_{m=1}^{M} b_m \prod_{k=1}^{K} J_{|\alpha_k|}\left(\frac{m\pi}{T}X_k\right), \tag{6}$$

$$\sum_{k=1}^{K} |\alpha_k| = \text{odd},$$

where $J_{|\alpha_k|}(.)$ is the Bessel function of order $|\alpha_k|$. Using (6), the odd-order intermodulation performance of any nonlinear amplifier can be investigated when the normalized input is formed of the multisinusoidal signal of (5). A similar expression can be obtained for the amplitude of the normalized even-order output components by combining (1) and (5).

III. Special Case

To illustrate the use of (6) in predicting the gain expansion and intermodulation performance of nonlinear amplifiers, consider the nonlinear amplifier with the normalized input-output characteristic shown in Fig. 1(a). Obviously, this characteristic does not represent a real amplifier. However, it represents a worst case with the amplifier exhibiting gain expansion in addition to weak and strong nonlinearities. In fact, this amplifier characteristic cannot be represented by a single analytical model that covers the full range of the input. Thus, it can be used as a vehicle to illustrate the applicability of the proposed analysis. Following the procedure described in the previous section, a_0=a_m=0 and T=2.0. Also the parameter b_m can be expressed as

$$b_m = \frac{8A}{\gamma}\frac{1}{(m\pi)^2}\sin\left(\frac{m\pi}{2}\gamma\right) + \frac{8B}{\beta-\gamma}\frac{1}{(m\pi)^2}\left(\sin\left(\frac{m\pi}{2}\beta\right) - \sin\left(\frac{m\pi}{2}\gamma\right)\right), \tag{7}$$

where $m = 1,3,5,\cdots$. In fact, (7) was obtained by decomposing the characteristic of Fig. 1(a) as shown in Fig. 1(b) and (c) and using the results readily available in [17]. The parameters a_0=a_m=0 are equal to zero because of the odd-symmetry of the nonlinear amplifier characteristic.

To illustrate the use of (6) and (7) for predicting the gain expansion and the intermodulation performance of the nonlinear amplifier characterized by Fig. 1(a), the special case of an input signal formed of two equal-amplitude sinusoids will be considered in detail. For the special case of K=2, the input signal is formed of two sinusoids each of amplitude X. Using (6), the amplitudes of the output fundamental and intermodulation products of the normalized output of the idealized nonlinear amplifier characteristic of Fig. 1(a) can be expressed as

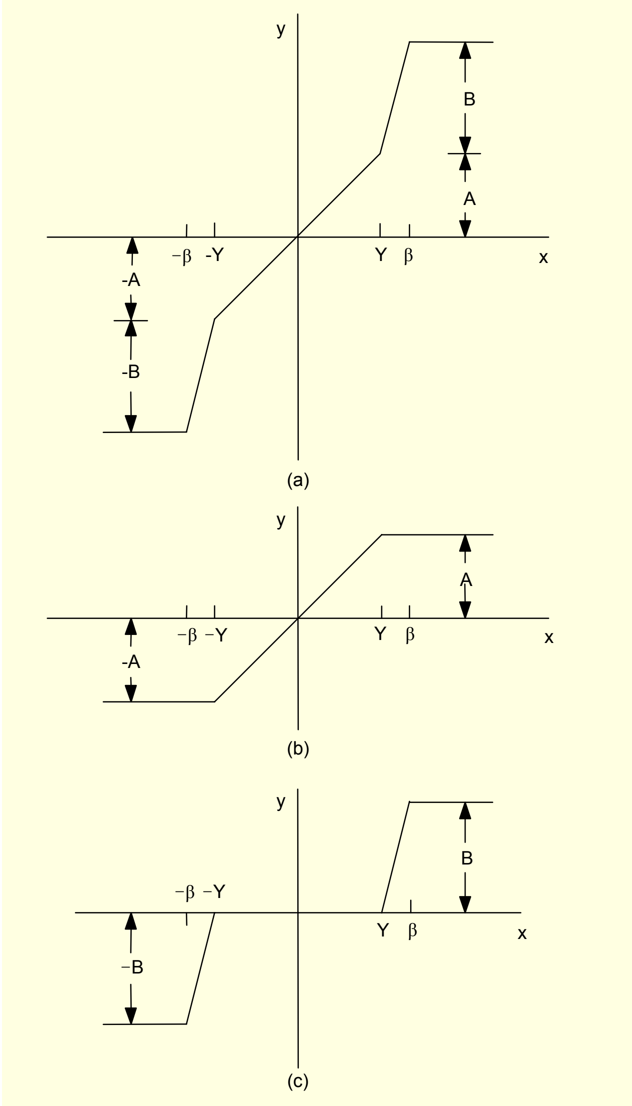

Fig. 1. (a) Input-output characteristic of a nonlinear amplifier exhibiting gain expansion and weak and strong nonlinearities. The characteristics of (a) are the summation of the characteristics in (b) and (c). In all cases, the maximum normalized input $=\pm 1.0$.

$$Y_1 = 2\sum_{m=1}^{M} b_m J_1\left(\frac{m\pi}{T}X\right)J_0\left(\frac{m\pi}{T}X\right), \tag{8}$$

and

$$Y_{n,j} = 2\sum_{m=1}^{M} b_m J_{|n|}\left(\frac{m\pi}{T}X\right)J_{|j|}\left(\frac{m\pi}{T}X\right), \tag{9}$$

$$|n| + |j| = \text{odd integer},$$

where n and j are either positive or negative integers or zero. Using (8) and (9) the gain and the relative third-order intermodulation can be expressed as

$$G = \frac{Y_1}{X}, \tag{10}$$

and

$$IMD3 = \frac{Y_{2,1}}{Y_1}. \tag{11}$$

Using (10) and (11), the gain and intermodulation performance of the nonlinear amplifier exhibiting the gain expansion characteristic of Fig. 1(a) and weak and strong nonlinearities can be studied for different scenarios of the parameters β, γ, A , and B. The results are shown in Figs. 2 to 4.

Figure 2 clearly shows that, as expected, for the set of values of the parameters $\gamma = 0.2$, $\beta = 0.8$, $A = 0.5$, and B=0.5, as well as $\gamma = 0.0$, $\beta = 0.5$, $A = 0.0$, and $B = 1.0$, the gain starts at a constant value, corresponding to the initial linear characteristic, and then starts to decrease monotonically with no gain expansion. For all other sets of the parameters β, γ, A, and B, as expected, the gain starts at a constant value, followed by a range of increase of gain expansion, then followed by monotonically decreasing gain.

Figures 3 and 4 show that the intermodulation performance does not change monotonically and it suffers from discontinuities, especially for normalized input amplitudes that are not sufficient to drive the amplifier into the nonlinear part of its characteristic. For example, with $\gamma = 0.0$, $\beta = 0.5$, $A = 0.0$, and $B = 1.0$, the intermodulation performance suffers discontinuities for values of $X < 0.25$. Similarly, for $\gamma = 0.5$, $\beta = 0.55$, $A = 0.5$, and $B = 0.5$, the intermodulation

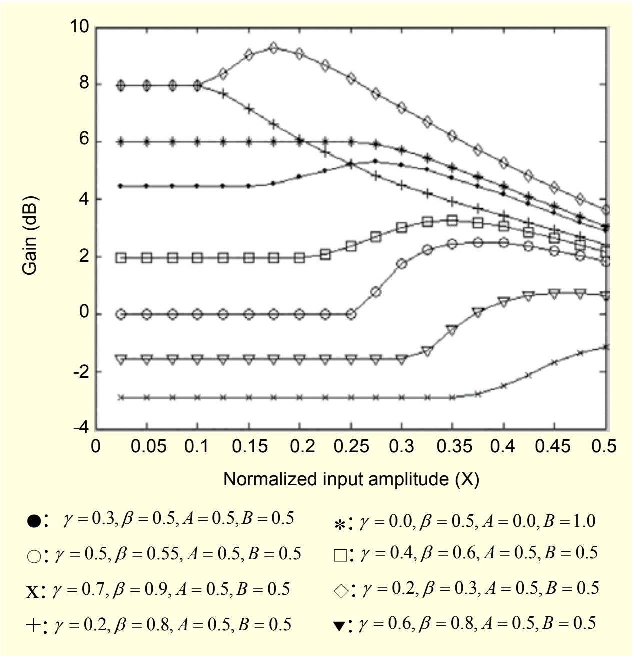

Fig. 2. Variation of the gain with normalized input amplitude X.

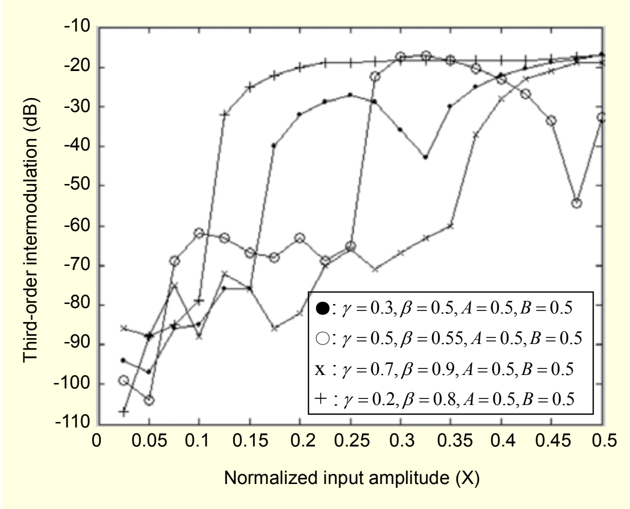

Fig. 3. Variation of the third-order intermodulation with normalized input amplitude X (I).

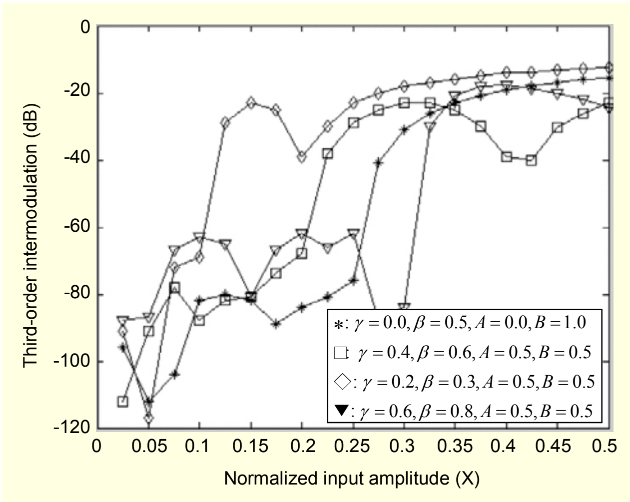

Fig. 4. Variation of the third-order intermodulation with normalized input amplitude X (II).

performance suffers discontinuities for values of X<0.25. This can be attributed to the fact that if the amplifier is not driven outside the linear part of its characteristic, the intermodulation products are expected to be very small, ideally zero; thus, their calculated magnitudes may be close to or inside the error range of the Fourier series approximation used especially when the relative intermodulation is below -70 dB. However, for normalized input amplitudes sufficiently large to drive the amplifier into its nonlinear characteristics, the amplifiers with gain expansion show a minimum in their intermodulation performance. For example, with $\gamma = 0.2$, $\beta = 0.8$, $A = 0.5$, and $B = 0.5$, as well as $\gamma = 0.0$, $\beta = 0.5$, $A = 0.0$, and $B = 1.0$, there is no minimum in the intermodulation performance as these amplifiers do not exhibit gain expansion as shown in Fig. 2. However, for $\gamma = 0.3$, $\beta = 0.5$, $A = 0.5$, and $B = 0.5$, there is a minimum at X=0.325; for $\gamma = 0.5$, $\beta = 0.55$, $A = 0.5$, and $B = 0.5$, there is a minimum at X=0.475; and for $\gamma = 0.2$, $\beta = 0.3$, $A = 0.5$, and $B = 0.5$, there is a minimum at X=0.2. These observations are in excellent agreement with the experimental observations reported in [2], [6], [8].

IV. Discussion and Conclusion

In this paper a simple, yet powerful technique has been presented for obtaining a tractable mathematical model for the input-output characteristics of a nonlinear amplifier exhibiting gain expansion in addition to weak and strong nonlinearities. The parameters of the model can be extracted easily using simple calculations and without recourse to sophisticated software or computer programming. Using this model, closed-form series expressions can be obtained for the gain and intermodulation components resulting from exciting the nonlinear amplifier by a multisinusoidal input signal.

The special case of an equal-amplitude two-tone input signal amplified by an amplifier exhibiting gain expansion in addition to weak and strong nonlinearities was considered in detail. The results obtained clearly show that when the amplifier exhibits gain expansion it can also achieve the minimum in relative intermodulation performance. However, the location of this minimum is strongly dependent on the shape of the nonlinear amplifier characteristics.

It is worth mentioning here that the analysis presented in this paper assumes that the nonlinear amplifier under consideration exhibits only AM/AM nonlinear characteristics. However, bandpass amplifiers exhibiting both AM/AM and AM/PM nonlinear characteristics are often found in practical applications. Extension of the analysis presented in this paper to accommodate the AM/AM and AM/PM nonlinearities is simple and straightforward using the quadrature model proposed in [18]-[20]. This model combines the AM/AM nonlinearity $F(A)$ and AM/PM nonlinearity $\phi(A)$ into two new quadrature-nonlinearities expressed as

$$P(A) = F(A)\cos\phi(A) \tag{12}$$

and

$$Q(A) = F(A)\sin\phi(A). \tag{13}$$

In (12) and (13), $P(A)$ and $Q(A)$ represent the inphase and quadrature nonlinearities. Given the AM/AM and AM/PM nonlinear characteristics of an amplifier, the inphase and quadrature nonlinearities can be deduced using (12) and (13). These two characteristics are then modeled by the mathematical model of (1). Equation (6) can be used to predict the amplitudes of the intermodulation components resulting from these two nonlinearities. The resultant amplitude of the intermodulation product at any frequency is the vector summation of the two quadrature components resulting from the two quadrature nonlinearities.

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