On Diagonal Loading for Robust Adaptive Beamforming Based on Worst-Case Performance Optimization

Jing-ran Lin, Qi-cong Peng, and Huai-zong Shao

Robust adaptive beamforming based on worst-case performance optimization is investigated in this paper. It improves robustness against steering vector mismatches by the approach of diagonal loading. A closed-form solution to optimal loading is derived after some approximations. Besides reducing the computational complexity, it shows how different factors affect the optimal loading. Based on this solution, a performance analysis of the beamformer is carried out. As a consequence, approximated closed-form expressions of the source-of-interest power estimation and the output signal-to-interference-plus-noise ratio are presented in order to predict its performance. Numerical examples show that the proposed closed-form expressions are very close to their actual values.

Keywords: Robust adaptive beamforming (RABF), optimal diagonal loading, worst-case performance optimization, performance analysis.

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I. Introduction

It is well known that the performance of adaptive beamforming is very sensitive to steering vector mismatches [1], [2]. In the presence of such errors, the beamformer tends to misinterpret the desired source as interference and suppress it. This phenomenon is the so-called signal cancellation. So far many approaches, referred to as robust adaptive beamforming (RABF), have been proposed to improve the performance of adaptive beamforming [2]-[13]. Among these approaches, diagonal loading is widely used for its simplicity [5]-[13]. However, how to select the loading level remains a crucial and open problem. If this parameter is not chosen properly, the robustness of the diagonal loading approach may be insufficient.

In traditional opinion, the loading level chosen should be higher than the noise power but lower than the lowest interference eigenvalue. Most of the early suggested methods are rather ad hoc in choosing the loading. For example, it can be set at 5 to 10 dB above the noise power or it can be fixed equal to the standard deviation of the diagonal entries of the covariance matrix. In another study, the loading is calculated simply according to the eigenvalues of the covariance matrix (see [5] and the references therein). These methods definitely provide improved robustness. However, since the chosen loadings are not directly related to the steering vector uncertainty, they are not guaranteed to be always optimal when the uncertainty changes. As a result, the robustness improvement may be insufficient.

Recently, a few methods [6]-[11] have been proposed which determine the optimal loading by defining the so-called uncertainty set and optimizing the worst-case performance. For this reason, these methods are referred to as worst-case RABFs

(W-RABF) in this paper. Adaptively choosing the loading according to the steering vector uncertainty, W-RABF tends to outperform the ad hoc approaches previously mentioned. However, optimal loading is still solved mainly by iteration at present; using, for example, the second-order cone program (SOCP) method [6] and Newton's method [7]-[9]. The iterative methods may suffer from slow convergence or non-convergence unless the initial point for searching is selected very carefully. In these slow convergence or non-convergence cases, a heavy computational burden is inevitable. Moreover, the iterative methods help little in revealing what factors can affect the optimal loading and how to affect it. Some advances in this field have been achieved in [10], in which a closed-form solution to the optimal weight vector of W-RABF, instead of the optimal loading of it, is presented in the general-rank signal case. In [10], the diagonal loading problem is turned into a generalized eigenvalue problem. Although it is very simple and computationally efficient, it does not tell clearly how different factors affect the optimal loading. Since the essence of W-RABF is to improve robustness by imposing a diagonal matrix onto the covariance matrix, it is still useful and necessary to study how to determine the optimal loading according to such parameters as the steering vector uncertainty, the noise power, the source power, and so on.

In this paper, W-RABF is investigated further and a closed-form solution to optimal loading is suggested after some approximations. Besides its simplicity and low computational cost, the solution reveals how different factors affect optimal loading. Then, a performance analysis of W-RABF is carried out, concentrating mainly on the source of interest (SOI) power estimation and the signal-to-interference-plus-noise ratio (SINR). Approximated closed-form expressions are also presented to predict its performance. Numerical examples demonstrate that the proposed closed-form expression of optimal loading is very close to its actual value. Moreover, the results of the performance analysis, based on the approximated solution to optimal loading, predict the behavior of W-RABF very accurately.

It should be pointed out here that, besides steering vector mismatches, errors in the covariance matrix, namely, the so-called finite sample effects, may lead to a clear performance degradation of W-RABF. Diagonal loading can also improve robustness against this kind of error [5], [10], [11]. Therefore, when determining the optimal loading, both the steering vector mismatch and the finite sample effect can be considered in order to achieve better performance. However, as shown in [10]-[12] and the studies referenced therein, this makes sense only when the two errors are of the same order of magnitude. Since the magnitude of finite sample effects is typically $O(1/N)$, where N is the number of snapshots, it can be ignored in most practical situations, compared with steering vector mismatches. For this reason, only steering vector mismatches are considered in this paper, under the assumption that the true covariance matrix is available. The problem of how to determine the loading when both errors are considered will be addressed in our next work [11].

The paper is organized as follows. The background of the signal model and W-RABF is presented in section II. In section III, a closed-form solution to optimal loading is suggested after some approximations. A performance analysis concentrating on SOI power estimation and SINR is presented in section IV. Section V presents numerical simulation results, and conclusions are drawn in section VI.

II. Background

Consider an array of M sensors and let $\mathbf{R}$ denote the covariance matrix of the array output vector.

$$\mathbf{R} = \sigma_s^2 \mathbf{a}\mathbf{a}^H + \sum_{k=1}^{K} \sigma_{Jk}^2 \mathbf{a}_{Jk}\mathbf{a}_{Jk}^H + \sigma_n^2 \mathbf{I} = \sigma_s^2 \mathbf{a}\mathbf{a}^H + \mathbf{J}, \tag{1}$$

where σ_s^2 and σ_{Jk}^2, $k = 1, 2, \cdots, K$, are the powers of the SOI and the K uncorrelated interference signals, $\mathbf{a}$ and $\mathbf{a}_{Jk} \in \mathbf{C}^{M\times 1}$ denote the corresponding steering vectors, σ_n^2 is the power of the spatially white noise, $\mathbf{J}$ is the interference-plus-noise matrix, and $(\cdot)^H$ stands for the complex conjugate transpose. In practice, $\mathbf{R}$ is generally approximated by

$$\mathbf{R} \approx \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}(n)\mathbf{x}^H(n), \tag{2}$$

where N is the number of snapshots and $\mathbf{x}(n) \in \mathbf{C}^{M\times 1}$ is the array output vector at the time index n. Here, assume that N is large enough that the finite sample effect can be ignored.

The W-RABF can be described as [6], [7]

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{R}\mathbf{w} \quad \text{s.t.} \quad \min_{\|\mathbf{c}-\tilde{\mathbf{a}}\|^2 \le \varepsilon^2} \left|\mathbf{w}^H\mathbf{c}\right| \ge 1, \tag{3}$$

where $\tilde{\mathbf{a}} \in \mathbf{C}^{M\times 1}$ is the presumed value of $\mathbf{a}$ and ε is the steering vector distortion bound with $\varepsilon^2 < \|\tilde{\mathbf{a}}\|^2$ to avoid a trivial solution.

It is easy to prove [6], [7] that (3) is equivalent to

$$\begin{aligned} &\min_{\mathbf{w}} \mathbf{w}^H\mathbf{R}\mathbf{w} \\ &\text{s.t.} \ \ \mathbf{w}^H\tilde{\mathbf{a}} = \varepsilon\|\mathbf{w}\| + 1 \text{ and } \operatorname{Im}\{\mathbf{w}^H\tilde{\mathbf{a}}\} = 0, \end{aligned} \tag{4}$$

and the corresponding optimal weight vector is

$$\mathbf{w}_o = \frac{\xi}{\xi\tilde{\mathbf{a}}^H(\mathbf{R}+\xi\mathbf{I})^{-1}\tilde{\mathbf{a}} - \varepsilon^2}(\mathbf{R}+\xi\mathbf{I})^{-1}\tilde{\mathbf{a}}. \tag{5}$$

Obviously, it belongs to the diagonal loading approach in which ξ is the loading level. In (5), only the parameter of ξ is unknown and it can be determined by inserting (5) into the constraints in (4) as

$$\left|\mathbf{w}_o^H \tilde{\mathbf{a}} - 1\right|^2 = \varepsilon^2 \left\|\mathbf{w}_o\right\|^2 . \quad (6)$$

Unfortunately, this task is by no means easy. Many existing methods [6]-[9] solve it by iteration. The iterative methods, however, are computationally demanding and give little information regarding how different factors affect optimal loading. In the next section, an approximated closed-form solution will be suggested.

III. Optimal Loading of W-RABF

Applying the matrix inverse lemma and using the result of (1), it follows that

$$(\mathbf{R} + \xi\mathbf{I})^{-1} = (\mathbf{J} + \xi\mathbf{I})^{-1}\left[\mathbf{I} - \frac{\sigma_s^2 \mathbf{a}\mathbf{a}^H (\mathbf{J} + \xi\mathbf{I})^{-1}}{1 + \sigma_s^2 \mathbf{a}^H (\mathbf{J} + \xi\mathbf{I})^{-1}\mathbf{a}}\right]. \quad (7)$$

Perform the eigenvalue decomposition of $\mathbf{J}$ as

$$\mathbf{J} = \mathbf{V}\boldsymbol{\Gamma}\mathbf{V}^H = \mathbf{V}_J \boldsymbol{\Gamma}_J \mathbf{V}_J^H + \sigma_n^2 \mathbf{V}_n \mathbf{V}_n^H , \quad (8)$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_M)$ is the eigenvalue matrix of $\mathbf{J}$ with the eigenvalues arranged in descending order: $\gamma_1 > \gamma_2 > \cdots > \gamma_K > \gamma_{(K+1)} = \gamma_{(K+2)} = \cdots = \gamma_M = \sigma_n^2$, $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_M]$ with $\mathbf{v}_m \in \mathbf{C}^{M\times 1}$ the eigenvector corresponding to γ_m, $m = 1, 2, \cdots, M$, respectively, $\boldsymbol{\Gamma}_J = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_K)$, $\mathbf{V}_J = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_K]$, and $\mathbf{V}_n = [\mathbf{v}_{(K+1)}, \mathbf{v}_{(K+2)}, \cdots, \mathbf{v}_M]$.

Assume that all the K interference signals are outside the main beam of the array. The projection of the SOI steering vector onto the interference subspace is small. Further assume that the powers of the interference signals are large compared with that of the spatially white noise ($\gamma_k >> \sigma_n^2$, $k = 1, 2, \cdots, K$). Under these assumptions, using methods similar to those in [12] and [14], it follows that

$$\begin{aligned} \mathbf{a}^H (\mathbf{J} + \xi\mathbf{I})^{-p}\mathbf{a} &\approx \sum_{m=K+1}^{M} \frac{\left|\mathbf{v}_m^H \mathbf{a}\right|^2}{(\gamma_m + \xi)^p} = \frac{\left\|\mathbf{a}_n\right\|^2}{(\sigma_n^2 + \xi)^p}, \\ \mathbf{a}^H (\mathbf{J} + \xi\mathbf{I})^{-p}\tilde{\mathbf{a}} &\approx \sum_{m=K+1}^{M} \frac{\mathbf{a}^H \mathbf{v}_m \mathbf{v}_m^H \tilde{\mathbf{a}}}{(\gamma_m + \xi)^p} = \frac{\mathbf{a}_n^H \tilde{\mathbf{a}}_n}{(\sigma_n^2 + \xi)^p}, \end{aligned} \quad (9)$$

where $\mathbf{a}_n = [\mathbf{v}_{(K+1)}, \cdots, \mathbf{v}_M]^H \mathbf{a} = \mathbf{V}_n^H \mathbf{a}$, $\tilde{\mathbf{a}}_n = [\mathbf{v}_{(K+1)}, \cdots, \mathbf{v}_M]^H \tilde{\mathbf{a}} = \mathbf{V}_n^H \tilde{\mathbf{a}}$, $\|\mathbf{a}_n\|^2 = \mathbf{a}_n^H \mathbf{a}_n$, and p is a positive integer.

Inserting (5), (7), and (9) into (6), after some direct derivations, it follows that

$$\left\|\tilde{\mathbf{a}}_n\right\|^2 - 2L\left|\mathbf{a}_n^H \tilde{\mathbf{a}}_n\right|^2 + L^2 \left|\mathbf{a}_n^H \tilde{\mathbf{a}}_n\right|^2 \left\|\mathbf{a}_n\right\|^2 = \left(\frac{\xi + \sigma_n^2}{\xi\varepsilon^{-1}}\right)^2, \quad (10)$$

where scale L is given by

$$L = \sigma_s^2 \Big/ (\xi + \sigma_n^2 + \sigma_s^2 \left\|\mathbf{a}_n\right\|^2).$$

Let $\mathbf{a}_n = \tilde{\mathbf{a}}_n + \boldsymbol{\Delta}_n$ and when the steering vector error, ε, is small, it follows that $\|\mathbf{a}_n\|^2 \approx \|\tilde{\mathbf{a}}_n\|^2$ and $\|\boldsymbol{\Delta}_n\|^2$ is negligible compared with $\|\mathbf{a}_n\|^2$ and $\|\tilde{\mathbf{a}}_n\|^2$. Consequently,

$$\begin{aligned} \left|\mathbf{a}_n^H \tilde{\mathbf{a}}_n\right|^2 &= \tilde{\mathbf{a}}_n^H \mathbf{a}_n \mathbf{a}_n^H \tilde{\mathbf{a}}_n = \tilde{\mathbf{a}}_n^H (\tilde{\mathbf{a}}_n + \Delta_n)\mathbf{a}_n^H (\mathbf{a}_n - \Delta_n) \\ &\approx \left\|\tilde{\mathbf{a}}_n\right\|^2 \left\|\mathbf{a}_n\right\|^2 + \left\|\mathbf{a}_n\right\|^2 \Delta_n^H \Delta_n - \tilde{\mathbf{a}}_n^H \Delta_n \mathbf{a}_n^H \Delta_n . \end{aligned} \quad (11)$$

Eliminating the second-order terms of $\boldsymbol{\Delta}_n$ in (11), it follows that

$$\left|\mathbf{a}_n^H \tilde{\mathbf{a}}_n\right|^2 \approx \left\|\tilde{\mathbf{a}}_n\right\|^2 \left\|\mathbf{a}_n\right\|^2 . \quad (12)$$

This approximation is reasonable in practice. First, RABF based on diagonal loading can no longer compensate for the steering vector mismatch when the mismatch exceeds a specific threshold [12]. Second, the larger the distortion bound ε is, the poorer capability RABF will possess in rejecting interference signals adaptively [2], [9]. In short, the use of diagonal loading in RABF is only advisable for small and middle-sized ε.

A simplified version of (10) can be obtained by inserting (12) into (10) as

$$\left(\frac{\left\|\tilde{\mathbf{a}}_n\right\|}{\xi + \sigma_n^2 + \sigma_s^2 \left\|\mathbf{a}_n\right\|^2}\right)^2 = \left(\frac{\varepsilon}{\xi}\right)^2 . \quad (13)$$

Thus, (6) is approximated by (13) from which two simple closed-form solutions (a positive one plus a negative one) to optimal loading can be easily found. Applying the eigenvalue decomposition of $\mathbf{R}$ and using similar methods to those in [8] and [9], it can be proved that (6) has a unique positive solution; therefore, the optimal loading is

$$\xi_0 = \frac{\varepsilon(\sigma_n^2 + \sigma_s^2 \left\|\mathbf{a}_n\right\|^2)}{\left\|\tilde{\mathbf{a}}_n\right\| - \varepsilon} . \quad (14)$$

The following points are straightforward. First, compared with the iterative methods, (14) reveals how different factors affect the optimal loading, namely, ξ_o depends on the noise level σ_n^2, the SOI power σ_s^2, the steering vector distortion bound ε, and the norm of the steering vector projection onto the subspace orthogonal to the interference subspace, that is, $\|\mathbf{a}_n\|^2$ and $\|\tilde{\mathbf{a}}_n\|^2$. Second, ξ_o increases as ε increases. When ε approaches $\|\tilde{\mathbf{a}}_n\|$, an infinite loading is required. Consequently, the W-RABF turns to a data-independent beamformer and no adaptive capability of rejecting interference signals is achieved. On the other hand, when ε decreases to zero, ξ_o approaches zero asymptotically and then the W-RABF turns to the standard adaptive beamforming (SABF).

It should be noted that, in (14), σ_n^2 and ε are known or can be estimated in advance. For example, σ_n^2 can be measured directly in the source and interference free case. When the source and interference are present, it still can be estimated simply from those eigenvalues associated with the noise subspace of $\mathbf{R}$ [2]. Where ε is concerned, a coarse knowledge regarding the propagation channel, which is usually available, can be exploited to determine it properly [10].

On the other hand, σ_s^2 is unknown in (14) and there seems to be no simple method to estimate σ_s^2 from the interference and noise contaminated data directly, without knowledge of the optimal weight vector in advance. So there may be some difficulties in applying (14) in practice if σ_s^2 is unavailable. Additionally, σ_s^2, $\|\mathbf{a}_n\|^2$ is unknown. However, when ε is small, as mentioned in (11), $\|\mathbf{a}_n\|^2$ can be replaced by $\|\tilde{\mathbf{a}}_n\|^2$. Combined with the fact that the projection of the SOI steering vector onto the interference subspace is small, further simplifications can be made, namely, $\|\tilde{\mathbf{a}}_n\|^2 \approx \|\tilde{\mathbf{a}}_n\|^2 \approx M$. Thus, the following simpler result can be found:

$$\xi_0 \approx \frac{\varepsilon(\sigma_n^2 + \sigma_s^2\|\mathbf{a}_n\|^2)}{\|\mathbf{a}_n\| - \varepsilon} \approx \frac{\varepsilon(\sigma_n^2 + \sigma_s^2 M)}{\sqrt{M} - \varepsilon}. \tag{15}$$

On the basis of (14) or (15), the corresponding closed-form optimal weight vector of W-RABF is

$$\mathbf{w}_o = \frac{\xi_o}{\xi_o \tilde{\mathbf{a}}^H (\mathbf{R} + \xi_o \mathbf{I})^{-1} \tilde{\mathbf{a}} - \varepsilon^2} (\mathbf{R} + \xi_o \mathbf{I})^{-1} \tilde{\mathbf{a}}. \tag{16}$$

Note that another closed-form solution to the optimal weight vector of W-RABF was proposed in [10]. Since it is the solution to a generalized eigenvalue problem, the vector is referred to as $\mathbf{w}_{GE}$ in this paper, which is given as

$$\mathbf{w}_{GE} = \mathcal{P}\{\mathbf{R}^{-1}(\tilde{\mathbf{a}}\tilde{\mathbf{a}}^H - \varsigma \mathbf{I})\}, \tag{17}$$

where $P\{\cdot\}$ denotes the principle eigenvector of the matrix between the brackets and ς is the distortion bound of the signal matrix associated with ε.

Regarding $\mathbf{w}_o$ and $\mathbf{w}_{GE}$, the following obvious conclusions can be drawn. First, in this paper, the traditional problem of solving the optimal loading ξ_o is studied, and then the optimal weight vector $\mathbf{w}_o$ is determined. However, in [10], the authors address the W-RABF as a generalized eigenvalue problem and they get another expression of the weight vector $\mathbf{w}_{GE}$. Second, $\mathbf{w}_o$ is an approximated closed-form solution while $\mathbf{w}_{GE}$ is an actual one. However, as will be shown in the next section, the performance of $\mathbf{w}_o$ almost coincides with that of $\mathbf{w}_{GE}$, implying that $\mathbf{w}_o$ approximates $\mathbf{w}_{GE}$ very accurately with some possible scale ambiguities. Third, $\mathbf{w}_o$ is suitable for rank-one signal models only, while $\mathbf{w}_{GE}$ is suitable for general-rank signal models. Fourth, although the expressions of $\mathbf{w}_o$ and $\mathbf{w}_{GE}$ are obviously different, they both belong to the diagonal loading approach. In $\mathbf{w}_o$, a positive loading $\xi_o\mathbf{I}$ is applied to $\mathbf{R}$, while in $\mathbf{w}_{GE}$, a negative loading $-\varepsilon\mathbf{I}$ is applied to $\tilde{\mathbf{a}}\tilde{\mathbf{a}}^H$. Finally, $\mathbf{w}_o$ and $\mathbf{w}_{GE}$ consume comparable computational complexity requiring $O(M^3)$ flops. Applying some fast online implementations such as those methods in [10] can reduce it to $O(M^2)$ flops. In contrast, the iterative methods are computationally more demanding. For example, the SOCP method requires $O(\rho M^3)$ flops with ρ the number of iterations. Newton's method, although computationally more efficient than the SOCP method, consumes at least $100M$ flops more than the proposed method.

IV. Performance Analysis

The SOI power estimation and the output SINR are used here to measure the performance of the W-RABF.

1. SOI Power Estimation

Usually the term $\mathbf{w}_o^H\mathbf{R}\mathbf{w}_o$ is used to estimate σ_s^2 as

$$\hat{\sigma}_s^2 = \mathbf{w}_o^H \mathbf{R} \mathbf{w}_o = \frac{1}{\hat{\mathbf{a}}^H (\mathbf{R} + \xi\mathbf{I})^{-1} \mathbf{R} (\mathbf{R} + \xi\mathbf{I})^{-1} \tilde{\mathbf{a}}}. \tag{18}$$

Applying the previously mentioned approximations and after some direct algebraic manipulations, it follows that

$$\hat{\sigma}_s^2 = \frac{(\xi_o + \sigma_n^2 + \sigma_s^2\|\mathbf{a}_n\|^2)^2}{\|\tilde{\mathbf{a}}_n\|^2 (\sigma_n^2 + \sigma_s^2 \|\mathbf{a}_n\|^2)}. \tag{19}$$

Inserting (14) into (19), it follows that

$$\hat{\sigma}_s^2 = \frac{(\sigma_n^2 + \sigma_s^2\|\mathbf{a}_n\|^2)}{(\|\tilde{\mathbf{a}}_n\| - \varepsilon)^2}. \tag{20}$$

Given $(\|\tilde{\mathbf{a}}_n\| - \varepsilon)^2 \approx (\|\mathbf{a}_n\| - \varepsilon)^2 \leq \|\mathbf{a}_n\|^2$ for small steering vector errors, it can be concluded that the W-RABF is likely to overestimate the SOI power: $\hat{\sigma}_s^2 > \sigma_s^2$. If there is no steering vector error, it follows that $\hat{\sigma}_s^2 \approx \sigma_s^2$. In the high signal-to-noise ratio (SNR) scenarios, $(\sigma_s^2 >> \sigma_n^2)$ is consistent with the fact that the W-RABF will then turn to SABF. When ε increases, $\hat{\sigma}_s^2$ increases, too.

On the basis of (20), the following new SOI power estimator can be presented to eliminate the overestimation:

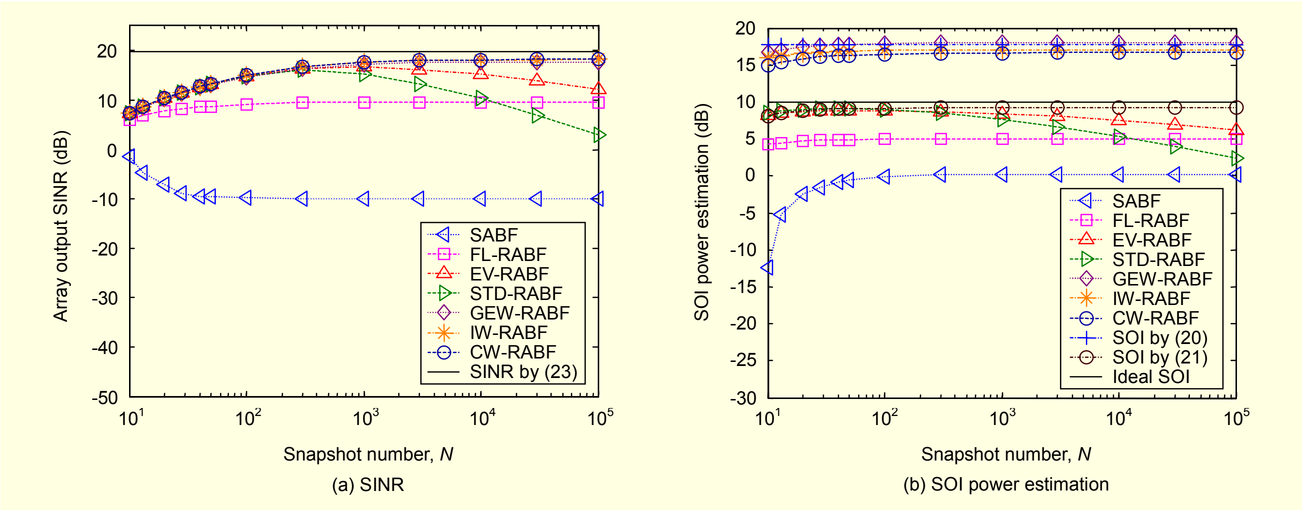


Fig. 1. Performance comparisons based on snapshot number N at $\Delta_s = +2^o$ and $\varepsilon = 1.8017$.

$$\hat{\hat{\sigma}}_s^2 = \frac{(\|\tilde{\mathbf{a}}_n\| - \varepsilon)^2 \hat{\sigma}_s^2 - \sigma_n^2}{\|\tilde{\mathbf{a}}_n\|^2} = \frac{(\|\tilde{\mathbf{a}}_n\| - \varepsilon)^2 \mathbf{w}_o^H \mathbf{R} \mathbf{w}_o - \sigma_n^2}{\|\mathbf{a}_n\|^2} \approx \sigma_s^2. \quad (21)$$

There may be some contradictions. That is, to estimate σ_s^2, one must calculate ξ_o and $\mathbf{w}_o$ first. On the other hand, if one wants to calculate ξ_o and $\mathbf{w}_o$ according to (14) and (16), respectively, the knowledge of σ_s^2 is required, so the practical value of (21) may be restricted. However, it still makes sense in theoretical analysis. For example, it can be used to show whether (14) is an accurate approximation of the actual optimal loading. If not, an obvious difference between $\hat{\hat{\sigma}}_s^2$ and σ_s^2 will be observed at least in simulation.

2. Signal-to-Interference-Plus-Noise Ratio

The SINR is generally computed as

$$\mathrm{SINR} = \frac{\sigma_s^2 |\mathbf{w}_o^H \mathbf{a}|^2}{\mathbf{w}_o^H \mathbf{J} \mathbf{w}_o} = \frac{\sigma_s^2 |\mathbf{a}^H (\mathbf{R} + \xi_o \mathbf{I})^{-1} \tilde{\mathbf{a}}|^2}{\tilde{\mathbf{a}}^H (\mathbf{R} + \xi_o \mathbf{I})^{-1} \mathbf{J} (\mathbf{R} + \xi_o \mathbf{I})^{-1} \tilde{\mathbf{a}}}. \quad (22)$$

Inserting (7), (9), and (12) into (22), it follows that

$$\mathrm{SINR} = \frac{\sigma_s^2 \|\mathbf{a}_n\|^2}{\sigma_n^2} \approx \frac{\sigma_s^2 \|\tilde{\mathbf{a}}_n\|^2}{\sigma_n^2}. \quad (23)$$

It is shown in [13] that (23) is the maximal SINR that the RABF based on diagonal loading can achieve. The corresponding loading is $\xi_m = -(\sigma_n^2 + \sigma_s^2\|\mathbf{a}_n\|^2)$, calculated by letting the derivative of (22) with respect to ξ be zero (see [13] for detail). It is surprising that, despite the apparent differences between the proposed ξ_o and the ξ_m in [13] in both sign and magnitude, the W-RABF can achieve almost the same SINR as its counterpart based on maximal-SINR criterion when ε is small. Another useful conclusion is that the output SINR of the W-RABF is nearly constant and independent of ε when it is small.

V. Numerical Examples

Several numerical examples are presented here to show the correctness and effectiveness of the proposed method. The SOI power estimation and the SINR are used here to measure the performance of the beamformers. The involved algorithms are the following: (a) SABF; (b) RABF with a fixed loading (FL) of 10 dB above the spatially white noise (FL-RABF); (c) RABF with a loading equal to the standard deviation of the diagonal entries of $\mathbf{R}$ (STD-RABF); (d) the RABF with a loading equal to $-\lambda_{\min} + [(\lambda_K - \lambda_{\min})(\lambda_{K+1} - \lambda_{\min})]^{-1/2}$, where $\lambda_{\max} = \lambda_1 \geq \cdots \geq \lambda_M = \lambda_{\min}$ are the eigenvalues of $\mathbf{R}$ (EV-RABF); (e) W-RABF with the optimal weight vector determined by (17), referred to as the GEW-RABF; (f) W-RABF with the optimal loading solved by iterative W-RABF (IW-RABF); (g) W-RABF with the optimal loading given by (14), the closed-form W-RABF (CW-RABF).

Consider a uniform linear array with $M = 10$ sensors spaced half a wavelength apart and the 0 dB spatially white Gaussian noise, $\sigma_n^2 = 1$. Three uncorrelated signals impinge upon the array: the SOI from the angle of 0^o and two interfering signals from 20^o and -30^o. The SOI power is 10 dB, while the powers of the two interfering signals are both 30 dB. For the sake of simplicity, assume the steering vector errors are mainly due to

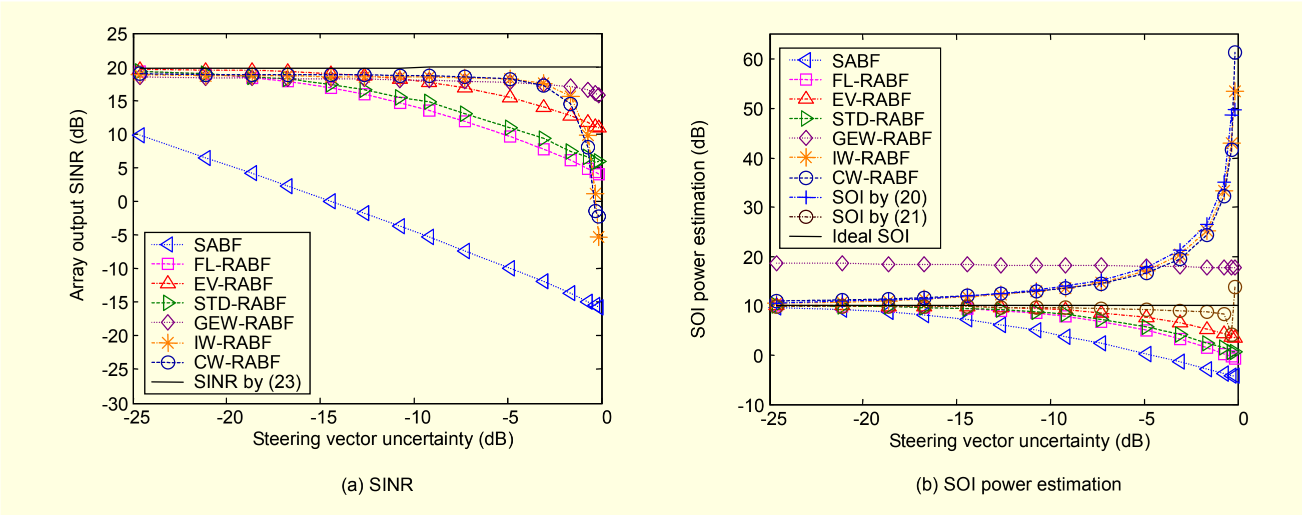


Fig. 2. Performance comparisons based on steering vector uncertainties at $N = 1000$.

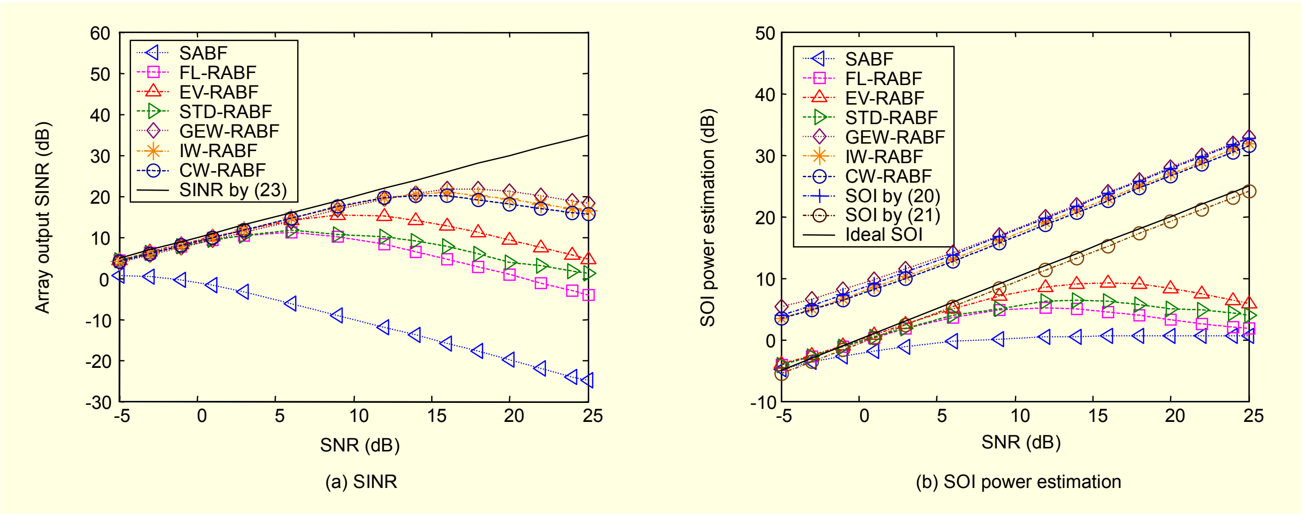
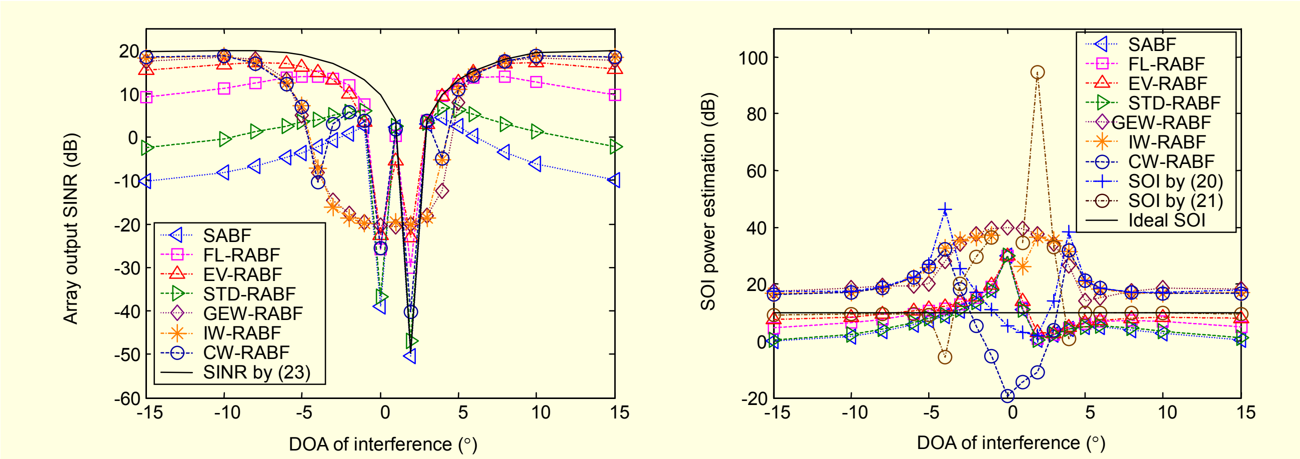


Fig. 3. Performance comparisons based on SNR at $N = 1000$, $\Delta_s = +2^{\circ}$, and $\varepsilon = 1.8017$.

DOA uncertainties here with the DOA estimate error $\Delta_s = 2^{\circ}$ and consequently $\varepsilon = 1.8017$.

The performance comparisons of these algorithms in relation to the number of snapshots N are illustrated in Fig. 1. The SINR comparison is shown in Fig. 1(a) and the SOI power estimation comparison is shown in Fig. 1(b). At each snapshot number, 100 Monte Carlo trials are carried out to get the averaged results. For SINR, it is clear that the performance of SABF is unacceptable in the presence of steering vector mismatches, for it tends to suppress the SOI as interference (signal cancellation). Although FL-RABF improves robustness, since its loading is fixed, its performance is still poor. Selecting diagonal loading adaptively according to the elements of **R**, STD-RABF and EV-RABF outperform FL-RABF. But the chosen loadings are not necessarily optimal, so the performance of STD-RABF and EV-RABF is still somewhat unsatisfactory. On the other hand, the three W-RABFs–GEW-RABF, IW-RABF, and CW-RABF–achieve the best performance among these algorithms. The SINR of CW-RABF almost coincides with that of IW-RABF, implying that (14) approximates the actual optimal loading very accurately. Moreover, the performance lines of IW-RABF and CW-RABF are very close to that of GEW-RABF, showing that $\mathbf{w}_o$ is a very accurate approximation of $\mathbf{w}_{GE}$ or the actual optimal weight vector of W-RABF. Note that when N exceeds 1000, the SINR of W-RABF can be predicted precisely. For SOI power estimation, the result of SABF is far below the actual value due to signal cancellation. The performance lines of FL-RABF, STD-RABF, and EV-RABF indicate that the SOI is still suppressed somewhat in the three algorithms because the chosen loadings are not optimal. Predicted accurately by (20), the three W-RABFs tend to overestimate σ_s^2. Equation (21)



(a) SINR (b) SOI power estimation

Fig. 4. Performance comparisons based on DOA of interference at $N = 1000$, $\Delta_s = +2^o$, and $\varepsilon = 1.8017$.

estimates the SOI power precisely, proving implicitly that (14) is an accurate approximation of the actual optimal loading. Furthermore, when N is small, ignoring finite sample effects causes an error between (14) and the actual optimal loading. As a result, the performance of W-RABF degrades and is similar to that of EV-RABF and STD-RABF. The problem of how finite sample effects affect the optimal loading is detailed in [11].

Changes in the DOA estimation error and the performance comparisons in relation to the steering vector uncertainty are illustrated in Fig. 2 with the snapshot number fixed at $N = 1000$. In this paper, the steering vector uncertainty is defined as $Un = 20 \times log_{10}(\|\mathbf{a}\|^{-1}\varepsilon)$ (dB). Considering the simulation results in Fig. 2, the following points are straightforward. First, even in the case of small steering vector uncertainties, SABF works very poorly compared with other algorithms. Second, FL-RABF works well when the steering vector uncertainty is small, but since its loading is fixed, its performance is still poor in the case of high uncertainties. Third, EV-RABF and STD-RABF outperform FL-RABF as expected. However, because the chosen loadings are not directly related to the steering vector uncertainty, their performance degrades when the uncertainty varies. Fourth, the three W-RABFs achieve the best robustness against steering vector mismatches. Note that CW-RABF, IW-RABF, and GEW-RABF achieve almost the same performance, confirming again that (14) approximates the actual optimal loading very accurately. It is beyond our expectation that although the analysis in this paper is based on the assumption of small or middle-sized ε (uncertainties), (14) still works well for uncertainties as high as -3 dB. When the steering vector uncertainty is below -3 dB, the SINR of W-RABF is nearly constant and independent of ε, in accordance with the analysis in the previous section. When the uncertainty exceeds this bound, the performance of IW-RABF and CW-RABF degrades abruptly. This is because larger uncertainties lead to larger loadings, as shown in (14), which make the robust beamformer act more like a data-independent one with poor capability to suppress interference adaptively [2]. For SOI power estimation, IW-RABF and CW-RABF overestimate σ_s^2 as expected. In addition, as predicted in (20), when the uncertainty increases, more obvious overestimation will occur. This overestimation is eliminated in (21), which predicts the SOI power far more precisely in a wider range of steering vector uncertainties. Furthermore, GEW-RABF acts differently from IW-RABF and CW-RABF in the SOI power estimation. This may be due to the fact that there is the possibility of some scale ambiguity between $\mathbf{w}_o$ and $\mathbf{w}_{GE}$ which affects the SOI power estimation but has no effect on the output SINR.

The performance comparisons in relation to the input SNR at the fixed snapshot number $N = 1000$ are illustrated in Fig. 3. The input SNR is changed by varying the SOI power σ_s^2 while the other parameters are the same as those in the first example. As expected, the three W-RABFs possess the best robustness against SNR variation. It is confirmed again that (14) and (16) accurately approximate the actual optimal loading and weight vector, respectively. Based on the two expressions, (20), (21), and (23) predict the performance of W-RABF effectively.

In the final simulation, only one interference source is reserved and its DOA varies from -15^o to +15^o. The snapshot number is fixed at $N = 1000$ and the other parameters are the same as those in the first simulation. It is obvious that when the interference is outside the main beam, that is, outside the range of about [-10^o, +10^o], the performance of W-RABF is nearly constant. On the other hand, when the interference moves into the main beam, the performance of W-RABF, whether solved by iteration or by (14) degrades dramatically, and even worse than the other four algorithms in this example. This indicates that W-RABF is only advised when the interference is outside the main beam. So the assumption on small projection of the SOI steering vector into the interference subspace is reasonable. Furthermore, the threshold of the difference between SOI and the interference is about 4^{o} in this example. When the difference exceeds this threshold, the approximated results proposed in this paper work well and the performance lines of CW-RABF, in both SINR and SOI power estimation, almost coincide with those of IW-RABF and GEW-RABF. When the difference is below 4^{o}, however, the assumption of small projection is no longer valid and the approximated results deviate from their actual values. As a result, CW-RABF acts strangely.

VI. Conclusion

The approach of W-RABF was investigated in this paper, which belongs to the class of diagonal loading approaches in which the loading determination is based on worst-case performance optimization. A simple closed-form solution to optimal loading was suggested after some approximations. Since the loading can be directly calculated, it is computationally more efficient than the iterative methods. More importantly, the solution reveals how difference factors affect the optimal loading. The optimal loading is determined by the noise level σ_n^2, the SOI power σ_s^2, the steering vector distortion bound ε, and the norm of the steering vector projection onto the subspace orthogonal to the interference subspace, that is, $\|\mathbf{a}_n\|^2$ and $\|\tilde{\mathbf{a}}_n\|^2$.

Based on the closed-form optimal loading, a performance analysis of W-RABF was performed, focusing mainly on the SOI power estimation and the output SINR. Corresponding closed-form expressions were presented to predict the behavior of W-RABF and some useful conclusions about W-RABF were drawn based on these expressions.

Since the optimal loading is chosen adaptively according to the steering vector uncertainty, simulation results show that W-RABF achieves excellent robustness and outperforms many related diagonal loading approaches, such as FL-RABF, EV-RABF, STD-RABF, and so on. It has also been confirmed that the proposed closed-form solution is an accurate approximation of the actual optimal loading and the results of the performance analysis predict the behavior of W-RABF precisely.

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