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Note on σ-derivations in Near-rings and Reduced Near-rings

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Abstract. We study σ-derivations on reduced near-rings and extend certain results on derivations to σ-derivations with some conditions.

1. Introduction

A (left) near-ring is a set with two binary operations $+$ and $\cdot$ such that

(i) $(N, +)$ is a group (not necessarily abelian) with identity 0,

(ii) $(N, \cdot)$ is a semigroup and

(iii) for all $x, y, z \in N$, $x(y + z) = xy + xz$.

Throughout this paper, N will denote a *zero symmetric* near-ring ($a0 = 0a = 0$ for all $a \in N$). A *derivation* d on a near-ring N is an additive endomorphism having $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ and a σ-*derivation* d on N is defined to be an additive endomorphism satisfying the product rule $d(xy) = \sigma(x)d(y) + d(x)y$ for all $x, y \in N$, where σ is an automorphism on N. Posner [7] defined derivations on prime rings. Herstein [3] derived commutativity property of prime rings with derivations. Recently in near-ring theory, Bell, Mason [1], Kamal [4] and Cho [2] researched in prime and semi prime near-rings. A near-ring N is said to be *reduced* if it has no nilpotent elements. N is said to have IFP(*Insertion Factor Property*) if whenever $ab = 0$ with $a, b \in N$ implies $anb = 0$ for all $n \in N$. A σ-*derivation* d on N is said to be *nilpotent* if there exist a positive integer k such that $d^k = 0$. The smallest k having the property is called the *index of nilpotency* and denoted by *nil(d)*. A σ-derivation is called *nil*, if for every $a \in N$, there exists a natural number k such that $d^k(a) = 0$. The smallest such number is denoted by *nil(d,a)*. Clearly, nilpotent derivation is nil but not the converse [2]. A non-empty subset U of N is called a *left* (resp. *right*) $N-$*subset* if $NU \subset U$ (resp. $UN \subset U$). If U is both right and left N-subset, it is called an $N-$*subset* of N. An $N-$ subset U of N, which is also a group with respective to $+$, is called an N-subgroup of N. Basic concepts on near-rings can be found in Meldrum [5] and Pilz [6].

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2. σ-derivations and annihilators

In this section, we see the role of σ-derivations on annihilator ideals of a reduced near-ring. The following lemmas are useful in this sequel.

Lemma 2.1 ([6]). *Let N be a reduced near-ring. If $ab = 0$ for all $a, b \in N$, then $ba = 0$. Moreover, for $s \in N$, $Ann(s) = \{x \in N \mid sx = 0\}$ is an ideal of N.*

Lemma 2.2 ([6]). *Let N be a reduced near-ring. Then N has IFP and for any subset S of N, $Ann(S)$ is an ideal of N.*

Lemma 2.3. *Let d be an additive endomorphism on a near-ring N. Then $d(ab) = \sigma(a)d(b) + d(a)b$ and $d(ab) = d(a)b + \sigma(a)d(b)$ are equivalent, for all $a, b \in N$ and any automorphism σ on N.*

Proof. Follows from the equality $d(a(b + b)) = d(ab + ab)$. □

Lemma 2.4. *Let N be a near-ring and S_1 and S_2 be subsets of N such that $S_1 \subset S_2$. Then $Ann(S_2) \subset Ann(S_1)$.*

Proof. Let $x \in Ann(S_2)$. Then $S_2x= 0$. That is, $s_2x = 0$ for all $s_2 \in S_2$. This means that x annihilates all elements of S_2. In particular, x annihilates all elements of S_1. Therefore, $x \in Ann(S_1)$. This completes the proof. □

Theorem 2.5. *Let d be a σ-derivation on a reduced near-ring N. Then for any subset S of N, $d(Ann(S)) \subset Ann(\sigma(S))$.*

Proof. Let $x \in Ann(S \cap \sigma(S))$. Then $sx = 0$ and $\sigma(s)x = 0$ for all $s \in S$. Since $sx = 0$, we get $d(sx) = 0$ and so by Lemma 2.3, $d(s)x + \sigma(s)d(x) = 0$. Multiplying by $\sigma(s)$ from the right, we get $d(s)x\sigma(s) + \sigma(s)d(x)\sigma(s) = 0\sigma(s) = 0$. Since $\sigma(s)x = 0$, we have $x\sigma(s) = 0$. Thus $d(s)x\sigma(s) = 0$. Therefore, from the above equation $\sigma(s)d(x)\sigma(s) = 0$ and so $d(x)\sigma(s)d(x)\sigma(s) = 0$. That is, $(d(x)\sigma(s))^2 = 0$. Since N is reduced, $d(x)\sigma(s) = 0$. Thus, we obtain $\sigma(s)d(x) = 0$, which means that $d(x) \in Ann(\sigma(S))$. Thus, $d(Ann(S \cap \sigma(S))) \subset Ann(\sigma(S))$. Clearly, $S \cap \sigma(S) \subset S$. By Lemma 2.4, $Ann(S) \subset Ann(S \cap \sigma(S))$. Since derivation d is a function from N to N, we obtain $d(Ann(S)) \subset d(Ann(S \cap \sigma(S)))$. Combining these two inclusions, we have $d(Ann(S)) \subset Ann(\sigma(S))$. □

The above Theorem 2.5 is valid, even if σ is an endomorphism. For the subsequent discussions, we require σ to be an automorphism. When σ is an identity automorphism, we have the following corollary.

Corollary 2.6 [2,Theorem 2.2]**.** *Let d be a derivation on a reduced near-ring. Then every annihilator ideal is invariant under d.*

Theorem 2.7. *Let N be a reduced near-ring and d be a σ-derivation on N. Then $Ann(S) \subset Ann(d(S))$ for any subset S of N.*

Proof. Let S be a subset of N and and d be a σ-derivation on N. Suppose $x \in Ann(S)$. Then $sx = 0$ for all $s \in S$. Since $sx = 0$, we have $d(sx) = 0$.

Therefore, $d(sx) = d(s)x + \sigma(s)d(x) = 0$. Multiplying by $\sigma(x)$ from the right, $d(s)x\sigma(x) + \sigma(s)d(x)\sigma(x) = 0$. Since $sx = 0$, $\sigma(s)\sigma(x) = 0$. By IFP property, $\sigma(s)d(x)\sigma(x) = 0$. Since σ is an automorphism, $d(s)x\sigma(x)\sigma^{-1}(x) = 0\sigma^{-1}(x) = 0$ and so $d(s)xx = 0$. That is, $(d(s)x)(x) = 0$. By IFP, $d(s)xd(s)x = 0$. So, we obtain $(d(s)x)^2 = 0$. Since N is reduced, $d(s)x = 0$. Hence $x \in Ann(d(S))$. Therefore, $Ann(S) \subset Ann(d(S))$. □

Remark 2.8. If we repeat the procedure mentioned in Theorem 2.7 continuously, we have the following ascending chain of annihilator ideals of N namely, $Ann(S) \subset Ann(d(S)) \subset Ann(d^2(S)) \subset \cdots$. In particular, for any $a \in N$, we get $Ann(a) \subset Ann(d(a)) \subset Ann(d^2(a)) \cdots$. Moreover, if the σ-derivation d is nil, there exists a positive integer k such that $d^k(a) = 0$. Thus we have $Ann(a) \subset Ann(d(a)) \subset \cdots \subset Ann(0) = N$. In other words, N has *ascending chain condition* on principal annihilator ideals of N by applying d.

3. Existence of N-subsets and N-subgroups

We conclude this paper, by showing the existence of N-subsets and N-subgroups using the σ-derivation d.

Theorem 3.1. *Let N be a near-ring with a σ-derivation d such that $d^2\sigma d \neq 0$. Then every subnear-ring B generated by $d(N)$ and satisfies $\sigma(B) \subset B$ has a two sided N-subset of N. Moreover, if $\sigma d = d\sigma$, then B has a two sided N-subgroup of N.*

Proof. Clearly, $d(N) \subset B$. So $d^2\sigma(d(N)) \subset d^2\sigma(B)$. Since $d^2\sigma d \neq 0$, we get $d^2\sigma(d(N)) \neq 0$ and there by $d^2\sigma(B) \neq 0$. Select $y \in B$ such that $d^2\sigma(y) \neq 0$. Let $x \in N$. From the definition of derivation, we get

$$d(x\sigma(y)) = d(x)\sigma(y) + \sigma(x)d(\sigma(y)).$$

Since $d(x\sigma(y)) \in B$ and $\sigma(y) \in \sigma(B) \subset B$, $d(x) \in B$ gives $\sigma(x)d(\sigma(y)) \in B$. Thus, we get $\sigma(x)d(\sigma(y)) \in N$ which implies $\sigma(N)d(\sigma(y)) \subset B$. Since σ is an automorphism, we have $\sigma(N) = N$. Therefore, $Nd(\sigma(y)) \subset B$. Also, $d(\sigma(y)\sigma(x)) = d(\sigma(y))\sigma(x) + \sigma(\sigma(y))d(\sigma(x))$. Since $\sigma(y) \in B$, we have $\sigma(\sigma(y)) \in B$. Also $d(\sigma(x)) \in d(N) \subset B$. Therefore, $\sigma(\sigma(y))d(\sigma(x)) \in B$. But $d(\sigma(y))\sigma(x) \in B$. Hence $d(\sigma(y))\sigma(N) \subset B$. This in turn implies that $d(\sigma(y))N \subset B$. Consider $a, b \in N$. Now,

$$\begin{aligned} d(ad(\sigma(y)b)) &= d(a)d(\sigma(y))b + \sigma(a)d(d(\sigma(y))b) \\ &= d(a)d(\sigma(y))b + \sigma(a)(d^2\sigma(y))b + \sigma(d(\sigma(y)))d(b)) \\ &= d(a)d(\sigma(y))b + \sigma(a)d^2(\sigma(y))b + \sigma(a)\sigma(d(\sigma(y)))d(b). \end{aligned}$$

Clearly, $d(ad(\sigma(y))b) \in B$. Since $d(a) \in B, d(\sigma(y))N \subset B$, and so $d(a)d(\sigma(y))b \in B$. Since σ is a homomorphism, $\sigma(a)\sigma(d(\sigma(y)))d(b) = \sigma(ad(\sigma(y)))d(b)$. But $ad(\sigma(y)) \in Nd(\sigma(y)) \subset B$. This means that $\sigma(ad(\sigma(y)) \in \sigma(B) \subset B$. Also $d(b) \in d(N) \subset B$.

Since B is a near-ring, $\sigma(a)\sigma(d(\sigma(y)))d(b)$ is in B. Therefore, $\sigma(a)d^2(\sigma(y))b \in B$. That is, $\sigma(N)d^2(\sigma(y))N \subset B$. Thus, $Nd^2(\sigma(y))N$ is the required two sided N-subset of N contained in B. Further, $d(ad(\sigma(y))) = d(a)d(\sigma(y)) + \sigma(a)d^2(\sigma(y))$. This implies that $\sigma(a)d^2(\sigma(y)) \in B$, which in turn implies that $\sigma(N)d^2(\sigma(y)) \subset B$. Thus, $Nd^2(\sigma(y)) \subset B$. Therefore, $Nd^2(\sigma(y))$ is a left N-subset of N contained in B. Also $d(d(\sigma(y))a) = d^2(\sigma(y))a + \sigma(d(\sigma(y)))d(a)$. Since σ and d commutes,

$$\sigma(d(\sigma(y)))d(a) = d(\sigma(\sigma(y)))d(a) \in B$$

and hence $d^2(\sigma(y))a \in B$. This in turn implies that, $d^2(\sigma(y))N \subset B$. Thus $d^2(\sigma(y))N$ is a right N-subgroup of N contained in B. Hence B has a non-zero two sided N-subgroup of N generated by $d^2(\sigma(y)) \neq 0$. □

Suppose N is any near-ring with a σ-derivation d and σ and d commutes, then the condition $d^2\sigma d \neq 0$ is equivalent to $d^3 \neq 0$. Since the identity homomorphism commutes with the derivation d, if we take σ as identity automorphism in the above Theorem 3.1, we get the following corollary.

Corollary 3.2 [2, Theorem 2.8]**.** *Let N be any near-ring with derivation d such that $d^3 \neq 0$. Then every subnear-ring B generated by $d(N)$ contains an N-Subgroup of N.*

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