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Metric and Spectral Geometric Means on Symmetric Cones

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Abstract. In a development of efficient primal-dual interior-points algorithms for self-scaled convex programming problems, one of the important properties of such cones is the existence and uniqueness of "scaling points". In this paper through the identification of scaling points with the notion of "(metric) geometric means" on symmetric cones, we extend several well-known matrix inequalities (the classical Löwner-Heinz inequality, Ando inequality, Jensen inequality, Furuta inequality) to symmetric cones. We also develop a theory of spectral geometric means on symmetric cones which has recently appeared in matrix theory and in the linear monotone complementarity problem for domains associated to symmetric cones. We derive Nesterov-Todd inequality using the spectral property of spectral geometric means on symmetric cones.

1. Introduction

By [15] and [32] the convex programming problems on a self-scaled convex cone turn into the problems on a symmetric cone (self-dual, homogeneous open convex cone in $\mathbb{R}^n$) which can be realized as an interior of the closed convex cone of square elements in the attached Euclidean Jordan algebra. See also [8], [9], [10], [11] for interior-point algorithms of optimization based on the theory of Euclidean Jordan algebras.

Let V be a Euclidean Jordan algebra and let Ω be the corresponding symmetric cone. Then for the canonical barrier functional $F(x) = -\log\det(x)$ on the symmetric cone Ω, $F'(x) = -x^{-1}, F''(x) = P(x)^{-1}$, where x^{-1} is the Jordan inverse of the element $x \in \Omega$ and the map P is the quadratic representation of the Jordan algebra V. Among other things, the existence and uniqueness (see Theorem 3.2 of [32] and Theorem 5.4 of [15]) of the "scaling point" corresponding to points a and b have played an important role in their development of a theoretical foundation for efficient primal-dual interior-point algorithms for problems of minimizing linear functionals over the intersection of an affine subspace on the cone. By definition, it is a unique element x in Ω with the property that the Hessian F'' at x maps a to b, or equivalently, it is a unique solution belonging to Ω of the quadratic equation

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$P(x)^{-1}a = b$.

On the other hand, over the past three decades the notion of "geometric mean" of positive semi-definite operators on a Hilbert space have played its role in operator theory; operator inequalities, monotone operator functions, operator means, Riccati equations, AGM(arithmetic-geometric mean) for operators, geometries in operator spaces ([1], [2], [3], [6], [18], [33], [39]). Pusz and Woronowicz [37] following Ando [1] generalized the notion of the geometric mean from the case that the variables are non-negative reals to the case that the variables are positive semi-definite operators in Hilbert spaces. The geometric mean $A\#B$ of positive semi-definite operators A and B in a Hilbert space is defined by the maximum among all Hermitian X for which $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ are positive semi-definite. It turns out that if A is invertible, then $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. One of the outstanding properties of the notion of the geometric mean is that if A and B are invertible then $A\#B$ is a midpoint of A and B for a natural Finsler metric (= Thompson's part metric) on the cone of positive definite operators ([34], [35]). In finite dimensional case, $A\#B$ is viewed as the (unique) midpoint of A and B with respect to a natural Riemannian metric on the cone of positive definite matrices [27]. One another remarkable property of the geometric mean $A\#B$ is that it can be regarded as a unique positive definite solution of the Riccati equation $XA^{-1}X = B$ (cf. [21]-[23], [30],[36]). Using the quadratic representations of Jordan algebras, it is quite natural to extend the notion $A\#B$ to any symmetric cones by solving the quadratic equation $P(x)a^{-1} = b$, which is exactly the same problem of scaling point. Although Rothaus [38], Nesterov and Todd [32] proved the existence and uniqueness of the scaling points, the author [27] and recently Faybusovich [11] obtained directly the same result via Jordan algebra theory (see also [40]). Motivated by the aim of expressing explicitly the unique fixed point of a strict monotone symplectic mapping which plays a key role in Kalman filtering theory [5], the author has developed a theory of the geometric mean on symmetric cones in the geometric viewpoint ([4], [27], [28]).

In [12] Fiedler and Pták introduced and studied a new positive definite geometric mean of two positive definite matrices which possesses some of important properties of a geometric mean. The "spectral" geometric mean, denoted by $F(A,B)$, of two positive definite matrices A and B is defined by $F(A,B) = (A^{-1}\#B)^{1/2}A(A^{-1}\#B)^{1/2}$. The most outstanding property of $F(A,B)$ is that its square is similar to AB; hence the eigenvalues of $F(A,B)$ coincide with the positive square roots of the eigenvalues of AB. A natural and possible extension of the notion $F(A,B)$ to any symmetric cones is to define $F(a,b) := P(a^{-1}\#b)^{1/2}a$. But this notion $F(a,b)$ has appeared in the study of a primal-dual potential-reduction algorithm on symmetric cones [11] which simplifies the one of Nesterov and Todd [32].

In the present paper, we provide a self-contained description of metric and spectral geometric means on symmetric cones in the context of Fiedler and Pták [12] toward convex programming problems on symmetric cones or other research areas related to symmetric cones. The plan of the paper is as follows. In Section 2 we provide necessary definitions and results from the theory of Euclidean Jordan algebras and we provide a key lemma (Lemma 2.3) related to the quadratic representation of Euclidean Jordan algebras which makes it possible to move developed matrix theories into symmetric cones. In Section 3 we review some geometric view-points of the geometric means on symmetric cones which give us a source to call it the "metric" geometric mean. We show that the geometric mean $a\#b$ of a and b is a midpoint of a and b with respect to the three invariant metrics, a natural Riemannian metric, Thompson's part metric, and Hilbert's projective (pseudo) metric. In Section 4 we introduce the notion of spectral geometric means on symmetric cones and develop various properties of the spectral geometric mean in the context of Fiedler and Pták [12]. In Section 5, we give an application of some properties of the spectral geometric mean obtained in Section 4 . We derive an inequality which turns out to be a very useful inequality in primal-dual interior-point method (Theorem 5.2 of [32], Proposition 3.5 of [11]). In Section 6, we extend some well-known matrix inequalities (e.g., the classical Löwner-Heinz inequality, Ando inequality, Jensen inequality, Furuta inequality) associated to the geometric mean of positive semi-definite matrices into symmetric cones using the results developed in Section 2 and Section 3.

2. Symmetric cones and the Löwner ordering

We recall certain basic notions and well-known facts concerning Jordan algebras from the book [7] by J. Faraut and A. Korányi. A *Jordan algebra* V over the field $\mathbb{R}$ or $\mathbb{C}$ is a finite-dimensional commutative algebra satisfying $x^2(xy) = x(x^2y)$ for all $x, y \in V$. Denote by L the regular representation $L(x)y = xy$, and set $P(x) = 2L(x)^2 - L(x^2)$ for $x \in V$. An element $x \in V$ is said to be invertible if there exists an element y in the subalgebra generated by x and e such that $xy = e$.

The following appears at Proposition II.3.1 and Proposition II.3.2 of [7].

Proposition 2.1. *Let V be a Jordan algebra.*

(i) *An element x in V is invertible if and only if $P(x)$ is invertible. In this case: $P(x)^{-1} = P(x^{-1})$.*

(ii) *If x and y are invertible, then $P(x)y$ is invertible and $(P(x)y)^{-1} = P(x^{-1})y^{-1}$.*

(iii) *For any elements x and y :*
$$P(P(x)y) = P(x)P(y)P(x).$$

(iv) $P(\exp x) = \exp 2L(x)$, *where* $\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

A real Jordan algebra V equipped with an inner product $\langle\cdot|\cdot\rangle$ is said to be *Euclidean* if

$$\langle xy|z\rangle = \langle y|xz\rangle \tag{2.1}$$

for all $x, y, z \in V$. The spectral theorem (Theorem III.1.2 of [7]) of a Euclidean Jordan algebra V states that for $x \in V$ there exist a Jordan frame (a complete system of orthogonal primitive idempotents) $c_1, \cdots, c_r$ (r is the rank of V) and real numbers $\lambda_1, \cdots, \lambda_r$ (eigenvalues of x) such that $x = \sum_{i=1}^r \lambda_i c_i$. Let $\mathrm{tr}(x) = \sum_{i=1}^r \lambda_i$ and $\det(x) = \prod_{i=1}^r \lambda_i$ be the trace and the determinant functional, respectively. Then it is known that $\mathrm{tr}(xy)$ is a positive definite bilinear form satisfying (2.1) (Proposition III.1.5 of [7]). Throughout this paper we will assume that V is a Euclidean Jordan algebra of rank r and with the associative inner product $\langle x|y\rangle = \mathrm{tr}(xy)$. Let Q be the set of all square elements of V. Then Q is a closed convex cone of V with $Q \cap -Q = \{0\}$, and is the set of element $x \in V$ such that $L(x)$ is positive semi-definite. It turns out that Q has non-empty interior Ω, and Ω is a symmetric cone, that is, the group

$$G(\Omega) = \{g \in \mathrm{GL}(V) | g(\Omega) = \Omega\}$$

acts transitively on it and Ω is a self-dual cone with respect to the inner product $\langle \cdot | \cdot \rangle$. Furthermore, for any a in Ω, $P(a) \in G(\Omega)$ and is positive definite with respect to the inner product. We remark that any symmetric cone (self-dual, homogeneous open convex cone) can be realized as an interior of squares in the appropriate Euclidean Jordan algebra [7].

The following lemma appears at Lemma 2.3 of [27].

Lemma 2.2. *The map $x \to P(x)$ is injective on Ω.*

For $x, y \in V$, put

$$\begin{aligned} x \leq y \quad &: \quad \Longleftrightarrow y - x \in \overline{\Omega}, \\ x < y \quad &: \quad \Longleftrightarrow y - x \in \Omega. \end{aligned}$$

For an element x in V, we let denote the i-th eigenvalue of x by $\alpha_i(x)$:

$$\alpha_1(x) \leq \alpha_2(x) \leq \cdots \leq \alpha_i(x) \leq \cdots \leq \alpha_r(x).$$

The next result will be useful to apply matrix theories to symmetric cones.

Lemma 2.3. *Let $a, b \in \overline{\Omega}$. Then $a \leq b$ if and only if $P(a) \leq P(b)$.*

Proof. Let u be an arbitrary element of V, and let $u = \sum_{i=1}^r \lambda_i c_i$ be the spectral decomposition of u, where $\{c_i\}_{i=1}^r$ is a Jordan frame on V. Let us consider the Peirce decomposition of V with respect to the Jordan frame $\{c_i\}_{i=1}^r$ (see Chapter IV of [7]):

$$V = \bigoplus_{i \leq j} V_{ij},$$

where

$$\begin{aligned} V_{ii} &= V(c_i, 1) = \mathbb{R}c_i, \\ V_{ij} &= \{x \in V \ : \ L(c_k)x = \frac{1}{2}(\delta_{ik} + \delta_{jk})x\}. \end{aligned}$$

It follows that the eigenvalues of $L(u)$ are of the form $\frac{1}{2}(\lambda_i + \lambda_j), i \leq j$ and hence the eigenvalues of $P(u) = 2L(u)^2 - L(u^2)$ are of the form $\lambda_i\lambda_j, i \leq j$.

With this observation, we prove the lemma. Suppose that $a \leq b$. Let n be a natural number. Then $a \leq b + \frac{1}{n}e$, and hence $P((b + \frac{1}{n}e)^{-1/2})a \leq e$. This implies that the eigenvalues of $P(b + \frac{1}{n}e)^{-1/2}a$ are less than equal to 1. Thus from the first paragraph we have $P(P(b + \frac{1}{n}e)^{-1/2}a) \leq I$, where I is the identity matrix on V. By Proposition 2.1 (iii),

$$P(P(b + \frac{1}{n}e)^{-1/2}a) = P(b + \frac{1}{n}e)^{-1/2}P(a)P(b + \frac{1}{n}e)^{-1/2} \leq I,$$

and thus

$$P(a) \leq P(b + \frac{1}{n}e) = P(b) + \frac{2}{n}L(b) + \frac{1}{n^2}I.$$

As $n \to \infty$, we get $P(a) \leq P(b)$. Conversely, suppose that $P(a) \leq P(b)$. Then since $b \leq b + \frac{1}{n}e$, $P(a) \leq P(b) \leq P(b + \frac{1}{n}e), \forall n > 0$. This implies that

$$P(P(b + \frac{1}{n}e)^{-1/2}a) = P(b + \frac{1}{n}e)^{-1/2}P(a)P(b + \frac{1}{n}e)^{-1/2} \leq I,$$

and hence the eigenvalues of $P(b + \frac{1}{n}e)^{-1/2}a$ are less than equal to 1 from the first paragraph. This implies that $P(b + \frac{1}{n}e)^{-1/2}a \leq e$, and hence $a \leq (b + \frac{1}{n}e)$. As $n \to \infty$, we get $a \leq b$. □

3. Geometric means on symmetric cones

The space $\mathrm{Sym}(n, \mathbb{R})$ of real symmetric $n \times n$ matrices under the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ is a simple Euclidean Jordan algebra with the bilinear form $\mathrm{tr}(XY)$. In this case the corresponding symmetric cone Ω is the cone of positive definite matrices, and the quadratic representation P of $\mathrm{Sym}(n, \mathbb{R})$ is given by $P(X)(Y) = XYX$ for $X, Y \in \mathrm{Sym}(n, \mathbb{R})$. The *geometric mean* of positive semi-definite matrices A and B is defined by the maximum, denoted by $A\#B$, of all $X \in \mathrm{Sym}(n, \mathbb{R})$ for which $\begin{pmatrix} A & X \\ X & B \end{pmatrix}$ are positive semi-definite [1]. It turns out that if A is invertible then

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

We remark that the geometric mean $A\#B$ of positive definite matrices A and B can be understood as the unique positive definite solution of Riccati equation ([12], [36], [39])

$$XA^{-1}X = B.$$

The following characteristic properties of the geometric mean of positive semi-definite operators are well-known [1].

(i) Symmetric property: $A\#B = B\#A$.

(ii) Positive homogeneity: $\alpha(A\#B) = (\alpha A)\#(\alpha B)$ for $\alpha \geq 0$.

(iii) Normalization: $A\#A = A$.

(iv) Monotonicity: $A\#B \geq A'\#B'$ whenever $A \geq A'$ and $B \geq B'$.

(v) Continuity from above: $A_k \downarrow A, B_k \downarrow B$ implies that $A_k\#B_k \downarrow A\#B$.

(vi) Transformer inequality: $C(A\#B)C \leq (CAC)\#(ABC)$ for all $C \geq 0$.

(vii) The harmonic-geometric-arithmetic inequality:

$$\{(A^{-1} + B^{-1})/2\}^{-1} \leq A\#B \leq (A + B)/2.$$

(viii) The inverse relation: $A^{-1}\#B^{-1} = (A\#B)^{-1}$.

The following result appears in [11] and [27].

Proposition 3.1. *Let $a, b \in \Omega$. Then $x := P(a^{1/2})(P(a^{-1/2})b)^{1/2}$ is a unique solution belonging to Ω of the following quadratic equation*

$$P(x)a^{-1} = b.$$

We call $a\#b := P(a^{1/2})(P(a^{-1/2})b)^{1/2}$ the *geometric mean* of a and b.

Proposition 3.2 ([27]). *Let $a, b \in \Omega$. Then*

(i) $P(a\#b) = P(a)\#P(b)$.

(ii) $a\#b = b\#a$.

(iii) $(a\#b)^{-1} = a^{-1}\#b^{-1}$.

(iv) $\{(a^{-1} + b^{-1})/2\}^{-1} \leq a\#b \leq (a + b)/2$.

Proposition 3.3. *The inversion $x \to x^{-1}$ on Ω is order reverting. That is, $a \leq b$ if and only if $b^{-1} \leq a^{-1}$ for any $a, b \in \Omega$.*

Proof. It follows from Proposition 3.2 (ii) and from the fact that $P(a^{-1}\#b^{-1})(a - b) = b^{-1} - a^{-1}$. □

A Riemannian metric on Ω. It turns out [7] that the symmetric cone Ω admits a $G(\Omega)$-invariant Riemannian metric γ_x defined by

$$\gamma_x(u, v) = \langle P(x)^{-1}u|v\rangle, x \in \Omega, u, v \in V$$

for which the inversion $j(x) = x^{-1}$ is an (unique) involutive isometry fixing the identity e. The unique geodesic curve joining a and b is

$$\gamma(t) = P(a^{1/2})(P(a^{-1/2})b)^t$$

and the Riemannian distance $d(a,b)$ is given by

$$d(a,b) = \left(\sum_{i=1}^{r} \log^2 \lambda_i \right)^{1/2}$$

where λ_i's are the eigenvalues of $P(a^{-1/2})b$ ([27], [40]). One may show that the geometric mean $a\#b$ is the unique (geodesic) midpoint of a and b for the Riemannian distance. The uniqueness of the midpoint of a and b follows from the fact that the symmetric cone is a Bruhat-Tits space for the Riemannian distance (see [19]).

Thompson's metric on Ω. The symmetric cone Ω admits a natural Finsler metric:

$$|x|_a := |P(a^{-1/2})x|_e, a \in \Omega, x \in V \equiv T_a(\Omega),$$

where $|\cdot|_e$ is the spectral norm on V. The Finsler distance $s(a,b)$ defined by the family of norms $|\cdot|_a, a \in \Omega$ is given by

$$s(a,b) = \max\{|\log \lambda| : \lambda \text{ is an eigenvalue of } P(a^{-\frac{1}{2}})b\},$$

It is known in [4], [30] that the Finsler distance $s(a,b)$ is invariant under $G(\Omega)$ and the inversion $j(x) = x^{-1}$. We remark that the Finsler distance s is exactly Thompson's part metric on Ω: Thompson's metric on Ω (or, on any pointed convex cone) is defined by

$$\overline{d}(a,b) = \max\{\log M(b/a), \log M(a/b)\},$$

where $M(a/b) = \inf\{\alpha > 0 : a \leq \alpha b\}$. Moreover, the unique (Riemannian) geodesic curve $\gamma(t) = P(a^{1/2})(P(a^{-1/2})b)^t$ joining a and b satisfies [4]

$$s(\gamma(t), \gamma(t')) = |t - t'| s(a,b), 0 \leq t, t' \leq 1.$$

This implies that the geometric mean $a\#b = \gamma(1/2)$ is a midpoint of a and b for the Finsler distance s. We remark that the Finsler structure on Ω has already been studied on even infinite dimensional symmetric cones ([31], [35], [41]).

Hilbert's projective metric on Ω. Hilbert's projective metric on Ω is defined by

$$p(a,b) := \log M(a/b)M(b/a), \quad a, b \in \Omega.$$

One can easily show that

$$\begin{aligned} M(a/b) &= \alpha_r(P(b^{-1/2})a), \\ M(b/a)^{-1} &= \sup\{\alpha : \alpha b \leq a\} = \alpha_1(P(b^{-1/2})a), \end{aligned}$$

and hence p is invariant under the group $G(\Omega)$ and under the inversion. It then follows that

$$p(a\#b, a) = \log \frac{\alpha_r(P(a^{-1/2})b)^{1/2})}{\alpha_1(P(a^{-1/2})b)^{1/2})} = \frac{1}{2} \log \frac{\alpha_r(P(a^{-1/2})b))}{\alpha_1(P(a^{-1/2})b))} = \frac{1}{2} p(a,b).$$

This implies that $a\#b$ is a midpoint of a and b for Hilbert's projective metric on Ω.

From the midpoint property of the geometric mean $a\#b$ for the Riemannian, Thompson and Hilbert's projective metrics, it is quite natural to call $a\#b$ the "metric" geometric mean of a and b.

For $\alpha \in \mathbb{R}$, we denote $a\#_\alpha b$ (the α-power mean of a and b) by the point at the time α of the unique geodesic curve passing a and b

$$a\#_\alpha b = P(a^{1/2})(P(a^{-1/2})b)^\alpha.$$

Proposition 3.4. *Let $a, b \in \Omega$. Then for any $\alpha \in \mathbb{R}$, $P(a\#_\alpha b) = P(a)\#_\alpha P(b)$ and $a\#_\alpha b = b\#_{1-\alpha} a$. Furthermore,*

$$\Big(P(b)a\Big)^\alpha = P(b)P(a^{1/2})\Big(P(a^{1/2})b^2\Big)^{\alpha-1}.$$

Proof. The property $P(a\#_\alpha b) = P(a)\#_\alpha P(b)$ appears in [28].

It turns out [13] that

$$(BAB)^\alpha = BA^{1/2}(A^{1/2}B^2A^{1/2})^{\alpha-1}A^{1/2}B$$

for any invertible positive operators A and B, and any real number α. Using this result, we have

$$\begin{aligned}
P\Big((P(b)a)^\alpha\Big) &= P(P(b)a)^\alpha \\
&= \Big(P(b)P(a)P(b)\Big)^\alpha \\
&= P(b)P(a)^{1/2}\Big(P(a)^{1/2}P(b)^2P(a)^{1/2}\Big)^{\alpha-1}P(a)^{1/2}P(b) \\
&= P(b)P(a)^{1/2}P\Big((P(a)^{1/2}b^2)^{\alpha-1}\Big)P(a)^{1/2}P(b) \\
&= P(b)P\Big(P(a^{1/2})(P(a)^{1/2}b^2)^{\alpha-1}\Big)P(b) \\
&= P\Big(P(b)\Big(P(a^{1/2})(P(a^{1/2})b^2)^{\alpha-1}\Big)\Big).
\end{aligned}$$

Since P is injective on Ω (Lemma 2.2), it then follows that

$$(P(b)a)^\alpha = P(b)\Big(P(a^{1/2})(P(a^{1/2})b^2)^{\alpha-1}\Big),\ \alpha \in \mathbb{R}.$$

This formula implies that

$$\begin{aligned}
a\#_\alpha b &= P(a^{1/2})(P(b)a)^\alpha \\
&= P(a^{1/2})\Big(P(a^{-1/2})P(b^{1/2})(P(b^{1/2})a^{-1})^{\alpha-1}\Big) \\
&= P(b^{1/2})(P(b^{-1/2})a)^{1-\alpha} \\
&= b\#_{1-\alpha} a
\end{aligned}$$

which completes the proof. $\square$

4. Spectral geometric means on symmetric cones

We begin this section with the following result.

Proposition 4.1. *Let $a, b \in \Omega$, and let $\alpha \in \mathbb{R}$. Then $x := P(a^{-1}\#b)^{\alpha}a$ is a unique solution belonging to Ω of the following equation*

$$(a^{-1}\#b)^{\alpha} = a^{-1}\#x.$$

Proof. Clearly, $x \in \Omega$. By Proposition 3.1, $a^{-1}\#x = (a^{-1}\#b)^{\alpha}$. If $y \in \Omega$ satisfies the equation, then $a^{-1}\#x = a^{-1}\#y$. Again by Proposition 3.1,

$$x = P(a^{-1}\#x)a = P(a^{-1}\#y)a = y,$$

which completes the proof. $\square$

We denote by $F_{\alpha}(a, b) = P(a^{-1}\#b)^{\alpha}a$. For convenience, put $F(a, b) = F_{1/2}(a, b)$. The following gives some characteristic properties of $F_{\alpha}(a, b)$.

Proposition 4.2. *Let $a, b \in \Omega$. Then*

(i) $F_0(a, b) = a, F_1(a, b) = b$. *Hence $F_{\alpha}(a, b)$ is a differentiable curve on Ω passing a and b.*

(ii) $F_{\alpha}(a, b)^{-1} = F_{\alpha}(a^{-1}, b^{-1})$.

(iii) $F_{\alpha}(a, b) = F_{1-\alpha}(b, a)$. *In particular,* $F(a, b) = F(b, a)$.

(iv) $a^{-1}\#F_{\alpha}(a, b) = b\#F_{\alpha}(a^{-1}, b^{-1}) = (b^{-1}\#F_{\alpha}(a, b))^{-1} = (a^{-1}\#b)^{\alpha}$.

(v) *The element $x = a^{-1}\#F_{\alpha}(a, b)$ is a unique element in Ω satisfying $F_{\alpha}(a, b) = P(x)a = P(x^{-1})b$.*

(vi) $P(F_{\alpha}(a, b)) = F_{\alpha}(P(a), P(b))$.

(vii) $F(k(a), k(b)) = k(F(a, b))$ *for any Jordan automorphism k.*

(viii) $\det(F_{\alpha}(a, b)) = \det(a\#_{\alpha}b) = [\det(a)]^{1-\alpha}\ [\det(b)]^{\alpha}$.

(ix) $\langle a|b\rangle = \langle F(a, b)|F(a, b)\rangle, \langle a^{-1}|b^{-1}\rangle = \langle F(a, b)^{-1}|F(a, b)^{-1}\rangle$.

Proof. (i) By definition of the geometric mean, we have $F_1(a, b) = P(a^{-1}\#b)a = b$.

(ii) It follows from Proposition 2.1 (ii)

$$F_{\alpha}(a, b)^{-1} = [P(a^{-1}\#b)^{\alpha}a]^{-1} = P(a\#b^{-1})^{\alpha}a^{-1} = F_{\alpha}(a^{-1}, b^{-1}).$$

(iii) By Proposition 2.1 (i) and Propositions 3.1, 3.2, we get

$$\begin{aligned} P(b^{-1}\#a)^{\alpha-1}P(a^{-1}\#b)^{\alpha}a &= P(a^{-1}\#b)^{1-\alpha}P(a^{-1}\#b)^{\alpha}a \\ &= P(a^{-1}\#b)a \\ &= b. \end{aligned}$$

Thus $F_\alpha(a,b) = P(a^{-1}\#b)^{\alpha}a = P(b^{-1}\#a)^{1-\alpha}b = F_{1-\alpha}(b,a)$. When $\alpha = 1/2$, it becomes $F(a,b) = F(b,a)$.

(iv) By Propositions 3.2, 4.1 and by (ii)

$$a^{-1}\#F_\alpha(a,b) = (a^{-1}\#b)^{\alpha} = (b\#a^{-1})^{\alpha} = b\#F_\alpha(b^{-1},a^{-1}) = (b^{-1}\#F_\alpha(a,b))^{-1}.$$

(v) By Proposition 4.1 and by (ii), (iv), we have

$$\begin{aligned} P(a^{-1}\#F_\alpha(a,b))a &= F_\alpha(a,b), \\ P((a^{-1}\#F_\alpha(a,b))^{-1})b &= P(a\#F_\alpha(a,b)^{-1})b \\ &= P(b^{-1}\#F_\alpha(a,b))b \\ &= F_\alpha(a,b). \end{aligned}$$

(vi) It follows from Proposition 2.1 (iii) and Proposition 3.2 that

$$\begin{aligned} P(F_\alpha(a,b)) &= P(P(a^{-1}\#b)^{\alpha}a) \\ &= P(a^{-1}\#b)^{\alpha}P(a)P(a^{-1}\#b)^{\alpha} \\ &= (P(a)^{-1}\#P(b))^{\alpha}P(a)(P(a)^{-1}\#P(b))^{\alpha} \\ &= F_\alpha(P(a),P(b)). \end{aligned}$$

(vii) Let k be a Jordan automorphism of V. Then $P(k(x)) = kP(x)k^{-1}$ for any $x \in \Omega$. Using this property, one may easily show that $k(x\#y) = k(x)\#k(y)$ for all $x, y \in \Omega$. This implies that

$$\begin{aligned} F_\alpha(k(a),k(b)) &= P(k(a^{-1})\#k(b))^{\alpha}k(a) \\ &= kP(a^{-1}\#b)^{\alpha}k^{-1}k(a) \\ &= k(F_\alpha(a,b)). \end{aligned}$$

(viii) It is known that $\det(P(y)x) = (\det(y))^2\det(x)$ (Proposition III. 4.2 of [7]). Thus

$$\begin{aligned} \det(a\#_\alpha b) &= \det(P(a^{1/2})(P(a^{-1/2})b)^{\alpha}) \\ &= \det(a)\ [\det(P(a^{-1/2})b)]^{\alpha} \\ &= \det(a)\ [\det(a^{-1})\ \det(b)]^{\alpha} \\ &= [\det(a)]^{1-\alpha}\ [\det(b)]^{\alpha}. \end{aligned}$$

Similarly, we have

$$\begin{aligned}\det F_\alpha(a,b) &= \det(P(a^{-1}\#b)^\alpha a)\\ &= [\det(a^{-1}\#b)]^\alpha \det(a)\\ &= [\det(a)]^{1-\alpha}\,[\det(b)]^\alpha.\end{aligned}$$

(ix) It follows from that

$$\begin{aligned}\langle F(a,b)|F(a,b)\rangle &= \langle P(a\#b^{-1})^{1/2}b|P(a\#b^{-1})^{1/2}b\rangle\\ &= \langle P(a\#b^{-1})b|b\rangle\\ &= \langle a|b\rangle.\end{aligned}$$

By (ii), we have $\langle F(a,b)^{-1}|F(a,b)^{-1}\rangle = \langle F(a^{-1},b^{-1})|F(a^{-1},b^{-1})\rangle = \langle a^{-1}|b^{-1}\rangle$. $\square$

It is known in [12] that for two positive definite matrices A and B, $A\#B = F(A,B)$ if and only if A and B commute. Using this fact, we have that for any elements a and b of the symmetric cone Ω,

$$\begin{aligned}a\#b = F(a,b) &\iff P(a\#b) = P(F(a,b)) \quad \text{by Lemma 2.2}\\ &\iff P(a)\#P(b) = F(P(a),P(b)) \quad \text{by Propositions 3.2 and 4.2}\\ &\iff P(a)P(b) = P(b)P(a).\end{aligned}$$

The condition $P(a)P(b) = P(b)P(a)$ is equivalent to that a and b lie in an associative subalgebra of V by Proposition 2.1 (iv), Lemma III. 1 and Theorem III. 3 of [16]. In this case, $a\#b = F(a,b) = a^{1/2}b^{1/2}$.

The next result provides the spectral property of $F(a,b)$ that forces us to call it the "spectral" geometric mean of a and b.

Theorem 4.3. *Let $a,b \in \Omega$. Then*

$$\alpha_i(F(a,b)) = \alpha_i^{1/2}(P(a^{1/2})b) = \alpha_i^{1/2}(P(b^{1/2})a)$$

for all $i = 1,\cdots,r$.

Proof. By definition of the spectral mean of a, and b and by Lemma XIV. 1.2. of [7]. $\square$

Corollary 4.4. *Let $a,b \in \Omega$. Then*

$$|\log F(a,b)|_e = \frac{1}{2}s(a^{-1},b).$$

Proof. Note that for any $x \in \Omega$, $s(e,x) = |\log x|_e = 2s(e,x^{1/2})$. Because the Finsler distance s is invariant under $G(\Omega)$ and the inversion, we have

$$\begin{aligned}|\log F(a,b)|_e &= s(e,F(a,b)) = s(e,P(a^{-1}\#b)^{1/2}a)\\ &= s(P(a\#b^{-1})^{1/2}e,a) = s(a\#b^{-1},a)\\ &= s(P(a^{1/2})(P(a^{-1/2})b^{-1})^{1/2},P(a^{1/2})e) = s((P(a^{-1/2})b^{-1})^{1/2},e)\\ &= \frac{1}{2}s(P(a^{-1/2})b^{-1},P(a^{-1/2})P(a^{1/2})e) = \frac{1}{2}s(b^{-1},a).\end{aligned}$$ $\square$

Corollary 4.5. *Let $a, b \in \Omega$. Then $M(a\#b/a) = M(b/a\#b) = \alpha_r(F(a^{-1}, b))$. In particular, $\alpha_1(F(a, b^{-1}))b \leq a\#b \leq \alpha_r(F(a^{-1}, b))a$.*

Proof. It follows from Theorem 4.3 that

$$\begin{aligned} M(a\#b/a) &= \inf\{\alpha > 0 : a\#b \leq \alpha a\} \\ &= \alpha_r(P(a^{-1/2})(a\#b)) \\ &= \alpha_r^{1/2}(P(a^{-1/2})b) \\ &= \alpha_r(F(a^{-1}, b)), \end{aligned}$$

and

$$\begin{aligned} M(b/a\#b)^{-1} &= \sup\{\alpha > 0 : \alpha b \leq a\#b = b\#a\} \\ &= \alpha_1(P(b^{-1/2})(b\#a)) \\ &= \alpha_1^{1/2}(P(b^{-1/2})a) \\ &= \alpha_1(F(a, b^{-1})). \end{aligned}$$

□

5. Nesterov-Todd inequality

Theorem 5.1. *For any $a \in \Omega$ we have :*

$$\langle a|a\rangle\langle a^{-1}|a^{-1}\rangle \geq r(r-1) + \frac{1}{4} + \frac{3}{4}\frac{\langle a|a\rangle}{\alpha_1^2(a)}.$$

Proof. Let $a \in \Omega$. By the spectral decomposition of a, it is enough to show that

$$\left(\sum_{i=1}^r \lambda_i\right)\left(\sum_{i=1}^r \frac{1}{\lambda_i}\right) \geq r(r-1) + \frac{1}{4} + \frac{3}{4}\sum_{i=1}^r \frac{\lambda_i}{\lambda_1},$$

for any positive real numbers λ_i such that $\lambda_1 \leq \lambda_i, i = 1, \cdots, r$. Note that

$$\left(\sum_{i=1}^r \lambda_i\right)\left(\sum_{i=1}^r \frac{1}{\lambda_i}\right) = r + \sum_{i=2}^r (\frac{\lambda_i}{\lambda_1} + \frac{\lambda_1}{\lambda_i}) + \sum_{2\leq i<j} (\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j}).$$

Since $\frac{\lambda_i}{\lambda_1} + \frac{\lambda_1}{\lambda_i} \geq 1 + \frac{3}{4}\frac{\lambda_i}{\lambda_1}$ and $\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} \geq 2$ for any i, j, we have

$$\left(\sum_{i=1}^r \lambda_i\right)\left(\sum_{i=1}^r \frac{1}{\lambda_i}\right) \quad \geq \quad r + \sum_{i=2}^r (1 + \frac{3}{4}\frac{\lambda_i}{\lambda_1}) + 2\sum_{2\leq i<j} 1$$

$$\begin{aligned}
&\geq && r+(r-1)-\frac{3}{4}+\frac{3}{4}\sum_{i=1}^{r}\frac{\lambda_i}{\lambda_1}+(r-1)(r-2)\\
&= && r(r-1)+\frac{1}{4}+\frac{3}{4}\sum_{i=1}^{r}\frac{\lambda_i}{\lambda_1}.
\end{aligned}$$ □

Corollary 5.2. *For any $a, b \in \Omega$ we have*:

$$\langle a^{-1}|b^{-1}\rangle \geq \frac{r(r-1)}{\langle a|b\rangle}+\frac{3}{4}\alpha_1^{-1}(P(a^{1/2})b).$$

Proof. Theorem 5.1 implies in particular that

$$\langle F(a,b)^{-1}|F(a,b)^{-1}\rangle \geq \frac{r(r-1)}{\langle F(a,b)|F(a,b)\rangle}+\frac{3}{4}\alpha_1^{-2}(F(a,b)).$$

By Proposition 4.2(ix) and by Theorem 4.3,

$$\begin{aligned}
\langle a|b\rangle &= \langle F(a,b)|F(a,b)\rangle,\\
\langle a^{-1}|b^{-1}\rangle &= \langle F(a,b)^{-1}|F(a,b)^{-1}\rangle,\\
\alpha_1^{-1}(P(a^{1/2})b) &= \alpha_1^{-2}(F(a,b)),
\end{aligned}$$

and hence we get the result. □

By Theorem 4.3 and Corollary 4.5, we have

$$\begin{aligned}
\alpha_1^{-2}(F(a,b)) &= \alpha_1^{-1}(P(a^{1/2})b)\\
&= \alpha_r(P(a^{-1/2})b^{-1})\\
&= M(a\#b^{-1}/a)^2.
\end{aligned}$$

Thus Corollary 5.2 leads Nesterov-Todd inequality (Theorem 5.2, [32])

$$\langle a^{-1}|b^{-1}\rangle \geq \frac{r(r-1)}{\langle a|b\rangle}+\frac{3}{4}M(a\#b^{-1}/a)^2.$$

For $\rho \geq r+\sqrt{r}$, since

$$r^2-r-\rho(2r-\rho)=(\rho-r-\sqrt{r})^2+2\sqrt{r}(\rho-r-\sqrt{r})\geq 0,$$

we have the following inequality (cf. Proposition 3.5, [11]) from Corollary 5.2

$$\begin{aligned}
\langle a^{-1}|b^{-1}\rangle-\frac{\rho(2r-\rho)}{\langle a|b\rangle} &\geq \langle a^{-1}|b^{-1}\rangle-\frac{r(r-1)}{\langle a|b\rangle}\\
&\geq \frac{3}{4}\alpha_1^{-2}(F(a,b)), \forall a,b\in\Omega.
\end{aligned}$$

6. On some inequalities on symmetric cones

In this section, we give some inequalities on symmetric cones that are well-known in matrix theory. The main tools we use are Lemmas 2.2-2.3 and Propositions 3.1-3.4.

The following result is an extension of the classical Löwner-Heinz inequality of positive definite matrices (cf. [17], [28]).

Theorem 6.1. *For $a, b \in \overline{\Omega}$,*

$$a \leq b \text{ implies } \; a^p \leq b^p \text{ for } 0 \leq p \leq 1.$$

Proof. Suppose that $a \leq b$. By Lemma 2.3, $a \leq b$ implies that $P(a) \leq P(b)$. By the classical Löwner-Heinz inequality of positive definite matrices, $P(a) \leq P(b)$ implies that $P(a)^p \leq P(b)^p$ for any $0 \leq p \leq 1$. Since $P(a)^p = P(a^p)$ and $P(b)^p = P(b^p)$, again by Lemma 2.3, $a^p \leq b^p$. □

Corollary 6.2. *The operation $\#$ can be extended to $\overline{\Omega}$. Furthermore, the operation $\#$ on $\overline{\Omega}$ satisfies*

(i) (*Monotonicity*) *$a_1 \leq a_2$ and $b_1 \leq b_2$ imply $a_1 \# b_1 \leq a_2 \# b_2$, for any $a_i, b_i \in \overline{\Omega}$.*

(ii) (*Transformer inequality*) *$P(x)(a\#b) \leq (P(x)a)\#(P(x)b)$, for any $a, b \in \overline{\Omega}$ and $x \in V$.*

(iii) (*Continuity from above*) *Let a_n and b_n be sequences in $\overline{\Omega}$ such that $a_n \downarrow a$ and $b_n \downarrow b$. Then $a_n \# b_n \downarrow a\#b$.*

In particular, the operation $\#$ is a "mean operation" in the sense of Kubo and Ando [18].

Proof. Note that if $a \in \Omega$ and $b \in \overline{\Omega}$, then $a\#b$ is naturally defined by $a\#b := P(a^{1/2})(P(a^{-1/2})b)^{1/2}$. Suppose that $a_1 \leq a_2$ and $b_1 \leq b_2$ for $b_1, b_2 \in \overline{\Omega}$ and $a_1, a_2 \in \Omega$. Then by Lemma 2.3 and by the monotonicity of the geometric mean of positive semi-definite matrices, we have $a_1\#b_1 \leq a_2\#b_2$.

Choose a basis $e_1, \cdots, e_n$ for the Jordan algebra V such that the pointed convex cone $\overline{\Omega}$ is contained in the convex cone $C^+ := \{\sum_{i=1}^n t_i e_i \mid t_i \geq 0, \forall i\}$. Then the order $\leq$ induced by $\overline{\Omega}$ is weaker than the order $\leq_+$ induced by the cone C^+. Let $a, b \in \overline{\Omega}$. Let $a_n = a + e/n$. Then $a_n \in \Omega$ converges to a. Since a_n is a decreasing sequence, by the first paragraph, $0 \leq a_n\#b$ is a decreasing sequence. By considering the order $\leq_+$, we conclude that the sequence $a_n\#b$ is a convergent sequence. We denote this limit point by $a\#b$. So we have defined the geometric mean on the closed convex cone $\overline{\Omega}$.

The remaining part of proof follows by using Lemma 2.3 and Proposition 3.2. □

Ando and Hiai showed the following inequality [3]: If B is positive semidefinite and A is positive definite then for each $\alpha \in [0, 1]$

(i) if $A\#_\alpha B \leq I$ then $A^r \#_\alpha B^r \leq A\#_\alpha B \leq I$ for all $r \geq 1$.

(ii) if $0 \leq B \leq A$ then $A^{-r}\#_{1/p}\Big(A^{-1/2}B^pA^{-1/2}\Big)^r \leq I$ for all $r, p \geq 1$.

By Lemma 2.3, Propositions 3.1 and 3.2,

Theorem 6.3. *Let $a \in \Omega$ and let $b \in \overline{\Omega}$. Then for each $\alpha \in [0,1]$*

$$a\#_\alpha b \leq e \text{ implies } a^r\#_\alpha b^r \leq a\#_\alpha b \leq e$$

for all $r \geq 1$. Furthermore, if $0 \leq b \leq a$ then

$$a^{-r}\#_{1/p}\left(P(a^{-1/2})b^p\right)^r \leq e$$

for all $r, p \geq 1$.

The next inequality is a version of Jensen inequality [14] on symmetric cones.

Theorem 6.4. *Let $a \geq e$. Then*

$$P(a^{-1/2})x^\alpha \leq \Big(P(a^{-1/2})x\Big)^\alpha$$

for any $x \geq 0$ and $\alpha \in [0,1]$. In particular,

$$x^\alpha \leq a\#_\alpha x.$$

Proof. Let $x \geq 0$. Let Δ be the set of $\alpha \in [0,1]$ for which the assertion is true. Then clearly Δ is a closed subset of the interval $[0,1]$ containing $0,1$. We claim that Δ is midpoint convex, that is, if $p_1, p_2 \in \Delta$ then $(p_1+p_2)/2 \in \Delta$. Suppose that $p_1, p_2 \in \Delta$. Then $P(a^{-1/2})x^{p_1} \leq \big(P(a^{-1/2})x\big)^{p_1}$ and $P(a^{-1/2})x^{p_2} \leq \big(P(a^{-1/2})x\big)^{p_2}$. By the monotonicity of the geometric mean (Corollary 6.2), we have

$$\begin{aligned}
\Big(P(a^{-1/2})x\Big)^{(p_1+p_2)/2} &= \Big(P(a^{-1/2})x\Big)^{p_1}\#\Big(P(a^{-1/2})x\Big)^{p_2} \\
&\geq P(a^{-1/2})x^{p_1}\#P(a^{-1/2})x^{p_2} \\
&= P(a^{-1/2})(x^{p_1}\#x^{p_2}) \\
&= P(a^{-1/2})x^{(p_1+p_2)/2}.
\end{aligned}$$

Therefore, $(p_1+p_2)/2 \in \Delta$.

Note that $P(a^{-1/2})x^\alpha \leq \big(P(a^{-1/2})x\big)^\alpha$ implies that

$$x^\alpha \leq P(a^{1/2})\left(P(a^{-1/2})x\right)^\alpha = a\#_\alpha x.$$

We finish the proof. □

In [2], Ando derived a quite useful result: For $X, Y \in \mathrm{Sym}(n, \mathbb{R})$, the following assertions are mutually equivalent:

(a) $Y \leq X$.

(b) $\exp(-tX)\#\exp(tY) \leq I$ for all $t \geq 0$.

(c) $t \to \exp(-tX)\#\exp(tY)$ is a decreasing map from $[0, \infty)$ to Ω.

By Proposition 2.1, Lemma 2.3, Proposition 3.2, and by using the fact that $y \leq x$ if and only if $L(x-y) \geq 0$, one may show

Theorem 6.5. *Let $x, y \in V$. Then the following assertions are mutually equivalent*:

(a) $y \leq x$.

(b) $\exp(-tx)\#\exp(ty) \leq e$ *for all* $t \geq 0$.

(c) $t \to \exp(-tx)\#\exp(ty)$ *is a decreasing map from* $[0, \infty)$ *to* Ω.

Similarly, the following Furuta inequality (an extension of the classical Löwner-Heinz inequality) [13]:

$$0 \leq B \leq A \text{ implies } B^{(p+2r)/q} \leq (B^r A^p B^r)^{1/q}$$

for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, holds on any symmetric cones.

Theorem 6.6. *If $0 \leq b \leq a$, then for each $r \geq 0$*

$$b^{(p+2r)/q} \leq (P(b^r)a^p)^{1/q}$$

holds for each p and q such that $p \geq 0, q \geq 1$ and $q \geq (p+2r)/(1+2r)$.

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