KYUNGPOOK Math. J. 47(2007), 127-132

PF-rings of Generalized Power Series

HWANKOO KIM
Department of Information Security, College of Engineering, Hoseo University, Asan 336-795, Korea
e-mail : hkkim@office.hoseo.ac.kr

TAE IN KWON
Department of Applied Mathematics, Changwon National University, Changwon 641-773, Korea
e-mail : taekwon@sarim.changwon.ac.kr

ABSTRACT. In this paper, we show that if R is a commutative ring with identity and $(S, \leq)$ is a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ of generalized power series is a PF-ring if and only if for any two S-indexed subsets A and B of R such that $B \subseteq \mathrm{ann}_R(A)$, there exists $c \in \mathrm{ann}_R(A)$ such that $bc = b$ for all $b \in B$, and that for a Noetherian ring R, $[[R^{S,\leq}]]$ is a PP ring if and only if R is a PP ring.

1. Introduction and preliminaries

Let R be a commutative ring with identity. Then R is called a *PF-ring* (resp., *PP-ring*) if every principal ideal of R is a flat (resp., projective) R-module. It is well-known that if R is Noetherian, then these two notions are equal (cf., [16, Corollary 4.3]). It is proved in [1] that a ring R is a PF-ring if and only if the annihilator of each element $r \in R$, $\mathrm{ann}_R(r)$, is a pure ideal; that is, for all $b \in \mathrm{ann}_R(r)$ there exists $c \in \mathrm{ann}_R(r)$ such that $bc = b$. It may be worth reminding the reader that for a commutative ring R, R is a PF-ring if and only if R is a locally integral domain (i.e., every localization R_P is an integral domain for any prime (resp., maximal) ideal P of R) ([3], [11]). It is also proved in [2] that the power series ring $R[[X]]$ is a PF-ring if and only if for any two countable subsets $A = \{a_0, a_1, \cdots\}$ and $B = \{b_0, b_1, \cdots\}$ of R such that $A \subseteq \mathrm{ann}_R(B)$, there exists $r \in \mathrm{ann}_R(B)$ such that $ar = a$ for all $a \in A$. In [7, Theorem 3, Theorem 4], J.-H. Kim proved that for a Noetherian ring R, $R[[X]]$ is a PF (resp., PP) ring if and only if R is a PF (resp., PP) ring. In recent years, Many researchers (for example, P. Ribenboim ([4], [12], [13], [14], [15]), Z. Liu ([8], [10], [9]), and the first author ([5], [6])) have carried out an extensive study of rings of generalized power series. In particular, Liu and

Received January 13, 2006.
2000 Mathematics Subject Classification: 13F25, 13C11.
Key words and phrases: PF-ring, PP-ring, generalized power series ring.
The first author was supported by Hoseo University Research Fund, 2005. The second author was supported by Changwon National University Research Fund, 2005.

Ahsan proved in [10] that the ring $[[R^{S,\leq}]]$ of generalized power series is a PP-ring if and only if R is a PP-ring and every S-indexed subset C of $B(R)$ (the set of all idempotents of R) has a least upper bound in $B(R)$.

In this paper, we will show that if R is a commutative ring with identity and $(S,\leq)$ is a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ of generalized power series is a PF-ring if and only if for any two S-indexed subsets A and B of R such that $B \subseteq \mathrm{ann}_R(A)$, there exists $c \in \mathrm{ann}_R(A)$ such that $bc = b$ for all $b \in B$, and that for a Noetherian ring R, $[[R^{S,\leq}]]$ is a PP ring if and only if R is a PP ring.

Let $(S,\leq)$ be an ordered set. Recall that $(S,\leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. It is easy to see that $(S,\leq)$ is artinian if and only if every non-empty subset of S has a minimal element. Moreover, if $\leq$ is a total order, then $(S,\leq)$ is artinian if and only if it is well-ordered. Recall that an ordered monoid $(S,\leq)$ is *strictly ordered* if $s, s' \in S$ with $s < s'$, then $s + t < s' + t$ for any $t \in S$. For example, if S is cancellative or the order is trivial, then $(S,\leq)$ is a strictly ordered monoid.

The following definition is due to P. Ribenboim [4]: Let $(S,\leq)$ be a strictly ordered monoid and let R be a commutative ring with 1. Let $[[R^{S,\leq}]]$ be the set of all functions $f : S \to R$ such that $Supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. We call $\{f(s) \mid s \in Supp(f)\}$ the set of all coefficients of f. It is clear that R is an additive abelian group with pointwise addition. For every $s \in S$ and $f_1, \cdots, f_n \in R$, let $X_s(f_1, \cdots, f_n) = \{(u_1, \cdots, u_n) \in S^n \mid s = u_1 + \cdots + u_n,\ u_i \in Supp(f_i) \text{ for each } i\}$. It follows from [4, (e) p. 368] that $X_s(f_1, \cdots, f_n)$ is finite. This fact allows one to define the operation of convolution $*$ as follows:

$$(f * h)(s) = \sum_{(u,v) \in X_s(f,h)} f(u)h(v).$$

With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes a commutative ring with identity element e, where

$$e(s) = \begin{cases} 1 & \text{if } \ s = 0 \\ 0 & \text{if } \ 0 \neq s \in S. \end{cases}$$

We call $[[R^{S,\leq}]]$ the *ring of generalized power series*. It should be noted that the definition of $[[R^{S,\leq}]]$ depends on the order $\leq$, for example, see [4, p. 371]. Following [12, 2.5], R is an integral domain if and only if D is an integral domain, and S is torsion-free and cancellative. It follows from [4, p. 368] that R is canonically embedded as a subring of $[[R^{S,\leq}]]$, and that S is canonically embedded as a submonoid of $([[R^{S,\leq}]] \setminus \{0\}, *)$. Numerous examples of rings of generalized power series are given in [12, 13].

In [4], [12], [13], [14], [15], there are many results on ordered monoids and the rings of generalized power series. The following result is well-known and will be frequently used in the sequel.

Lemma 1.1 ([14]).

(1) *If S has a compatible strict total order $\leq$, then S is torsion-free and cancellative.*

(2) *Let S be a torsion-free and cancellative monoid. If $\leq$ is any compatible order on S, then there exists a compatible total order $\leq'$ on S, which is finer than $\leq$ (i.e., if $s, t \in S$ such that $s \leq t$, then $s \leq' t$).*

General references for any undefined terminology or notation are [4], [12], [13], [14], [15].

2. Main results

Recall that a ring R is called a *PF-ring* if every principal ideal of R is a flat R-module. It is proved in [1] that a ring R is a PF-ring if and only if the annihilator of each element $r \in R$, $\mathrm{ann}_R(r)$, is a pure ideal; that is, for all $b \in \mathrm{ann}_R(r)$ there exists $c \in \mathrm{ann}_R(r)$ such that $bc = b$.

Lemma 2.1 ([12, 3.5])**.** *Let S be a torsion-free and cancellative monoid and $\leq$ a strict order on S. Then $[[R^{S,\leq}]]$ is reduced if and only if R is reduced.*

Lemma 2.2 ([2, Lemma 1])**.** *Any PF-ring is reduced.*

Lemma 2.3 ([9, Corollary 3.3])**.** *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a reduced ring. If $f_1, f_2, \cdots, f_n \in [[R^{S,\leq}]]$ are such that $f_1 f_2 \cdots f_n = 0$, then $f_1(s_1) f_2(s_2) \cdots f_n(s_n) = 0$ for all $s_1, s_2, \cdots, s_n \in S$.*

Let A be a subset of R. As in [10], we will say that A is *S-indexed* if there exists an artinian and narrow subset I of S such that A is indexed by I.

Theorem 2.4. *Let R be a commutative ring with identity and $(S, \leq)$ a strictly totally ordered monoid. Then $[[R^{S,\leq}]]$ is a PF-ring if and only if for any two S-indexed subsets A and B of R such that $B \subseteq ann_R(A)$, there exists $c \in ann_R(A)$ such that $bc = b$ for all $b \in B$.*

Proof. Note that S is torsion-free and cancellative by Lemma 1.1, since $(S, \leq)$ is a strictly totally ordered monoid.

($\Leftarrow$): Let $f, g \in [[R^{S,\leq}]]$ and let $g \in \mathrm{ann}_{[[R^{S,\leq}]]}(f)$. Then $gf = 0$. Note that, in particular, R is a PF-ring, since for all $b \in ann_R(a)$, there exists $c \in ann_R(a)$ such that $bc = b$. So by Lemma 2.2, R is reduced. Thus by Lemma 2.3, $g(t)f(s) = 0$ for all $s, t \in S$. Let $A = \{f(s) \mid s \in Supp(f)\}$ and $B = \{g(t) \mid t \in Supp(g)\}$. Then A and B are S-indexed and $B \subseteq \mathrm{ann}_R(A)$. So by hypothesis, there exists $c \in \mathrm{ann}_R(A)$ such that $g(t)c = g(t)$ for all $g(t) \in B$, and so $g(t)c = g(t)$ for all $t \in S$. Hence $gc = g$ and $c \in \mathrm{ann}_{[[R^{S,\leq}]]}(f)$. Therefore $[[R^{S,\leq}]]$ is a PF-ring.

($\Rightarrow$): Assume that $[[R^{S,\leq}]]$ is a PF-ring. Let $A = \{a_s \mid s \in I\}$ and $B = \{b_t \mid t \in J\}$ be two S-indexed subsets of R such that $B \subseteq \mathrm{ann}_R(A)$, where I and J are artinian and narrow subsets of S. Define $f : S \to R$ ($g : S \to R$ respectively) via

$$f(s) = \begin{cases} a_s & \text{if } s \in I \\ 0 & \text{if } s \notin I, \end{cases} \quad \text{and} \quad g(t) = \begin{cases} b_t & \text{if } t \in J \\ 0 & \text{if } t \notin J. \end{cases}$$

Then $Supp(f) = I$ and $Supp(g) = J$ are artinian and narrow, and so $f, g \in [[R^{S,\leq}]]$. It is easy to see that $gf = 0$. Therefore $g \in \text{ann}_{[[R^{S,\leq}]]}(f)$. Thus by assumption there exists $h \in \text{ann}_{[[R^{S,\leq}]]}(f)$ such that $gh = g$. Therefore, we have $hf = 0$ and $g(h - e) = 0$. Since, by Lemma 2.2 and Lemma 2.1, R is reduced, $h(u)f(s) = 0$ for all $u, s \in S$ and $g(t)(h(0) - 1) = 0$ for all $t \in S$. So $h(0) \in \text{ann}_R(A)$ and $bh(0) = b$ for all $b \in B$. Therefore the above condition holds. □

The following corollaries will give us other examples of PF-rings.

Corollary 2.5. *Let $\mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a \geq 0\}$ and $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$. Then the ring $[[\mathbb{Z}^{\mathbb{N},\leq}]]$, $[[\mathbb{Z}^{\mathbb{R}^+,\leq}]]$, $[[\mathbb{Z}^{\mathbb{Z},\leq}]]$, $[[\mathbb{Z}^{\mathbb{N},\leq}]]$, $[[\mathbb{Z}^{\mathbb{Q},\leq}]]$, and $[[\mathbb{Z}^{\mathbb{R},\leq}]]$ are PF-rings , where $\leq$ is the usual order.*

Corollary 2.6. *Let $(S_1, \leq_1), \cdots, (S_m, \leq_m)$ be strictly totally ordered monoids. Denote by $(lex \leq_i)$ and $(rev\ lex \leq_i)$ the lexicographic order and the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_m$. If R is a commutative ring satisfying property: for any two S-indexed subsets A and B of R such that $B \subseteq ann_R(A)$, there exists $c \in ann_R(A)$ such that $bc = b$ for all $b \in B$. Then $[[R^{S_1 \times \cdots \times S_m, (lex \leq_i)}]]$ and $[[R^{S_1 \times \cdots \times S_m, (rev\ lex \leq_i)}]]$ are PF-rings.*

Let R be a commutative ring, and consider the multiplicative monoid $\mathbb{N}_{\geq 1}$, endowed with the usual order $\leq$. Then $[[R^{\mathbb{N}_{\geq 1},\leq}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution: $fg(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$, for each $n \geq 1$.

Corollary 2.7. *Let R be a commutative ring. Then $[[R^{\mathbb{N}_{\geq 1},\leq}]]$ is a PF-ring if and only if for any two S-indexed subsets A and B of R such that $B \subseteq ann_R(A)$, there exists $c \in ann_R(A)$ such that $bc = b$ for all $b \in B$.*

Corollary 2.8. *Let R be a commutative ring and $(S, \leq)$ a strictly ordered monoid with S being cancellative and torsion-free. If for any two S-indexed subsets A and B of R such that $B \subseteq ann_R(A)$, there exists $c \in ann_R(A)$ such that $bc = b$ for all $b \in B$ and $(S, \leq)$ is narrow, then $[[R^{S,\leq}]]$ is a PF-ring.*

Recall that R called a *generalized PF-ring* (for short, *GPF-ring*) if, given any $a \in R$, then the principal ideal Ra^n is flat as an R-module for some $n \geq 1$. It is proved in [3] that a commutative ring R is a GPF-ring if and only if, given any $a \in R$, either a is a regular element in every prime localization R_P or for some $n \geq 1$, $a^n = 0$ in every R_P. Also note in [3] that a commutative ring R is a PF-ring if and only if R is a reduced GPF-ring.

Recall that a ring R is called a *PP-ring* if every principal ideal of R is a projective R-module. It is well-known that a ring R is a PP-ring if and only if the annihilator,

$ann_R(a)$, is generated by an idempotent for every $a \in R$ (cf., [1]). It is proved in [10, Theorem 2.3] that the ring $[[R^{S,\leq}]]$ of generalized power series is a PP-ring if and only if R is a PP-ring and every S-indexed subset C of $B(R)$ has a least upper bound in $B(R)$, where $B(R)$ is the set of all idempotents of R.

For each $f \in [[R^{S,\leq}]]$, let $C(f)$ denote the ideal of R generated by the coefficients of f: $C(f) = (\{f(s)|s \in S\})$. Let $r \in R$. Define a mapping $c_r \in [[R^{S,\leq}]]$ as follows:

$$c_r(s) = \begin{cases} r & \text{if } \ s = 0, \\ 0 & \text{if } \ 0 \neq s \in S. \end{cases}$$

Lemma 2.9 ([10, Lemma 2.2])**.** Let R be a reduced commutative ring and S a cancellative and torsion-free monoid. If $g^2 = g \in [[R^{S,\leq}]]$, then there exists an idempotent $e \in R$ such that $g = c_e$.

Theorem 2.10. *Let R be a Noetherian ring and let $(S, \leq)$ be a strictly totally ordered monoid. Then $[[R^{S,\leq}]]$ is a PP-ring if and only if R is a PP-ring.*

Proof. Suppose that $[[R^{S,\leq}]]$ is a PP-ring. Let $a \in R$. Then $ann_{[[R^{S,\leq}]]}(a) = g[[R^{S,\leq}]]$ for some $g \in [[R^{S,\leq}]]$ such that $g^2 = g$. By Lemma 2.9, there exists an idempotent $e \in R$ such that $g = c_e$. We claim that $ann_R(a) = eR$. If $b \in ann_R(a)$, then $ba = 0$. Then $b \in ann_{[[R^{S,\leq}]]}(a) = c_e[[R^{S,\leq}]]$, and so we have $b = c_e h$ for some $h \in [[R^{S,\leq}]]$. Thus $b = eh(0)$. Hence $ann_R(a) \subseteq eR$. For the opposite inclusion, suppose that $d \in eR$. Then $d = er$ for some $r \in R$. Since $e \in ann_R(a)$, we have $d \in ann_R(a)$. Thus $ann_R(a) \supseteq eR$, and so $ann_R(a) = eR$. Therefore R is a PP-ring.

Conversely, assume that R is a PP-ring. Let $h \in [[R^{S,\leq}]]$ and $f \in ann_{[[R^{S,\leq}]]}(h)$. Since R is reduced, $f(s)h(t) = 0$ for all $s, t \in S$. Since R is Notherian, $C(h)$ is finitely generated, say $c(h) = (h(t_0), h(t_1), \cdots, h(t_n))$. Let $N = ann_R(h(t_0), h(t_1), \cdots, h(t_n))$. Then $f(s) \in N$ for each $s \in S$. Therefore, $f \in [[N^{S,\leq}]]$ and $ann_{[[R^{S,\leq}]]}(h) \subseteq [[N^{S,\leq}]]$. If $g \in [[N^{S,\leq}]]$, then $C(g) \subseteq N = ann_R(C(h))$. Therefore, $g \in ann_{[[R^{S,\leq}]]}(h)$. Hence $ann_{[[R^{S,\leq}]]}(h) = [[N^{S,\leq}]]$ and $N = \bigcap_{i=0}^{n} ann_R(h(t_i))$. Since R is a PP-ring, $ann_R(h(t_i)) = e_i R$ for each $i = 0, 1, \cdots, n$, where e_i is an idempotent element of R. Then $N = \bigcap_{i=0}^{n} e_i R = (e_1 e_2 \cdots e_n)R = eR$, where e is an idempotent element of R. Therefore $ann_{[[R^{S,\leq}]]}(h) = e[[R^{S,\leq}]]$. Hence $[[R^{S,\leq}]]$ is a PP-ring. □

References

[1] H. Al-Ezeh, *On some properties of polynomial rings*, Int. J. Math. Math. Sci., **10**(1987), 311-314.

[2] H. Al-Ezeh, *Two properties of the power series ring*, Int. J. Math. Math. Sci., **11**(1988), 9-14.

[3] H. Al-Ezeh, *On generalized PF-rings*, Math. J. Okayama Univ., **31**(1989), 25-29.

[4] G. A. Elliott and P. Ribenboim, *Fields of generalized power series*, Arch. Math., **54**(1990), 365-371.

[5] H. Kim, *On t-closedness of generalized power series rings*, J. Pure Appl. Algebra, **166**(2002), 277-284.

[6] H. Kim and Y. S. Park, *Krull domains of generalized power series*, J. Algebra, **237**(2001), 292-301.

[7] J.-H. Kim, *A note on the quotient ring $R((X))$ of the power series ring $R[[X]]$*, J. Korean Math. Soc., **25**(1998), 265-271.

[8] Z. Liu, *On n-closedness of generalized power series rings over pairs of rings*, J. Pure Appl. Algebra, **144**(1999), 303-312.

[9] Z. Liu, *Special properties of rings of generalized power series*, Comm. Algebra, **32**(2004), 3215-3226.

[10] Z. Liu and J. Ahsan, *PP-rings of generalized power series*, Acta. Math. Sinica, **16**(2000), 573-578.

[11] E. Matlis, *The minimal spectrum of a reduced ring*, Ill. J. Math., **27**(1983), 353-391.

[12] P. Ribenboim, *Rings of generalized power series: Nilpotent elements*, Abh. Math. Sem. Univ. Hamburg, **61**(1991), 15-33.

[13] P. Ribenboim, *Noetherian rings of generalized power series*, J. Pure Appl. Algebra, **79**(1992), 293-312.

[14] P. Ribenboim, *Rings of generalized power series, II: Units and zero-divisors*, J. Algebra, **168**(1994), 71-89.

[15] P. Ribenboim, *Special properties of generalized power series*, J. Algebra, **173**(1995), 566-586.

[16] J. J. Rotman, "An Introduction to Homological Algebra", Academic Press, San Diego, 1993.