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# Linear Derivations Satisfying a Functional Equation on Semisimple Banach Algebras

Yong-Soo Jung*
*Research Institute for a Creative Future, Seowon University, Cheongju, Chungbuk 361-742, Korea*
*e-mail* : `jungys@cnu.ac.kr`

Ick-Soon Chang
*Department of Mathematics, Mokwon University, Daejeon 302-729, Korea*
*e-mail* : `ischang@mokwon.ac.kr`

Abstract. In this paper, we investigate the following: Let $A$ be a semisimple Banach algebra. Suppose that there exists a linear derivation $f : A \to A$ such that the functional equation $\langle f(x), x\rangle^2 = 0$ holds for all $x \in A$. Then we have $f = 0$ on $A$.

## 1. Introduction

Throughout this paper, $A$ will represent an algebra over a complex field $\mathbb{C}$ and the *Jacobson radical* of $A$ will be denoted by $rad(A)$, i.e., the intersection of all primitive ideals of $A$. $A$ is said to be semisimple if $rad(A) = \{0\}$. Recall that $A$ is *semiprime* if $xAx = \{0\}$ implies $x = 0$ and $A$ is *prime* if $xAy = \{0\}$ implies $x = 0$ or $y = 0$ An additive mapping $f : A \to A$ is called a *derivation* if $f(xy) = f(x)y + xf(y)$ holds for all $x, y \in A$. We write $[x, y]$ for the *Lie product* $xy - yx$ and $\langle x, y\rangle$ denotes the *Jordan product* $xy + yx$.

In 1955, Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical [6]. In the same paper they conjectured that the assumption of continuity is not necessary. In 1988, Thomas proved the (so-called) Singer-Wermer conjecture [7]. Obviously, the Singer-Wermer conjecture implies that every derivation on a commutative semisimple Banach algebra is identically zero. But, in the noncommutative setting, it is still an open question whether the above result is true or not.

Our main purpose in this paper is to supply a partial solution of the open question for noncommutative semisimple Banach algebras. That is, let $A$ be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $f : A \to A$ satisfying the functional equation $\langle f(x), x\rangle^2 = 0$ for all $x \in A$. Then we

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have $f = 0$ on $A$. By using this result, we also give a condition which characterizes commutative semisimple Banach algebras among all semisimple Banach algebras.

## 2. Results

For the purpose, we will need the lemmas below.

**Lemma 2.1.** *Let $R$ be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

*Proof.* See [8, Lemma 1]. □

**Lemma 2.2.** *Let $A$ be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $f : R \to R$ such that the functional equation $f(x)[f(x), x] = 0$ holds for all $x \in R$. Then we have $f = 0$ on $A$.*

*Proof.* See [3, Corollary 2.8]. □

**Lemma 2.3.** *Let $A$ be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $f : A \to A$ such that the functional equation $[[f(x), x], f(x)] = 0$ holds for all $x \in A$. Then we have $f = 0$ on $A$.*

*Proof.* By the result of Johnson and Sinclair [2], every linear derivation on a semisimple Banach algebra is continuous. Hence [4, Theorem 2.5] gives the result. □

**Lemma 2.4.** *Let $A$ be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $f : A \to A$ such that the functional equation $\langle\langle f(x), x\rangle, f(x)\rangle = 0$ holds for all $x \in A$. Then we have $f = 0$ on $A$.*

*Proof.* As the proof of Lemma 2.3, the conclusion is true in view of [4, Theorem 2.6]. □

Our main result is

**Theorem 2.5.** *Let $A$ be a semisimple Banach algebra. Suppose that there exists a linear derivation $f : A \to A$ such that the functional equation $\langle f(x), x\rangle^2 = 0$ holds for all $x \in A$. Then we have $f = 0$ on $A$.*

*Proof.* As above, every linear derivation on a semisimple Banach algebra is continuous. Also, following the result of Sinclair [5], every continuous linear derivation on a Banach algebra leaves the primitive ideals of $A$ invariant. Therefore for every primitive ideal $P \subseteq A$, we can define a linear derivation $f_P : A/P \to A/P$, where $A/P$ is a factor Banach algebra which is primitive, by $f_P(\hat{x}) = f(x) + P$, $\hat{x} = x + P$ for all $x \in A$. Observe that the assumption $\langle f(x), x\rangle^2 = 0$, $x \in A$ yields $\langle f(\hat{x}), \hat{x}\rangle^2 = \hat{0}$, $\hat{x} \in A/P$. Hence we see that there is no loss of generality in assuming that $A$ is primitive. In particular, $A$ is prime. Since a commutative Banach algebra is isomorphic to the complex field $\mathbb{C}$, we may assume that $A$ is noncommutative.

Now suppose that the functional equation

$$\langle f(x),\ x\rangle^2 = 0 \tag{1}$$

holds for all $x \in A$. The linearization of (1) leads to

$$P_1(x,y) + P_2(x,y) + P_3(x,y) = 0, \quad x,\ y \in A, \tag{2}$$

where $P_k(x,y)$ is the sum of terms involving $x$ and $y$ such that $P_k(x,my) = m^k P_k(x,y)$, $k = 1,2,3$ and $m \in \mathbb{Z}$. Substituting $-y$ for $y$ in (2), we obtain by comparing the result with (2) that

$$P_1(x,y) + P_3(x,y) = 0, \quad x,\ y \in A. \tag{3}$$

Substituting $2y$ for $y$ in (3), we get

$$2P_1(x,y) + 8P_3(x,y) = 0, \quad x,\ y \in A. \tag{4}$$

Multiplying by 8 in (3) and subtracting (4) from the result, we obtain

$$\begin{aligned} 0 &= P_1(x,y) \\ &= f(x)xf(x)y + f(x)xf(y)x + f(x)yf(x)x + f(y)xf(x)x \\ &\quad + f(x)x^2f(y) + f(x)xyf(x) + f(x)yxf(x) + f(y)x^2f(x) \\ &\quad + xf(x)^2y + xf(x)f(y)x + xf(y)f(x)x + yf(x)^2x \\ &\quad + xf(x)xf(y) + xf(x)yf(x) + xf(y)xf(x) + yf(x)xf(x), \quad x,\ y \in A. \end{aligned} \tag{5}$$

Putting $xy$ instead of $y$ in (5), we obtain

$$\begin{aligned} &f(x)xf(x)xy + f(x)x^2f(y)x + f(x)xf(x)yx + f(x)xyf(x)x \\ &+xf(y)xf(x)x + f(x)yxf(x)x + f(x)x^2f(x)y + f(x)x^3f(y) \\ &+f(x)x^2yf(x) + f(x)xyxf(x) + xf(y)x^2f(x) + f(x)yx^2f(x) \\ &+xf(x)^2xy + xf(x)xf(y)x + xf(x)^2yx + x^2f(y)f(x)x + xf(x)yf(x)x \\ &+xyf(x)^2x + xf(x)x^2f(y) + xf(x)xf(x)y + xf(x)xyf(x) \\ &+x^2f(y)xf(x) + xf(x)yxf(x) + xyf(x)xf(x) = 0, \quad x,\ y \in A. \end{aligned} \tag{6}$$

Left-multiplying by $x$ in (5) and subtracting the result from (6), we have

$$\begin{aligned} &[f(x),x^2]f(y)x + [f(x),x^2]xf(y) + \langle f(x),x\rangle f(x)yx \\ &+[f(x),x^2]yf(x) + f(x)xy\langle f(x),x\rangle + f(x)yx\langle f(x),x\rangle \\ &+x[f(x)^2,x]y + f(x)x\langle f(x),x\rangle y = 0, \quad x,\ y \in A. \end{aligned} \tag{7}$$

Substituting $yx$ for $y$ in (7), we arrive at

$$\begin{aligned} &[f(x),x^2]yf(x)x + [f(x),x^2]f(y)x^2 \\ &+[f(x),x^2]xyf(x) + [f(x),x^2]xf(y)x \\ &+\langle f(x),x\rangle f(x)yx^2 + [f(x),x^2]yxf(x) \\ &+f(x)xyx\langle f(x),x\rangle + f(x)yx^2\langle f(x),x\rangle \\ &+x[f(x)^2,x]yx + f(x)x\langle f(x),x\rangle yx = 0, \quad x,\ y \in A. \end{aligned} \tag{8}$$

Right-multiplying by $x$ in (7) and subtracting the result from (8), we obtain

$$\begin{aligned}(9)\qquad & [f(x),x^2]yxf(x)+[f(x),x^2]xyf(x)\\ & -f(x)xy[f(x),x^2]-f(x)yx[f(x),x^2]=0, \quad x,\ y\in A.\end{aligned}$$

Replacing $f(x)y$ for $y$ in (9), we have

$$\begin{aligned}(10)\qquad & [f(x),x^2]f(x)yxf(x)+[f(x),x^2]xf(x)yf(x)\\ & -f(x)xf(x)y[f(x),x^2]-f(x)^2yx[f(x),x^2]=0, \quad x,\ y\in A.\end{aligned}$$

Left-multiplying by $f(x)$ in (9) and subtracting the result from (10), we get

$$\begin{aligned}(11)\qquad & [[f(x),x^2],f(x)]yxf(x)+[[f(x),x^2]x,f(x)]yf(x)\\ & +f(x)[f(x),x]y[f(x),x^2]=0, \quad x,\ y\in A.\end{aligned}$$

Putting $yf(x)$ instead of $y$ in (11), we obtain

$$\begin{aligned}(12)\qquad & [[f(x),x^2],f(x)]yf(x)xf(x)+[[f(x),x^2]x,f(x)]yf(x)^2\\ & +f(x)[f(x),x]yf(x)[f(x),x^2]=0, \quad x,\ y\in A.\end{aligned}$$

Right-multiplying by $f(x)$ in (11) and subtracting the result from (12), we have

$$[[f(x),x^2],f(x)]y[f(x),x]f(x)-f(x)[f(x),x]y[[f(x),x^2],f(x)]=0, \quad x,\ y\in A.$$

¿From Lemma 2.1, it follows that for any $y\in A$,

$$[[f(x),x],f(x)]y[[f(x),x^2],f(x)]=0$$

and hence for any $x\in A$, either $[[f(x),x],f(x)]=0$ or $[[f(x),x^2],f(x)]=0$. That is, $A$ is the union of its subsets $D=\{x\in A:[[f(x),x],f(x)]=0\}$ and $E=\{x\in A:[[f(x),x^2],f(x)]=0\}$. Suppose that $f\neq 0$. Then we see from Lemma 2.3 that $D\neq A$. We also assert that $E\neq A$. Assume that $E=A$, i.e., $[[f(x),x^2],f(x)]=0$ for all $x\in A$. Replacing $y$ by $y\langle f(x),x\rangle$ in (9), we obtain

$$\begin{aligned}(13)\qquad & [f(x),x^2]y\langle f(x),x\rangle xf(x)+[f(x),x^2]xy\langle f(x),x\rangle f(x)\\ & -f(x)xy\langle f(x),x\rangle[f(x),x^2]-f(x)y\langle f(x),x\rangle x[f(x),x^2]=0, \quad x,\ y\in A.\end{aligned}$$

Right-multiplying by $\langle f(x),x\rangle$ in (9) and adding the result to (13), we have

$$\begin{aligned}(14)\quad & [f(x),x^2]y\langle\langle f(x),x\rangle,xf(x)\rangle+[f(x),x^2]xy\langle\langle f(x),x\rangle,f(x)\rangle\\ & -f(x)xy\langle\langle f(x),x\rangle,[f(x),x^2]\rangle-f(x)y\langle\langle f(x),x\rangle,x[f(x),x^2]\rangle=0, \quad x,\ y\in A.\end{aligned}$$

Since $\langle f(x),x\rangle^2=0$, we get

$$\begin{aligned} & \langle\langle f(x),x\rangle,[f(x),x^2]\rangle\\ = & \langle\langle f(x),x\rangle,[\langle f(x),x\rangle,x]\rangle\\ = & \langle f(x),x\rangle^2x-\langle f(x),x\rangle x\langle f(x),x\rangle\\ & +\langle f(x),x\rangle x\langle f(x),x\rangle-x\langle f(x),x\rangle^2=0, \quad x,\ y\in A.\end{aligned}$$

Therefore, the relation (14) can be reduced to

$$\begin{aligned}(15)\qquad &[f(x),x^2]y\langle\langle f(x),x\rangle, xf(x)\rangle + [f(x),x^2]xy\langle\langle f(x),x\rangle, f(x)\rangle \\ &-f(x)y\langle\langle f(x),x\rangle, x[f(x),x^2]\rangle = 0, \quad x,\ y\in A.\end{aligned}$$

Substituting $f(x)y$ for $y$ in (15), we arrive at

$$\begin{aligned}(16)\qquad &[f(x),x^2]f(x)y\langle\langle f(x),x\rangle, xf(x)\rangle + [f(x),x^2]xf(x)y\langle\langle f(x),x\rangle, f(x)\rangle \\ &-f(x)^2y\langle\langle f(x),x\rangle, x[f(x),x^2]\rangle = 0, \quad x,\ y\in A.\end{aligned}$$

Left-multiplying by $f(x)$ in (15) and subtracting the result from (16), we obtain

$$\begin{aligned}0 &= [[f(x),x^2],f(x)]y\langle\langle f(x),x\rangle, xf(x)\rangle + [[f(x),x^2]x,f(x)]y\langle\langle f(x),x\rangle, f(x)\rangle \\ &= [[f(x),x^2]x,f(x)]y\langle\langle f(x),x\rangle, f(x)\rangle, \quad x,\ y\in A.\end{aligned}$$

Namely, we see that

$$[[f(x),x^2]x,f(x)]y\langle\langle f(x),x\rangle, f(x)\rangle = 0, \quad x,\ y\in A.$$

¿From primeness of $A$, it follows that for any $x\in A$, either $[[f(x),x^2]x,f(x)] = 0$ or $\langle\langle f(x),x\rangle, f(x)\rangle = 0$. Hence $A$ is the union of its subsets $F = \{x\in A : \langle\langle f(x),x\rangle, f(x)\rangle = 0\}$ and $G = \{x\in A : [[f(x),x^2]x,f(x)] = 0\}$. Because of $f\neq 0$, we see from Lemma 2.4 that $F\neq A$. We claim that $G\neq A$. Assume that $G=A$, i.e., $[[f(x),x^2]x,f(x)]=0$ for all $x\in A$. Since both $E=A$ and $G=A$ are valid, it follows from (11) that

$$f(x)[f(x),x]y[f(x),x^2] = 0, \quad x,\ y\in A.$$

Again using primeness of $A$, we see that for any $x\in A$, either $f(x)[f(x),x]=0$ or $[f(x),x^2]=0$. Hence $A$ is the union of its subsets $H=\{x\in A : f(x)[f(x),x]=0\}$ and $I=\{x\in A : [f(x),x^2]=0\}$. Since $f\neq 0$, we obtain from Lemma 2.2 and [1, the proof of Theorem 2] that $H\neq A$ and $I\neq A$, respectively. This implies that there exist $x_0, y_0\in A$ such that $x_0\notin H$ and $y_0\notin I$. Hence, $y_0\in H$ and $x_0\in I$. Now consider $x_0+\lambda y_0,\ \lambda\in\mathbb{C}$. Then we see that either $x_0+\lambda y_0\in H$ or $x_0+\lambda y_0\in I$. If $x_0+\lambda y_0\in H$, then we have

$$(17)\qquad f(x_0)[f(x_0),x_0] + \lambda P_1(x_0,y_0) + \lambda^2 P_2(x_0,y_0) = 0$$

and also if $x_0+\lambda y_0\in I$, then we get

$$(18)\qquad \lambda P_1(x_0,y_0) + \lambda^2 P_2(x_0,y_0) + \lambda^3[f(y_0),y_0^2] = 0,$$

where $P_k(x_0,y_0)$ is the sum of terms involving $x_0$ and $y_0$ such that

$$P_k(x_0,my_0) = m^kP_k(x_0,y_0), \quad k=1,2 \text{ and } m\in\mathbb{Z}.$$

Therefore, for every $\lambda \in \mathbb{C}$ one of these two possibilities holds. But either (17) has more than two solutions or (18) has more than three solutions. And this contradicts the assumption that $f(x_0)[f(x_0), x_0] \neq 0$ and $[f(y_0), y_0^2] \neq 0$.

We now see that $G \neq A$, as claimed. Namely, $F \neq A$ and $G \neq A$. So there exist $x_1, y_1 \in A$ such that $x_1 \notin F$ and $y_1 \notin G$. Thus $y_1 \in F$ and $x_1 \in G$. Then we see that either $x_1 + \lambda y_1 \in F$ or $x_1 + \lambda y_1 \in G$. If $x_1 + \lambda y_1 \in F$, then we have

$$\langle\langle f(x_1), x_1\rangle, f(x_1)\rangle + \lambda P_1(x_1, y_1) + \lambda^2 P_2(x_1, y_1) = 0 \tag{19}$$

and also if $x_1 + \lambda y_1 \in G$, then we get

$$\begin{aligned} &\lambda P_1(x_1, y_1) + \lambda^2 P_2(x_1, y_1) + \lambda^3 P_3(x_1, y_1) \\ &+\lambda^4 P_4(x_1, y_1) + \lambda^5 [[f(y_1), y_1^2]y_1, f(y_1)] = 0, \end{aligned} \tag{20}$$

where $P_k(x_1, y_1)$ is the sum of terms involving $x_1$ and $y_1$ such that

$$P_k(x_1, my_1) = m^k P_k(x_1, y_1), \quad k = 1, 2, 3, 4 \text{ and } m \in \mathbb{Z}.$$

Therefore, for every $\lambda \in \mathbb{C}$ one of these two possibilities holds. But, since either (19) has more than two solutions or (20) has more than five solutions, this contradicts the assumption that $\langle\langle f(x_1), x_1\rangle, f(x_1)\rangle \neq 0$ and $[[f(y_1), y_1^2]y_1, f(y_1)] \neq 0$. Because this contradiction comes from the hypothesis $E = A$, it gives $E \neq A$ which was the first assertion. Hence we conclude that $D \neq A$ and $E \neq A$. This means that there exist $x_2, y_2 \in A$ such that $x_2 \notin D$ and $y_2 \notin E$. Thus $y_2 \in D$ and $x_2 \in E$. We also obtain that either $x_2 + \lambda y_2 \in D$ or $x_2 + \lambda y_2 \in E$. If $x_2 + \lambda y_2 \in D$, then we have

$$[[f(x_2), x_2], f(x_2)] + \lambda P_1(x_2, y_2) + \lambda^2 P_2(x_2, y_2) = 0 \tag{21}$$

and if $x_2 + \lambda y_2 \in E$, then we get

$$\lambda P_1(x_2, y_2) + \lambda^2 P_2(x_2, y_2) + \lambda^3 P_3(x_2, y_2) + \lambda^4 [[f(y_2), y_2^2], f(y_2)] = 0, \tag{22}$$

where $P_k(x_2, y_2)$ is the sum of terms involving $x_2$ and $y_2$ such that

$$P_k(x_2, my_2) = m^k P_k(x_2, y_2), \quad k = 1, 2, 3 \text{ and } m \in \mathbb{Z}.$$

Therefore, for every $\lambda \in \mathbb{C}$ one of these two possibilities holds. But either (21) has more than two solutions or (22) has more than four solutions. And this contradicts the assumption that $f(x_2)[f(x_2), x_2] \neq 0$ and $[[f(y_2), y_2^2], f(y_2)] \neq 0$. Consequently, we conclude that $f = 0$ which completes the proof. □

As a special case of Theorem 2.5, we obtain the next result which characterizes commutative semisimple Banach algebras among all semisimple Banach algebras.

**Corollary 2.6.** *Let $A$ be a semisimple Banach algebra. Suppose that $\langle [x, y], x\rangle^2 = 0$ holds for all $x, y \in A$. Then $A$ is commutative.*

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