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New Two-Weight Imbedding Inequalities for $\mathcal{A}$-Harmonic Tensors

Gao Hongya and Chen Yanmin
College of Mathematics and Computer Science, Hebei University, Baoding, 071002, P. R. China
e-mail : hongya-gao@sohu.com

Chu Yuming
Department of Mathematics, Huzhou Normal College, Huzhou, 313000, P. R. China

Abstract. In this paper, we first define a new kind of two-weight-$A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weight, and then prove the imbedding inequalities for $\mathcal{A}$-harmonic tensors. These results can be used to study the weighted norms of the homotopy operator T from the Banach space $L^p(D, \bigwedge^l)$ to the Sobolev space $W^{1,p}(D, \bigwedge^{l-1})$, $l = 1, 2, \cdots, n$, and to establish the basic weighted L^p-estimates for $\mathcal{A}$-harmonic tensors.

1. Introduction

Throughout this paper, we always assume that Ω is a connected open subset of R^n, $n \geq 2$. We write $\mathrm{R} = \mathrm{R}^1$. Balls are denoted by B and σB (σ is a real positive number) is the ball with the same center as B and with $diam(\sigma B) = \sigma diam(B)$. The n-dimensional Lebesgue measure of a set $E \subseteq \mathrm{R}^n$ is denote by $|E|$. We call $w(x)$ a weight if $w \in L^1_{loc}(\mathrm{R}^n)$ and $w > 0$ a.e.. For $1 \leq p < \infty$ and a weight $w(x)$, we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,\, E,\, w^\alpha} = \left(\int_E |f(x)|^p w^\alpha dx \right)^{1/p},$$

where α is a real number.

Let $e_1, e_2, \cdots, e_n$ be the standard unit basis of R^n. Let $\bigwedge^l = \bigwedge^l(\mathrm{R}^n)$ be the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 0, 1, \cdots, n$. The Grassman algebra $\bigwedge = \bigoplus \bigwedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \bigwedge$ and $\beta = \sum \beta^I e_I \in \bigwedge$, the inner product in $\bigwedge$ is given by

$$\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$$

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with summation over all l-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the Hodge star operator $* : \bigwedge \to \bigwedge$ by the rule $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1)$ for all $\alpha, \beta \in \bigwedge$. The norm of $\alpha \in \bigwedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \bigwedge^0 = \mathrm{R}$. The Hodge star is an isometric isomorphism on $\bigwedge$ with $* : \bigwedge^l \to \bigwedge^{l-1}$ and $* * (-1)^{l(n-l)} : \bigwedge^l \to \bigwedge^l$.

A differential l-form ω on Ω is a *de Rham* current (see [13, Chapter III]) on Ω with values in $\bigwedge^l(\mathrm{R}^n)$. We use $D'(\Omega, \bigwedge^l)$ to denote the space of all differential l-forms and $L^p(\Omega, \bigwedge^l)$ to denote the l-forms

$$\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$$

with $\omega_I(x) \in L^p(\Omega, \mathrm{R})$ for all ordered l-tuples I. Thus, $L^p(\Omega, \bigwedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.$$

For $\omega \in D'(\Omega, \bigwedge^l)$, the vector-valued differential form

$$\nabla \omega = \left(\frac{\partial \omega}{\partial x_1}, \cdots, \frac{\partial \omega}{\partial x_n} \right)$$

consists of differential forms $\partial \omega / \partial x_i \in D'(\Omega, \bigwedge^l)$, where the partial differentiation is applied to the coefficients of ω.

Similarly, $W^{1,p}(\Omega, \bigwedge^l)$ is used to denote the Sobolev space of l-forms which equals $L^p(\Omega, \bigwedge^l) \cap L_1^p(\Omega, \bigwedge^l)$ with norm

$$\|\omega\|_{W^{1,p}(\Omega,\, \bigwedge^l)} = \mathrm{diam}(\Omega)^{-1} \|\omega\|_{p,\,\Omega} + \|\nabla \omega\|_{p,\,\Omega}$$

The notations $W^{1,p}_{loc}(\Omega, \mathrm{R})$ and $W^{1,p}_{loc}(\Omega, \bigwedge^l)$ are self-explanatory. For $1 \le p < \infty$ and a weight $w(x)$, the weighted norm of $\omega \in W^{1,p}(\Omega, \bigwedge^l)$ over Ω is denoted by

$$\|\omega\|_{W^{1,p}(\Omega,\, \bigwedge^l),\, w^\alpha} = \mathrm{diam}(\Omega)^{-1} \|\omega\|_{p,\,\Omega,\, w^\alpha} + \|\nabla \omega\|_{p,\,\Omega,\, w^\alpha}, \tag{1.1}$$

where α is a real number. We denote the exterior derivative by $d : D'(\Omega, \bigwedge^l) \to D'(\Omega, \bigwedge^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $d^* : D'(\Omega, \bigwedge^{l+1}) \to D'(\Omega, \bigwedge^l)$ is given by $d^* = (-1)^{nl+1} * d *$ on $D'(\Omega, \bigwedge^{l+1})$, $l = 0, 1, \cdots, n$.

Consider the following $\mathcal{A}$-harmonic equation

$$d^* \mathcal{A}(x, du(x)) = 0 \tag{1.2}$$

for differential forms,where $\mathcal{A} : \Omega \times \bigwedge^l(\mathrm{R}^n) \to \bigwedge^l(\mathrm{R}^n)$ satisfies the conditions

$$|\mathcal{A}(x, \xi)| \le a|\xi|^{p-1} \ \text{ and } \ \langle \mathcal{A}(x, \xi), \xi \rangle \ge |\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \bigwedge^l(\mathrm{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.2). A solution to (1.2) is an element of the Sobolev space $W^{1,\,p}_{loc}(\Omega, \bigwedge^{l-1})$ such that

$$\int_\Omega \langle \mathcal{A}(x, du), d\varphi \rangle = 0$$

for all $\varphi \in W^{1,p}(\Omega, \bigwedge^{l-1})$ with compact support.

Definition 1.1. We call u an $\mathcal{A}$-harmonic tensor in Ω if u satisfies the $\mathcal{A}$-harmonic equation (1.2) in Ω.

A differential l-form $u \in D'(\Omega, \bigwedge^l)$ is called closed if $du = 0$ in Ω. Similarly, a differential $(l+1)$-form $v \in D'(\Omega, \bigwedge^{l+1})$ is called coclosed if $d^* v = 0$. Clearly, the $\mathcal{A}$-harmonic equation is not affected by adding a closed form to ω. Therefore, any type of estimates about u must be modulo a closed form.

If $\omega : \Omega \to \bigwedge^l$, then the value of $\omega(x)$ at the vectors $\xi_1, \xi_2, \cdots, \xi_l \in \mathrm{R}^n$ will be denoted by $\omega(x; \xi_1, \cdots, \xi_l)$. The following lemma appears in [11].

Lemma 1.2. *Let $D \subset R^n$ be a bounded, convex domain. To each $y \in D$, there corresponds a linear operator $K_y : C^\infty(D, \bigwedge^l) \to C^\infty(D, \bigwedge^{l-1})$ defined by*

$$(K_y \omega)(x; \xi_1, \cdots, \xi_l) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \cdots, \xi_{l-1}) dt$$

and the decomposition

$$\omega = d(K_y \omega) + K_y(d\omega).$$

A homotopy operator $T : C^\infty(D, \bigwedge^l) \to C^\infty(D, \bigwedge^{l-1})$ is defined by averaging K_y over all points y in D

$$T\omega = \int_D \varphi(y) K_y \omega dy, \tag{1.4}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y) dy = 1$. Then we have the following decomposition:

$$\omega = d(T\omega) + T(d\omega).$$

If we define the l-form $\omega_D \in D'(D, \bigwedge^l)$ by

$$\omega_D = |D|^{-1} \int_D \omega(y) dy, \quad \text{for } l = 0, \quad \text{and} \quad \omega_D = d(T\omega), \quad \text{for} \quad l = 1, 2, \cdots, n$$

for all $\omega \in L^p(D, \bigwedge^l), 1 \le p < \infty$, then

$$\omega_D = \omega - T(d\omega).$$

By substituting $z = tx + y - ty$, (1.4) reduces to

$$T\omega(x, \xi) = \int_D \omega(z, \zeta(z, x - z), \xi) dz, \tag{1.5}$$

where the vector function $\zeta : D \times \mathrm{R}^n \to \mathrm{R}^n$ is given by

$$\zeta(z, h) = h \int_0^\infty s^{l-1} (1+s)^{n-1} \varphi(z - sh) ds.$$

Integral (1.5) defines a bounded operator

$$T : L^s(D, \bigwedge^l) \to W^{1,s}(D, \bigwedge^{l-1}), \quad l = 1, 2, \cdots, n$$

with norm estimated by

$$\|Tu\|_{W^{1,\,s}(D)} \leq C|D|\|u\|_{s,\,D}.$$

In recent years many interesting results concerning geometric and analytic properties of solutions to the $\mathcal{A}$-harmonic equation (1.2) has been established, see [1]-[9]. The purpose of this paper is to prove some new weighted imbedding inequalities for solutions to the $\mathcal{A}$-harmonic equation (1.2) and establish some weighted norm estimates for the homotopy operator T. These inequalities are important tools in generalizing the theory of Sobolev functions to differential forms and estimating the upper bounds of L^p-norms of differential forms. These results can also be used to study the integrability of differential forms and estimate the integrals for them.

2. Local $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weighted imbedding inequalities

Definition 2.1. We say a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-condition for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$, write $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, if $w_1(x) > 0, w_2(x) > 0$ a.e., and

$$\sup_B \quad \frac{1}{|B|}\int_B w_1^{\lambda_1}dx \qquad \frac{1}{|B|}\int_B \quad \frac{1}{w_2} \quad ^{\lambda_2/(r-1)} dx\Bigg)^{\lambda_3(r-1)} < \infty \tag{2.1}$$

for any ball $B \subset \Omega$.

If $w_1 = w_2 = w$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda$ in definition 2.1, we then obtain the usual $A_r^{\lambda}(\Omega)$-weight introduced in [10], [18]. See [10], [18] for properties about $A_r^{\lambda}(\Omega)$-weight. If $w_1 = w_2$, $\lambda_2 = \lambda_3 = 1$ and $\lambda_1 = \lambda$ in definition 2.1, we then obtain the usual $A_r(\lambda, \Omega)$-weight introduced in [14]. See [1], [16]-[17] for results about $A_r(\lambda, \Omega)$-weight. If $w_1 = w_2$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in definition 2.1, we then obtain the usual $A_r(\Omega)$-weight introduced in [15]. See [2]-[5] and [15], [16] for results about $A_r(\Omega)$-weight. We will also need the following generalized Hölder's inequality.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on R^n, then*

$$\|fg\|_{s,\,\Omega} \leq \|f\|_{\alpha,\,\Omega} \cdot \|g\|_{\beta,\,\Omega}$$

for any $\Omega \subset R^n$.

From results appearing in [11], we have the following lemma.

Lemma 2.3. *Let $u \in L^s_{loc}(B, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form in a ball $B \subset R^n$, then*

$$\|\nabla(Tu)\|_{s,\,B} \leq C|B|\|u\|_{s,\,B}$$

and

$$\|Tu\|_{s,\,B} \leq C|B|\mathrm{diam}(B)\|u\|_{s,\,B}.$$

The following weak reverse Hölder inequality appears in [8].

Lemma 2.4. *Let u be a differential form satisfying* (1.2) *in a domain Ω, $\rho > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st}\|u\|_{t,\,\rho B}$$

for all balls or cubes B with $\rho B \subset \Omega$.

Theorem 2.5. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in (1.4). Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |Tu|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \le C|B|\mathrm{diam}(B)\left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s} \tag{2.2}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Note that (2.2) can be written as

$$\|Tu\|_{s,\, B,\, w_1^{\alpha\lambda_1}} \le C|B|\mathrm{diam}(B)\|u\|_{s,\, \rho B,\, w_2^{\alpha\lambda_2\lambda_3}}. \tag{2.2$'$}$$

Proof. Let $t = s/(1-\alpha)$. Applying Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}\left(\int_B |Tu|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} &= \left(\int_B \left(|Tu| w_1^{\alpha\lambda_1/s}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B |Tu|^t dx\right)^{1/t} \left(\int_B w_1^{\alpha t\lambda_1/(t-s)} dx\right)^{(t-s)/st} \\ &\le C_1|B|\mathrm{diam}(B)\|u\|_{t,\, B}\left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s}.\end{aligned} \tag{2.3}$$

Choose $m = s/(1 + \alpha\lambda_3(r-1))$, then $m < s$. Using Lemma 2.4, we have

$$\|u\|_{t,\, B} \le C_2|B|^{(m-t)/mt}\|u\|_{m,\, \rho B}. \tag{2.4}$$

Where $\rho > 1$. Substituting (2.4) in (2.3), we have

$$\left(\int_B |Tu|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \le C_3|B|\mathrm{diam}(B)|B|^{(m-t)/mt}\|u\|_{m,\, \rho B}\left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s}. \tag{2.5}$$

Using Hölder's inequality again with $1/m = 1/s + (s-m)/sm$, we have

$$\begin{aligned}&\|u\|_{m,\, \rho B} \\ &= \left(\int_{\rho B} |u|^m dx\right)^{1/m} \\ &= \left(\int_{\rho B} \left(|u| w_2^{\alpha\lambda_2\lambda_3/s} w_2^{-\alpha\lambda_2\lambda_3/s}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s}\left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\alpha\lambda_2\lambda_3 m/(s-m)} dx\right)^{(s-m)/sm} \\ &= \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s}\left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s}\end{aligned} \tag{2.6}$$

for all balls B with $\rho B \subset \Omega$. Combining (2.5) and (2.6), we find that

$$
\begin{aligned}
(2.7)\qquad & \left(\int_B |Tu|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \\
\leq\ & C_3 |B| \operatorname{diam}(B) |B|^{(m-t)/mt} \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s} \\
& \times \left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s}.
\end{aligned}
$$

Using the condition that $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, we derive that

$$
\begin{aligned}
(2.8)\qquad & \left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s} \\
\leq\ & \left(\left(\int_{\rho B} w_1^{\lambda_1} dx\right) \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)} \right)^{\alpha/s} \\
=\ & \left(|\rho B|^{\lambda_3(r-1)+1} \left(\frac{1}{|\rho B|}\int_{\rho B} w_1^{\lambda_1} dx\right) \left(\frac{1}{|\rho B|}\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)} \right)^{\alpha/s} \\
\leq\ & C_4 |B|^{\alpha(\lambda_3(r-1)+1)/s}.
\end{aligned}
$$

Finally, substituting (2.8) in (2.7) and using the fact that $(m-t)/mt = -\alpha(\lambda_3(r-1)+1)/s$, we obtain

$$
(2.9)\qquad \left(\int_B |Tu|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leq C|B| \operatorname{diam}(B) \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s}
$$

for all balls B with $\rho B \subset \Omega$. This ends the proof of Theorem 2.5.

Theorem 2.6. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in* (1.4). *Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$
(2.10)\qquad \left(\int_B |\nabla(Tu)|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leq C|B| \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s}
$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Note that (2.10) can be written as

$$
(2.10)'\qquad \|\nabla(Tu)\|_{s,\, B,\, w_1^{\alpha\lambda_1}} \leq C|B| \|u\|_{s,\, \rho B,\, w_2^{\alpha\lambda_2\lambda_3}}.
$$

Proof. Let $t = s/(1-\alpha)$. Using Lemma 2.2, we obtain

$$
\begin{aligned}
(2.11)\qquad \left(\int_B |\nabla(Tu)|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} &= \left(\int_B (|\nabla(Tu)| w_1^{\alpha\lambda_1/s})^s dx\right)^{1/s} \\
&\le \|\nabla(Tu)\|_{t,\,B} \left(\int_B w_1^{\alpha t\lambda_1/(t-s)} dx\right)^{(t-s)/st} \\
&= \|\nabla(Tu)\|_{t,\,B} \left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s}.
\end{aligned}
$$

By lemma 2.3, we have

$$
(2.12)\qquad \|\nabla(Tu)\|_{t,\,B} \le C_1|B|\|u\|_{t,\,B}.
$$

Choose $m = s/(1+\alpha\lambda_3(r-1))$, then $m < s$. Substituting (2.12) into (2.11) and using Lemma 2.4, we have

$$
\begin{aligned}
(2.13)\qquad & \left(\int_B |\nabla(Tu)|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \\
\le\ & C_1|B|\|u\|_{t,\,B} \left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s} \\
\le\ & C_2|B||B|^{(m-t)/mt}\|u\|_{m,\,\rho B} \left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s}.
\end{aligned}
$$

Using Hölder's inequality again with $1/m = 1/s + (s-m)/sm$, we arrive at

$$
\begin{aligned}
(2.14)\ \|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m dx\right)^{1/m} \\
&= \left(\int_{\rho B} (|u| w_2^{\alpha\lambda_2\lambda_3/s} w_2^{-\alpha\lambda_2\lambda_3/s})^m dx\right)^{1/m} \\
&\le \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s} \left(\int_{\rho B} (\frac{1}{w_2})^{\alpha\lambda_2\lambda_3 m/(s-m)} dx\right)^{(s-m)/sm} \\
&= \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s} \left(\int_{\rho B} (\frac{1}{w_2})^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s}
\end{aligned}
$$

for all balls B with $\rho B \subset \Omega$. Combining (2.13) and (2.14), we obtain

$$
\begin{aligned}
(2.15)\qquad & \left(\int_B |\nabla(Tu)|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \\
\le\ & C_3|B||B|^{(m-t)/mt} \left(\int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s} \\
& \times \left(\int_B w_1^{\lambda_1} dx\right)^{\alpha/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s}.
\end{aligned}
$$

Since $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, we then have

$$
\begin{aligned}
(2.16)\qquad & \int_B w_1^{\lambda_1}dx\ {}^{\alpha/s} \ \int_{\rho B} \frac{1}{w_2}\ {}^{\lambda_2/(r-1)} dx\Big)^{\alpha\lambda_3(r-1)/s} \\
\le\ & \Bigg(\int_{\rho B} w_1^{\lambda_1}dx \ \int_{\rho B}\frac{1}{w_2}\ {}^{\lambda_2/(r-1)}dx\Big)^{\lambda_3(r-1)}\Bigg)^{\alpha/s} \\
=\ & \Bigg(|\rho B|^{\lambda_3(r-1)+1}\ \frac{1}{|\rho B|}\int_{\rho B} w_1^{\lambda_1}dx \ \ \frac{1}{|\rho B|}\int_{\rho B}\frac{1}{w_2}\ {}^{\lambda_2/(r-1)}dx\Big)^{\lambda_3(r-1)}\Bigg)^{\alpha/s} \\
\le\ & C_4|B|^{\alpha(\lambda_3(r-1)+1)/s}.
\end{aligned}
$$

Combining (2.16) and (2.15), we find that

$$
(2.17)\qquad \int_B |\nabla(Tu)|^s w_1^{\alpha\lambda_1}dx\ {}^{1/s} \le C_4|B| \ \int_{\rho B} |u|^s w_2^{\alpha\lambda_2\lambda_3}dx\ {}^{1/s}
$$

for all balls B with $\rho B \subset \Omega$. The proof of Theorem 2.6 has been completed.

Now, we prove the following local weighted imbedding inequality for differential forms under the homotopy operator T.

Theorem 2.7. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$ and $T : L^s(\Omega, \bigwedge^l) \to W^{1,s}(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in (1.5). Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$
(2.18)\qquad \|Tu\|_{W^{1,s}(B,\bigwedge^l),\, w_1^{\alpha\lambda_1}} \le C|B|\|u\|_{s,\,\rho B,\, w_2^{\alpha\lambda_2\lambda_3}}
$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Proof. Form (1.1), $(2.2)'$ and $(2.10)'$, we obtain

$$
\begin{aligned}
\|Tu\|_{W^{1,s}(B,\bigwedge^l),\, w_1^{\alpha\lambda_1}} &= \operatorname{diam}(B)^{-1}\|Tu\|_{s,\,B,\,w_1^{\alpha\lambda_1}} + \|\nabla(Tu)\|_{s,\,B,\,w_1^{\alpha\lambda_1}} \\
&\le \operatorname{diam}(B)^{-1}\Big[C_1|B|\operatorname{diam}(B)\|u\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}}\Big] + C_2|B|\|u\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}} \\
&= C_1|B|\|u\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}} + C_2|B|\|u\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}} \\
&\le C_3|B|\|u\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}}
\end{aligned}
$$

which is equivalent to (2.18). The proof of Theorem 2.7 has been completed.

Using Theorem 2.7, we prove the following Sobolev-Poincaré imbedding inequality for differential forms.

Theorem 2.8. *Let $du \in L^s_{loc}(\Omega, \bigwedge^{l+1})$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$. Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$
(2.19)\qquad \|u - u_B\|_{W^{1,s}(B,\bigwedge^l),\, w_1^{\alpha\lambda_1}} \le C|B|\|du\|_{s,\,\rho B,\, w_2^{\alpha\lambda_2\lambda_3}}
$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Proof. Since $\omega = du \in L^s_{loc}(\Omega, \bigwedge^{l+1})$ satisfies (1.2), using (2.18) and $u_B = u - T(du)$ we have

$$\begin{aligned} \|u - u_B\|_{W^{1,s}(B,\bigwedge^l),\, w_1^{\alpha\lambda_1}} &= \|T(du)\|_{W^{1,s}(B,\bigwedge^l),\, w_1^{\alpha\lambda_1}} \\ &\le C|B|\|du\|_{s,\rho B,\, w_2^{\alpha\lambda_2\lambda_3}}. \end{aligned}$$

This ends the proof of Theorem 2.8.

Note that the parameters $\lambda_1, \lambda_2, \lambda_3$ and α in the above Theorems are any real numbers with $0 < \alpha < 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Therefore, we will have different versions of the weighted imbedding inequalities by choosing $\lambda_1, \lambda_2, \lambda_3$ and α to take different values. Choosing $\lambda_1 = 1$, $\alpha = 1/s$ and $\alpha = 1/\lambda_1$ with $\lambda_1 > 1$ in theorems 2.5 and 2.6, we have the following Corollaries 2.9-2.11, respectively.

Corollary 2.9. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in (1.4). Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{s,B,w_1^\alpha} \le C|B|\mathrm{diam}(B)\|u\|_{s,\rho B,\, w_2^{\alpha\lambda_2\lambda_3}} \tag{2.20}$$

$$\|\nabla(Tu)\|_{s,B,w_1^\alpha} \le C|B|\|u\|_{s,\rho B,\, w_2^{\alpha\lambda_2\lambda_3}} \tag{2.21}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Corollary 2.10. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in (1.4). Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{s,B,w_1^{\lambda_1/s}} \le C|B|\mathrm{diam}(B)\|u\|_{s,\rho B,\, w_2^{\lambda_2\lambda_3/s}} \tag{2.22}$$

$$\|\nabla(Tu)\|_{s,B,w^{\lambda_1/s}} \le C|B|\|u\|_{s,\rho B,\, w^{\lambda_2\lambda_3/s}} \tag{2.23}$$

for all balls B with $\rho B \subset \Omega$.

Corollary 2.11. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in (1.4). Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $\lambda_1 > 1$ and $0 < \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{s,B,w_1} \le C|B|\mathrm{diam}(B)\|u\|_{s,\rho B,\, w_2^{\lambda_2\lambda_3/\lambda_1}} \tag{2.24}$$

$$\|\nabla(Tu)\|_{s,B,w_1} \le C|B|\|u\|_{s,\rho B,\, w_2^{\lambda_2\lambda_3/\lambda_1}} \tag{2.25}$$

for all balls B with $\rho B \subset \Omega$.

If we set $\lambda_3 = 1/\lambda_2$ in Corollary 2.9, we have the following symmetric imbedding inequality.

Corollary 2.12. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in* (1.4). *Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{1/\lambda_2}(1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_2 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{s,\,B,\,w_1^\alpha} \leq C|B|\mathrm{diam}(B)\|u\|_{s,\,\rho B,\,w_2^\alpha} \tag{2.26}$$

$$\|\nabla(Tu)\|_{s,\,B,\,w_1^\alpha} \leq C|B|\|u\|_{s,\,\rho B,\,w_2^\alpha} \tag{2.27}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Putting $\lambda_1 = \lambda_2\lambda_3$ in Corollary 2.10 and Corollary 2.11, we also obtain some symmetric imbedding inequalities. Considering the length of the paper, we do not list these similar results here.

If we choose $\lambda_1 = 1$ in Theorem 2.7, we then have

$$\|Tu\|_{W^{1,\,s}(B,\,\bigwedge^l),\,w_1^\alpha} \leq C|B|\|u\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}}. \tag{2.28}$$

Putting $\alpha = 1/s, s > 1$ in Theorem 2.7, we find that

$$\|Tu\|_{W^{1,\,s}(B,\,\bigwedge^l),\,w_1^{\lambda_1/s}} \leq C|B|\|u\|_{s,\,\rho B,\,w_2^{\lambda_2\lambda_3/s}}. \tag{2.29}$$

Putting $\alpha = 1/\lambda_1, \lambda_1 > 1$ in Theorem 2.7, we have

$$\|Tu\|_{W^{1,\,s}(B,\,\bigwedge^l),\,w_1} \leq C|B|\|u\|_{s,\,\rho B,\,w_2^{\lambda_2\lambda_3/\lambda_1}}. \tag{2.30}$$

Similarly, from Theorem 2.8, we have the following Sobolev-Poincaré imbedding inequalities for differential forms.

Corollary 2.13. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$. Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u - u_B\|_{W^{1,\,s}(B,\,\bigwedge^l),\,w_1^\alpha} \leq C|B|\|du\|_{s,\,\rho B,\,w_2^{\alpha\lambda_2\lambda_3}} \tag{2.31}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Corollary 2.14. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$. Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u - u_B\|_{W^{1,\,s}(B,\,\bigwedge^l),\,w_1^{\lambda_1/s}} \leq C|B|\|du\|_{s,\,\rho B,\,w_2^{\lambda_2\lambda_3/s}} \tag{2.32}$$

for all balls B with $\rho B \subset \Omega$.

Corollary 2.15. *Let $u \in L^s_{loc}(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$. Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in$*

$A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $\lambda_1 > 1$ and $0 < \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that

$$\|u - u_B\|_{W^{1,\, s}(B, \bigwedge^l),\, w_1} \le C|B|\|du\|_{s,\, \rho B,\, w_2^{\lambda_2\lambda_3/\lambda_1}} \tag{2.33}$$

for all balls B with $\rho B \subset \Omega$.

Putting $\lambda_3 = 1/\lambda_2$, $\lambda_1 = \lambda_2\lambda_3$ in (2.28)-(2.30), Corollaries 2.13, 2.14 and 2.15, we also obtain similar symmetric imbedding inequalities. Considering the length of the paper, we leave it to the reader to find the similar results.

3. Global $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weighted imbedding inequalities

As applications of the local results, we prove the global $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weighted imbedding inequalities in this section. We shall need the following Lemma about the Whitney covers appearing in [8]. See [19] for more properties of Whitney cubes.

Lemma 3.1. *Each Ω has a modified Whitney cover of cubes $\nu = \{Q_i\}$ such that*

$$\begin{aligned} \bigcup_i Q_i &= \Omega, \\ \sum_{Q \in \nu} \chi_{\sqrt{5/4}Q}(x) &\le N\chi_\Omega(x), \end{aligned}$$

for all $x \in R^n$ and $N > 1$, where χ_E is the characteristic function for a set E. Moreover, if $Q_i \cap Q_j \neq \phi$, there exists a cube R (this cube does not need to be a member of ν) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Also, if Ω is a δ-John domain, then there is a distinguished cube $Q_0 \in \nu$ which can be connected with every cube $Q \in \nu$ by a chain of cubes $Q_0, Q_1, \cdots, Q_k = Q$ from ν and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \cdots, k$, for some $\rho = \rho(n, \delta)$.

Now, we prove the following global $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weighted imbedding inequalities in a bounded domain Ω for differential forms satisfying the $\mathcal{A}$-harmonic equation.

Theorem 3.2. *Let $u \in L^s(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying (1.2) in a bounded domain $\Omega \subset R^n$ and $T : C^\infty(\Omega, \bigwedge^l) \to C^\infty(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in (1.4). Assume that $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{s,\, \Omega,\, w_1^{\alpha\lambda_1}} \le C\|u\|_{s,\, \Omega,\, w_2^{\alpha\lambda_2\lambda_3}} \tag{3.1}$$

and

$$\|\nabla(Tu)\|_{s,\, \Omega,\, w_1^{\alpha\lambda_1}} \le C\|u\|_{s,\, \Omega,\, w_2^{\alpha\lambda_2\lambda_3}} \tag{3.2}$$

for any real number α with $0 < \alpha < 1$.

Proof. Using (2.2) and Lemma 3.1, we find that

$$\begin{aligned}\left(\int_\Omega |Tu|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} &\le \sum_{Q\in\nu} C_1|Q|\mathrm{diam}(Q)\left(\int_{\rho Q}|u|^s w_2^{\alpha\lambda_2\lambda_3}dx\right)^{1/s}\\ &\le C_1|\Omega|\mathrm{diam}(\Omega)\sum_{Q\in\nu}\left(\int_{\rho Q}|u|^s w_2^{\alpha\lambda_2\lambda_3}dx\right)^{1/s}\\ &\le C_2\left(\int_\Omega |u|^s w_2^{\alpha\lambda_2\lambda_3}dx\right)^{1/s}\end{aligned}$$

since Ω is bounded. We have proved the inequality (3.1). Similarly, using Lemma 3.1 and (2.10), we can prove (3.2). The proof of Theorem 3.2 has been completed. □

From Theorem 2.7 and Theorem 3.2, we have the following global imbedding inequality.

Theorem 3.3. *Let $u \in L^s(\Omega, \bigwedge^l)$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$ and $T : L^s(\Omega, \bigwedge^l) \to W^{1,s}(\Omega, \bigwedge^{l-1})$ be a homotopy operator defined in* (1.5). *Assume that $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|Tu\|_{W^{1,s}(\Omega,\bigwedge^l),\, w_1^{\alpha\lambda_1}} \le C\|u\|_{s,\Omega,\, w_2^{\alpha\lambda_2\lambda_3}} \tag{3.3}$$

for any real number α with $0 < \alpha < 1$.

Proof. Combining (1.1), (3.1) and (3.2), we derive that

$$\begin{aligned}\|Tu\|_{W^{1,s}(\Omega,\bigwedge^l),\, w_1^{\alpha\lambda_1}} &= \mathrm{diam}(\Omega)^{-1}\|Tu\|_{s,\Omega,\, w_1^{\alpha\lambda_1}} + \|\nabla(Tu)\|_{s,\Omega,\, w_1^{\alpha\lambda_1}}\\ &\le \mathrm{diam}(\Omega)^{-1}\left[C_1\|u\|_{s,\Omega,\, w_2^{\alpha\lambda_2\lambda_3}}\right] + C_2\|u\|_{s,\Omega,\, w_2^{\alpha\lambda_2\lambda_3}}\\ &\le C_3\|u\|_{s,\Omega,\, w_2^{\alpha\lambda_2\lambda_3}}.\end{aligned}$$

Hence (3.3) is true. The proof of Theorem 3.3 has been completed. □

Theorem 3.4. *Let $du \in L^s(\Omega, \bigwedge^{l+1})$, $l = 1, 2, \cdots, n$, $1 < s < \infty$, be a differential form satisfying* (1.2) *in a bounded domain $\Omega \subset R^n$. Assume that $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|u - u_\Omega\|_{W^{1,s}(\Omega,\bigwedge^l),\, w_1^{\alpha\lambda_1}} \le C\|du\|_{s,\Omega,\, w_2^{\alpha\lambda_2\lambda_3}} \tag{3.4}$$

for any real number α with $0 < \alpha < 1$.

Proof. Since $\omega = du \in L^s(\Omega, \bigwedge^{l+1})$ satisfies (1.2), using (3.3) and $u_\Omega = u - T(du)$ we have

$$\begin{aligned} \|u - u_\Omega\|_{W^{1,s}(\Omega, \bigwedge^l), w_1^{\alpha\lambda_1}} &= \|T(du)\|_{W^{1,s}(\Omega, \bigwedge^l), w_1^{\alpha\lambda_1}} \\ &\leq C\|du\|_{s, \Omega, w_2^{\alpha\lambda_2\lambda_3}} \end{aligned}$$

Therefore, (3.4) is true. The proof of Theorem 3.4 has been completed. □

Remark. If we choose $\lambda_1, \lambda_2, \lambda_3$ and α to take some special values in Theorems 3.2-3.4, respectively, we shall have some global results similar to the local case. For example, if we let $w_1 = w_2 = w$ and $\lambda_1 = \lambda_2\lambda_3 = 1$, then (3.1) and (3.2) become

$$\|Tu\|_{s, \Omega, w^\alpha} \leq C\|u\|_{s, \Omega, w^\alpha} \tag{3.5}$$

$$\|\nabla(Tu)\|_{s, \Omega, w^\alpha} \leq C\|u\|_{s, \Omega, w^\alpha} \tag{3.6}$$

respectively. Considering the length of the paper, we leave it to the reader to find the similar global results.

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