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# A Generalization of the Hyers-Ulam-Rassias Stability of the Pexiderized Quadratic Equations, II

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Abstract. In this paper we prove the Hyers–Ulam–Rassias stability by considering the cases that the approximate remainder $\varphi$ is defined by $f(x*y)+f(x*y^{-1})-2g(x)-2g(y) = \varphi(x,y)$, $f(x*y)+g(x*y^{-1})-2h(x)-2k(y) = \varphi(x,y)$, where $(G,*)$ is a group, $X$ is a real or complex Hausdorff topological vector space and $f,g,h,k$ are functions from $G$ into $X$.

## 1. Introduction

In 1940, S. M. Ulam [23] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, D. H. Hyers [7] proved that if $f : V \to X$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in V$, where $V$ and $X$ are Banach spaces and $\delta$ is a given positive number, then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in V$.

Th.M. Rassias [17] gave a generalization of the Hyers' result(see also [5], [17], [20], [21]). This is the first theorem that has been proved in the subject of stability of functional equations which allows the Cauchy difference to be unbounded. P. Găvruta [6] following Th. M. Rassias's approach for the Cauchy difference to be unbounded, obtained a generalization of the Hyers-Rassias theorem. (see also [8], [15],[ 16]).
Lee and Jun [13], [14] also obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$(see also [12]).

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In 1983, the stability theorem for the quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0$$

was proved F. Skof [22] for the function $f : V \to X$. In 1984, P. W. Cholewa [1] extended the Skof's result to the case where $V$ is an Abelian group $G$. In 1992, S. Czerwik [3] gave a generalization of the Skof-Cholewa's result. Since then, the stability problem of the quadratic equation has been extensively investigated by a number of mathematician([2],[4],[18],[19]). In 2001, the authors [11] proved the stability of the Pexiderized quadratic inequalities :

$$\begin{aligned} \|f(x+y)+f(x-y)-2g(x)-2g(y)\| &\leq \varphi(x,y), \\ \|f(x+y)+g(x-y)-2h(x)-2k(y)\| &\leq \varphi(x,y). \end{aligned}$$

Throughout this paper, we denote by $G$ a group and by $X$ a real or complex Hausdorff topological vector space. By $\mathbb{N}$ we denote the set of positive integers. $e$ stands for the unit of $G$, while it is 0 instead of $e$ if $G$ is an abelian group. W. Jian [9] obtained the Hyers-Ulam-Rassias stability theory by considering the cases where the approximate remainder $\varphi$ is defined by

$$\begin{aligned} f(x*y)-f(x)-f(y) &= \varphi(x,y) \text{ for all } x,y \in G, \\ f(x*y)-g(x)-h(y) &= \varphi(x,y) \text{ for all } x,y \in G, \end{aligned}$$

where $f, g, h$ are functions from $G$ into $X$. In 2004, the authors [10] obtained the Hyers-Ulam-Rassias stability theory by considering the cases where the approximate remainder $\varphi$ is defined by

$$\begin{aligned} f(x*y)+f(x*y^{-1})-2g(x)-2g(y) &= \varphi(x,y) \text{ for all } x,y \in G\backslash\{e\}, \\ f(x*y)+g(x*y^{-1})-2h(x)-2k(y) &= \varphi(x,y) \text{ for all } x,y \in G\backslash\{e\}, \end{aligned}$$

where $f, g, h, k$ are functions from $G$ into $X$. In this paper, using the direct method, we obtain some generalization of the Hyers-Ulam-Rassias stability theory by considering the cases where the approximate remainder $\varphi$ is defined by

$$f(x*y)+f(x*y^{-1})-2g(x)-2g(y) = \varphi(x,y) \text{ for all } x,y \in G, \tag{1.1}$$

$$f(x*y)+g(x*y^{-1})-2h(x)-2k(y) = \varphi(x,y) \text{ for all } x,y \in G. \tag{1.2}$$

A function $Q : G \to X$ is called quadratic on $G$ if $Q(x*y)+Q(x*y^{-1})-2Q(x)-2Q(y)=0$.

## 2. Stability of the equation

In this section, we prove the stability of the functional equation (1.1).

**Theorem 2.1.** *Let $\varphi : G \times G \to X$ be a mapping satisfying the conditions*

$$\lim_{n\to\infty} \frac{\varphi(x^{2^n}, y^{2^n})}{4^n} = 0, \tag{2.1}$$

$$\tilde{\varphi}(x^i, x^j) := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{4^{k+1}} \varphi(x^{i\cdot 2^k}, x^{j\cdot 2^k}) \in X \tag{2.2}$$

*for all* $x, y \in G$ *and for any fixed* $i, j = 0, 1, 2, 3, \cdots$. *Suppose that the functions* $f, g : G \to X$ *satisfy*

$$f(x * y) + f(x * y^{-1}) - 2g(x) - 2g(y) = \varphi(x, y), \tag{2.3}$$

$$f((x * y)^{2^n}) = f(x^{2^n} * y^{2^n}), g((x * y)^{2^n}) = g(x^{2^n} * y^{2^n}) \tag{2.4}$$

*for all* $x, y \in G$ *and* $n \in \mathbb{N}$. *Then the limit* $Q(x) = \lim_{n\to\infty} f(x^{2^n})/4^n = \lim_{n\to\infty} g(x^{2^n})/4^n$ *exists for all* $x \in G$ *and* $Q$ *is quadratic. In this case, the equations*

$$f(x) - f(e) - Q(x) = -\tilde{\varphi}(x, x) + 2\tilde{\varphi}(x, e) - \frac{\varphi(e, e)}{3}, \tag{2.5}$$

$$g(x) - g(e) - Q(x) = \frac{1}{2}\tilde{\varphi}(x^2, e) - \tilde{\varphi}(x, x) + \frac{1}{6}\varphi(e, e) \tag{2.6}$$

*hold for all* $x \in G$.

*Proof.* Let $x$ be an arbitrary fixed element of $G$. From (2.3), we have

$$\frac{f(e)}{2} - g(e) = \frac{1}{4}\varphi(e, e), \tag{2.7}$$

$$f(x) - g(x) - g(e) = \frac{1}{2}\varphi(x, e), \tag{2.8}$$

$$\frac{1}{4}(f(x^2) + f(e)) - g(x) = \frac{1}{4}\varphi(x, x) \tag{2.9}$$

for all $x \in G$. From (2.7), (2.8) and (2.9), we get

$$\begin{aligned} f(x) - f(e) - \frac{1}{4}(f(x^2) - f(e)) &= f(x) - g(x) - g(e) \\ &- [\frac{1}{4}f(x^2) + f(e) - g(x)] - [\frac{f(e)}{2} - g(e)] \\ &= -\frac{1}{4}\varphi(x, x) + \frac{1}{2}\varphi(x, e) - \frac{1}{4}\varphi(e, e). \end{aligned} \tag{2.10}$$

for all $x \in G$. Induction argument implies

$$f(x) - f(e) - \frac{1}{4^n}(f(x^{2^n}) - f(e)) = \sum_{i=0}^{n-1} \frac{-\varphi(x^{2^i}, x^{2^i}) + 2\varphi(x^{2^i}, e) - \varphi(e, e)}{4^{i+1}}. \tag{2.11}$$

for all $x \in G$ and for all $n \in N$. From (2.1) and (2.11), $\lim_{n\to\infty} \frac{f(x^{2^n}) - f(e)}{4^n}$ exists for any $x \in G$. From this, we can define $Q : G \to X$ by

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n}$$

for any $x \in G$ and the equation (2.5) holds for all $x \in G$. Replacing $x$ by $x^{2^n}$ and dividing by $4^n$ in (2.8), we get

$$\frac{1}{4^n}f(x^{2^n}) - \frac{1}{4^n}g(x^{2^n}) - \frac{1}{4^n}g(e) = \frac{\varphi(x^{2^n}, e)}{2 \cdot 4^n} \tag{2.12}$$

for all $n \in N$. Taking the limit in (2.12) as $n \to \infty$, the equation $Q(x) = \lim_{n\to\infty} g(x^{2^n})/4^n$ holds. From (2.5) and (2.8), we have the equation (2.6). Replacing $x$, $y$ by $x^{2^n}$, $y^{2^n}$, respectively and dividing by $4^n$ in (2.3), we have

$$\frac{f(x^{2^n} * y^{2^n})}{4^n} + \frac{f(x^{2^n} * y^{-2^n})}{4^n} - \frac{2g(x^{2^n})}{4^n} - \frac{2g(y^{2^n})}{4^n} = \frac{\varphi(x^{2^n}, y^{2^n})}{4^n},$$

for all $x, y \in G$ and for all $n \in N$. Taking the limit in the above equation as $n \to \infty$, we easily obtain

$$Q(x * y) + Q(x * y^{-1}) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in G$. From (2.5), (2.7) and (2.8), we easily get (2.6).

**Corollary 2.1.** *Let $V$ be a vector space and $X$ a Banach space. Let $\varphi : V \times V \to [0, \infty)$ be a mapping such that*

$$\tilde{\varphi}(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

*for all $x, y \in V$. Suppose that the functions $f, g : V \to X$ satisfy*

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \le \varphi(x, y) \quad \text{for all} \quad x, y \in V.$$

*Then there exists exactly one quadratic function $Q : V \to X$ such that*

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\le \tilde{\varphi}(x, x) + 2\tilde{\varphi}(x, 0) + \frac{1}{3}\varphi(0, 0) \\ \|g(x) - g(0) - Q(x)\| &\le \frac{1}{2}\tilde{\varphi}(2x, 0) + \tilde{\varphi}(x, x) + \frac{1}{6}\varphi(0, 0) \end{aligned}$$

*for all $x \in V$. The function $Q$ is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} = \lim_{n\to\infty} \frac{g(2^n x)}{4^n}$$

*for all $x \in V$.*

*Proof.* Let $f(x+y) + f(x-y) - 2g(x) - 2g(y) = \varphi_1(x, y)$. Since $V$ is a vector space, the equation (2.4) holds for any $x, y \in V$. Since $\|\varphi_1(x, y)\| \le \varphi(x, y)$ and $X$ is a Banach space, $\varphi_1 : V \times V \to X$ is a mapping satisfying the two conditions

$$\lim_{n\to\infty} \frac{\varphi_1(2^n x, 2^n y)}{4^n} = 0$$

for all $x, y \in V$ and

$$\tilde{\varphi}(ix, jx) := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{4^{k+1}} \varphi_1(2^k \cdot ix, 2^k \cdot jx) \in X$$

for all $x \in V$ and for any fixed $i, j = 0, 1, 2, 3, \cdots$. By Theorem 2.1, the limit $Q(x) = \lim_{n\to\infty} f(2^n x)/4^n$ exists for any $x \in V$ and $Q$ satisfies

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in V$. In this case, the equations

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &= \| - \tilde{\varphi}_1(x, x) + 2\tilde{\varphi}_1(x, 0) - \frac{\varphi_1(0, 0)}{3}\| \\ &\le \tilde{\varphi}(x, x) + 2\tilde{\varphi}(x, 0) + \frac{1}{3}\varphi(0, 0), \\ \|g(x) - g(0) - Q(x)\| &= \|\frac{1}{2}\tilde{\varphi}_1(2x, 0) - \tilde{\varphi}_1(x, x) + \frac{1}{6}\varphi_1(0, 0)\| \\ &\le \frac{1}{2}\tilde{\varphi}(2x, 0) + \tilde{\varphi}(x, x) + \frac{1}{6}\varphi(0, 0) \end{aligned}$$

hold for all $x \in V$. It remains to show that $Q$ is uniquely determined. Let $Q' : V \to X$ be another function satisfying (). Then

$$\begin{aligned}
\|Q(x) - Q'(x)\| &\leq \|\frac{f(2^n x) - f(0)}{4^n} - Q(x)\| + \|\frac{f(2^n x) - f(0)}{4^n} - Q'(x)\| \\
&= \|\frac{f(2^n x) - f(0) - Q(4^n x)}{4^n}\| + \|\frac{f(2^n x) - f(0) - Q'(2^n x)}{4^n}\| \\
&\leq \frac{1}{4^n}[\tilde{\varphi}(2^{n+1}x, 0) + 2\tilde{\varphi}(2^n x, 2^n x) + \frac{1}{3}\varphi(0,0)]
\end{aligned}$$

for every $x \in V$ and $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \to \infty$, we obtain

$$Q(x) = Q'(x) \quad \text{for all } x \in V.$$

## 3. Stability of the equation (1.2)

In this section, we prove the stability of the Pexiderized quadratic equation (1.2). If a function $f : G \to X$ satisfies $f(x) = f(x^{-1})$ for all $x \in G$, then the function $f$ is called an even function. If a function $f : G \to X$ satisfies $f(x) = -f(x^{-1})$ for all $x \in G$, then the function $f$ is called an odd function.

**Theorem 3.1** (even function)**.** *Let $\varphi : G \times G \to X$ be a mapping satisfying the conditions in Theorem 2.1. Suppose that the even functions $f, g, h, k : G \to X$ satisfy*

$$f(x * y) + g(x * y^{-1}) - 2h(x) - 2k(y) = \varphi(x, y) \tag{3.1}$$

*and the condition* (2.4)*. Then there exists exactly one quadratic function $Q : G \to X$ such that*

$$\begin{aligned}
f(x^2) - f(e) - Q(x^2) &= M(x^2) + \frac{1}{2}[\varphi(x,x) - \varphi(x,x^{-1})] - \frac{1}{3}\varphi(e,e) \\
g(x^2) - g(e) - Q(x^2) &= M(x^2) - \frac{1}{2}[\varphi(x,x) - \varphi(x,x^{-1})] - \frac{1}{3}\varphi(e,e) \\
h(x) - h(e) - Q(x) &= M(x) + \frac{1}{6}\varphi(e,e) - \frac{1}{2}\varphi(x,e) \\
k(x) - k(e) - Q(x) &= M(x) + \frac{1}{6}\varphi(e,e) - \frac{1}{2}\varphi(e,x)
\end{aligned}$$

*for all $x \in G$, where*

$$M(x) = \frac{1}{2}[-\tilde{\varphi}(x,x) + 2\tilde{\varphi}(e,x) + 2\tilde{\varphi}(x,e) - \tilde{\varphi}(x,x^{-1})].$$

*The function $Q$ is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{g(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{h(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{k(x^{2^n})}{4^n}.$$

*for all $x \in G$.*

*Proof.* Since $f, g, h, k$ are the even functions, from (3.1), we can easily obtain

$$f(x) + g(x) - 2h(x) - 2k(e) = \varphi(x, e) \tag{3.2}$$
$$f(x^2) + g(e) - 2h(x) - 2k(x) = \varphi(x, x) \tag{3.3}$$
$$f(x) + g(x) - 2h(e) - 2k(x) = \varphi(e, x) \tag{3.4}$$
$$f(e) + g(x^2) - 2h(x) - 2k(x) = \varphi(x, x^{-1}) \tag{3.5}$$
$$f(e) + g(e) - 2h(e) - 2k(e) = \varphi(e, e) \tag{3.6}$$

for all $x \in G$. From (3.2), (3.3) and (3.4), we get

$$\begin{aligned} & f(x^2) - 2f(x) - 2g(x) + g(e) + 2h(e) + 2k(e) \\ = \ & [f(x^2) + g(e) - 2h(x) - 2k(x)] - [f(x) + g(x) - 2h(e) - 2k(x)] \\ & - [f(x) + g(x) - 2h(x) - 2k(e)] = \varphi(x, x) - \varphi(e, x) - \varphi(x, e) \end{aligned} \tag{3.7}$$

for all $x \in G$. From (3.2), (3.4) and (3.5), we get

$$\begin{aligned} & g(x^2) - 2f(x) - 2g(x) + f(e) + 2h(e) + 2k(e) \\ = \ & [f(e) + g(x^2) - 2h(x) - 2k(x)] - [f(x) + g(x) - 2h(x) - 2k(e)] \\ & - [f(x) + g(x) - 2h(e) - 2k(x)] = \varphi(x, x^{-1}) - \varphi(x, e) - \varphi(e, x) \end{aligned} \tag{3.8}$$

for all $x \in G$. From (3.6), (3.7) and (3.8), we get

$$\begin{aligned} & 4(f(x) + g(x) - f(e) - g(e)) - (f(x^2) + g(x^2) - f(e) - g(e)) \\ = \ & -[f(x^2) - 2f(x) - 2g(x) + g(e) + 2h(e) + 2k(e)] \\ & -[g(x^2) - 2f(x) - 2g(x) + f(e) + 2h(e) + 2k(e)] \\ & -2[f(e) + g(e) - 2h(e) - 2k(e)] \\ = \ & -\varphi(x, x) + 2\varphi(e, x) + 2\varphi(x, e) - \varphi(x, x^{-1}) - 2\varphi(e, e) \end{aligned}$$

for all $x \in G$. Induction argument implies

$$\begin{aligned} & f(x) + g(x) - f(e) - g(e) - \frac{f(x^{2^n}) + g(x^{2^n}) - f(e) - g(e)}{4^n} \\ = \ & \sum_{i=0}^{n-1} \frac{-\varphi(x^{2^i}, x^{2^i}) + 2\varphi(e, x^{2^i}) + 2\varphi(x^{2^i}, e) - \varphi(x^{2^i}, x^{-2^i}) - 2\varphi(e, e)}{4^{i+1}} \end{aligned} \tag{3.9}$$

for all $n \in N$ and $x \in G$. From (2.2) and the above equation, we can define $Q : G \to X$ by

$$2Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n}) + g(x^{2^n}) - f(e) - g(e)}{4^n}$$

for all $x \in G$. From (3.9) and the definition of $Q$, we have

$$\begin{aligned} & f(x) + g(x) - f(e) - g(e) - 2Q(x) \\ = \ & -\tilde{\varphi}(x, x) + 2\tilde{\varphi}(e, x) + 2\tilde{\varphi}(x, e) - \tilde{\varphi}(x, x^{-1}) - \frac{2}{3}\varphi(e, e) \end{aligned} \tag{3.10}$$

for all $x \in G$. Form (3.3), (3.5) and (3.10), we get

$$\begin{aligned} & 2f(x^2) - 2f(e) - 2Q(x^2) = f(x^2) + g(x^2) - f(e) - g(e) - 2Q(x^2) \\ & + [f(x^2) + g(e) - 2h(x) - 2k(x)] - [f(e) + g(x^2) - 2h(x) - 2k(x)] \\ = \ & 2M(x^2) - \frac{2}{3}\varphi(e, e) + \varphi(x, x) - \varphi(x, x^{-1}) \end{aligned}$$

and

$$\begin{aligned}
& 2g(x^2) - 2g(e) - 2Q(x^2) = f(x^2) + g(x^2) - f(e) - g(e) - 2Q(x^2) \\
& \quad - [f(x^2) + g(e) - 2h(x) - 2k(x)] + [f(e) + g(x^2) - 2h(x) - 2k(x)] \\
= \; & 2M(x^2) - \frac{2}{3}\varphi(e,e) - \varphi(x,x) + \varphi(x, x^{-1})
\end{aligned}$$

for all $x \in G$. From (3.2), (3.6) and (3.10), we get

$$\begin{aligned}
& 2h(x) - 2h(e) - 2Q(x) \\
= \; & f(x) + g(x) - f(e) - g(e) - 2Q(x) - [f(x) + g(x) - 2h(x) - 2k(e)] \\
& \quad + [f(e) + g(e) - 2k(e) - 2h(e)] = M(x) + \frac{1}{3}\varphi(e,e) - \varphi(x,e)
\end{aligned}$$

for all $x \in G$. From (3.4), (3.6) and (3.10), we get

$$\begin{aligned}
& 2k(x) - 2k(e) - 2Q(x) \\
= \; & f(x) + g(x) - f(e) - g(e) - 2Q(x) - [f(x) + g(x) - 2h(e) - 2k(x)] \\
& \quad + [f(e) + g(e) - 2k(e) - 2h(e)] \\
= \; & -\tilde{\varphi}(x,x) + 2\tilde{\varphi}(e,x) + 2\tilde{\varphi}(x,e) - \tilde{\varphi}(x,x^{-1}) + \frac{1}{3}\varphi(e,e) - \varphi(e,x)
\end{aligned}$$

for all $x \in G$. Replacing $x$ by $x^{2^n}$ and dividing by $4^n$ in (3.2), we have

$$\frac{f(x^{2^n}) + g(x^{2^n})}{4^n} - \frac{2h(x^{2^n}) + 2k(e)}{4^n} = \frac{\varphi(x^{2^n}, e)}{4^n} \tag{3.11}$$

for all $n \in N$ and $x \in G$. Taking the limit in (3.11), we have

$$Q(x) = \lim_{n\to\infty} \frac{h(x^{2^n})}{4^n}$$

for all $x \in G$. By the similar method, we obtain

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{g(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{k(x^{2^n})}{4^n}$$

for all $x \in G$. Replacing $x$ by $x^{2^n}$ and $y$ by $y^{2^n}$ and dividing $4^n$ on both sides, the equation (3.1) implies

$$\frac{f(x^{2^n} * y^{2^n})}{4^n} + \frac{g(x^{2^n} * y^{-2^n})}{4^n} - \frac{2h(x^{2^n})}{4^n} - \frac{2k(y^{2^n})}{4^n} = \frac{\varphi(x^{2^n}, y^{2^n})}{4^n} \quad (\forall x, y \in G).$$

Taking the limit in the above equation, we have

$$Q(x * y) + Q(x * y^{-1}) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in G$.

**Corollary 3.1** (even function)**.** *Let $\varphi$ be a mapping as in Corollary 2.1. Suppose that the even functions $f, g, h, k : V \to X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \le \varphi(x,y) \quad \textit{for all} \quad x, y \in V.$$

*Then there exists exactly one quadratic function $Q : V \to X$ such that*

$$\begin{aligned}
\|f(x)-f(0)-Q(x)\| &\leq M(x)+\frac{1}{2}[\varphi(\frac{x}{2},\frac{x}{2})+\varphi(\frac{x}{2},-\frac{x}{2})]+\frac{1}{3}\varphi(0,0),\\
\|g(x)-g(0)-Q(x)\| &\leq M(x)+\frac{1}{2}[\varphi(\frac{x}{2},\frac{x}{2})+\varphi(\frac{x}{2},-\frac{x}{2})]+\frac{1}{3}\varphi(0,0),\\
\|h(x)-h(0)-Q(x)\| &\leq M(x)+\frac{1}{2}\varphi(x,0)+\frac{1}{6}\varphi(0,0) \quad \textit{and}\\
\|k(x)-k(0)-Q(x)\| &\leq M(x)+\frac{1}{2}\varphi(0,x)+\frac{1}{6}\varphi(0,0)
\end{aligned}$$

*for all $x \in V$, where*

$$M(x)=\frac{1}{2}[\tilde{\varphi}(x,x)+2\tilde{\varphi}(0,x)+2\tilde{\varphi}(x,0)+\tilde{\varphi}(x,-x)].$$

*The function $Q$ is given by*

$$Q(x)=\lim_{n\to\infty}\frac{f(2^nx)}{4^n}=\lim_{n\to\infty}\frac{g(2^nx)}{4^n}=\lim_{n\to\infty}\frac{h(2^nx)}{4^n}=\lim_{n\to\infty}\frac{k(2^nx)}{4^n}$$

*for all $x \in V$.*

**Theorem 3.2** (odd function)**.** *Let $\varphi : G \times G \to X$ be a mapping satisfying the conditions*

$$\lim_{n\to\infty}\frac{\varphi(x^{2^n},y^{2^n})}{2^n}=0 \tag{3.12}$$

$$\hat{\varphi}(x^i,x^j):=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{2^k}\varphi(x^{i\cdot 2^k},x^{j\cdot 2^k})\in X \tag{3.13}$$

*for all $x, y \in G$ and for any fixed $i, j = 0, 1, 2, 3, \cdots$. Suppose that the odd functions $f, g, h, k : G \to X$ satisfy*

$$f(x*y)+g(x*y^{-1})-2h(x)-2k(y)=\varphi(x,y) \tag{3.14}$$

*and the condition* (2.4) *for all $x, y \in G$. Then the limits $T(x) = \lim_{n\to\infty} f(x^{2^n})/2^n$ and $T'(x) = \lim_{n\to\infty} g(x^{2^n})/2^n$ exist for any $x \in G$, and $T, T'$ satisfy the equation*

$$T(x*y)+T(y*x)=2T(x)+2T(y) \tag{3.15}$$

*for all $x, y \in G$. In this case the equations*

$$f(x)-T(x) = \frac{1}{2}[-\hat{\varphi}(x,x)+\hat{\varphi}(x,e)+\hat{\varphi}(e,x)] \tag{3.16}$$

$$g(x)-T'(x) = \frac{1}{2}[-\hat{\varphi}(x,x^{-1})+\hat{\varphi}(x,e)-\hat{\varphi}(e,x)] \tag{3.17}$$

$$h(x)-\frac{T(x)+T'(x)}{2} = \frac{1}{4}[-\hat{\varphi}(x,x)-\hat{\varphi}(x,x^{-1})+2\hat{\varphi}(x^2,e)] \tag{3.18}$$

$$k(x)-\frac{T(x)-T'(x)}{2} = \frac{1}{4}[-\hat{\varphi}(x,x)+\hat{\varphi}(x,x^{-1})+2\hat{\varphi}(e,x^2)] \tag{3.19}$$

*hold for all $x \in G$.*

*Proof.* From (3.14), we can easily obtain

$$f(x*y^{-1})+g(x*y)-2h(x)+2k(y)=\varphi(x,y^{-1}), \tag{3.20}$$
$$f(y*x^{-1})+g(y*x)-2h(y)+2k(x)=\varphi(y,x^{-1}), \tag{3.21}$$
$$f(y*x)-g(x*y^{-1})-2h(y)-2k(x)=\varphi(y,x) \tag{3.22}$$
$$f(x^2)-2h(x)-2k(x)=\varphi(x,x), \tag{3.23}$$
$$f(x)+g(x)-2h(x)=\varphi(x,e), \tag{3.24}$$
$$f(x)-g(x)-2k(x)=\varphi(e,x), \tag{3.25}$$
$$g(x^2)-2h(x)+2k(x)=\varphi(x,x^{-1}) \tag{3.26}$$

for all $x, y \in G$. From (3.23), (3.24), (3.25) and (3.26), we obtain the equations

$$\begin{aligned} f(x)-\frac{f(x^2)}{2} &= \frac{-\varphi(x,x)+\varphi(x,e)+\varphi(e,x)}{2} \\ g(x)-\frac{g(x^2)}{2} &= \frac{-\varphi(x,x^{-1})+\varphi(x,e)-\varphi(e,x)}{2} \end{aligned}$$

for all $x \in G$. Applying the similar method as in the Theorem 3.1 to the above equations, we easily see that the limits $T(x)=\lim_{n\to\infty} f(x^{2^n})/2^n$ and $T'(x)=\lim_{n\to\infty} g(x^{2^n})/2^n$ exist for all $x \in G$ and the equations (3.16) and (3.17) hold for all $x \in G$. From (3.24), (3.25) and the definition of $T$ and $T'$, the limits

$$\lim_{n\to\infty}\frac{h(x^{2^n})}{2^n}=\frac{T(x)+T'(x)}{2}, \quad \lim_{n\to\infty}\frac{k(x^{2^n})}{2^n}=\frac{T(x)-T'(x)}{2}$$

exist for all $x \in G$. From (3.16), (3.17), (3.24) and (3.25), we easily see that the equations (3.18) and (3.19) hold for all $x \in G$. From (3.14) and (3.22), we obtain

$$f(x*y)+f(y*x)-2h(x)-2h(y)-2k(x)-2k(y)=\varphi(x,y)+\varphi(y,x) \tag{3.27}$$

for all $x, y \in G$. From (3.20) and (3.21), we obtain

$$g(x*y)+g(y*x)-2h(x)-2h(y)+2k(x)+2k(y)=\varphi(x,y^{-1})+\varphi(y,x^{-1}) \tag{3.28}$$

for all $x, y \in G$. Replacing $x, y$ by $x^{2^n}, y^{2^n}$ and dividing $2^n$ on both sides in the equation (3.27) and (3.28) implies

$$\begin{aligned} &f(x^{2^n}*y^{2^n})+f(y^{2^n}*x^{2^n})-2h(x^{2^n})-2h(y^{2^n}) \\ &\quad -2k(x^{2^n})-2k(y^{2^n})=\varphi(x^{2^n},y^{2^n})+\varphi(y^{2^n},x^{2^n}), \\ &g(x^{2^n}*y^{2^n})+g(y^{2^n}*x^{2^n})-2h(x^{2^n})-2h(y^{2^n}) \\ &\quad +2k(x^{2^n})+2k(y^{2^n})=\varphi(x^{2^n},(y^{-1})^{2^n})+\varphi(y^{2^n},(x^{-1})^{2^n}) \end{aligned}$$

for all $x, y \in G$. Taking the limit in the above equations as $n \to \infty$, we see that $T$ and $T'$ satisfy the equation (3.15).

**Corollary 3.2** (odd function)**.** *Let $\varphi : V \times V \to [0,\infty)$ be a mapping such that*

$$\hat{\varphi}(x,y) := \sum_{i=0}^{\infty}\frac{1}{2^i}\varphi(2^i x, 2^i y) < \infty$$

*for all $x, y \in V$. Suppose that the odd functions $f, g, h, k : V \to X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \le \varphi(x,y) \quad \text{for all} \quad x, y \in V.$$

*Then there exist two unique additive functions $T, T' : V \to X$ such that*

$$\begin{aligned}
\|f(x) - T(x)\| &\le \frac{1}{2}[\hat{\varphi}(x,x) + \hat{\varphi}(x,0) + \hat{\varphi}(0,x)] \\
\|g(x) - T'(x)\| &\le \frac{1}{2}[\hat{\varphi}(x,-x) + \hat{\varphi}(x,0) + \hat{\varphi}(0,x)] \\
\|h(x) - \frac{T(x)+T'(x)}{2}\| &\le \frac{1}{4}[\hat{\varphi}(x,x) + \hat{\varphi}(x,-x) + 2\hat{\varphi}(2x,0)] \\
\|k(x) - \frac{T(x)-T'(x)}{2}\| &\le \frac{1}{4}[\hat{\varphi}(x,x) + \hat{\varphi}(x,-x) + 2\hat{\varphi}(0,2x)]
\end{aligned}$$

*for all $x \in V$.*

Now we prove the stability of the general Pexiderized quadratic equation.

From Theorem 3.1 and Theorem 3.2, we can easily obtain the following theorem.

**Theorem 3.3.** *Let $\varphi_1, \varphi_2 : G \times G \to X$ be mappings satisfying the conditions* (2.1), (2.2) *in Theorem 2.1 and the conditions* (3.12), (3.13) *in Theorem 3.2. Suppose that the functions $f, g, h, k : G \to X$ satisfy*

$$f(x*y) + g(x*y^{-1}) - 2h(x) - 2k(y) = \varphi_1(x,y), \tag{3.29}$$

$$f(y^{-1}*x^{-1}) + g(y*x^{-1}) - 2h(x^{-1}) - 2k(y^{-1}) = \varphi_2(x,y) \tag{3.30}$$

*for all $x, y \in G$. Then there exist exactly one quadratic function $Q : G \to X$ and the two limits $T(x) = \lim_{n\to\infty}[f(x^{2^n}) - f(x^{-2^n})]/(2\cdot 2^n)$, $T'(x) = \lim_{n\to\infty}[g(x^{2^n}) - g(x^{-2^n})]/(2\cdot 2^n)$ satisfying* (3.15) *for all $x \in G$. The equations*

$$\begin{aligned}
&(3.31)\ \ f(x^2) - f(e) - Q(x^2) - 2T(x) \\
&= M(x^2) + \frac{1}{2}[\varphi(x,x) - \varphi(x,x^{-1})] - \frac{1}{3}\varphi(e,e) + \frac{1}{2}[-\hat{\psi}(x^2,x^2) + \hat{\psi}(x^2,e) + \hat{\psi}(e,x^2)], \\
&\quad g(x^2) - g(e) - Q(x^2) - 2T'(x) \\
&= M(x^2) - \frac{1}{2}[\varphi(x,x) - \varphi(x,x^{-1})] - \frac{1}{3}\varphi(e,e) + \frac{1}{2}[-\hat{\psi}(x^2,x^{-2}) + \hat{\psi}(x^2,e) - \hat{\psi}(e,x^2)], \\
&\quad h(x) - h(e) - Q(x) - \frac{T(x)+T'(x)}{2} \\
&= M(x) + \frac{1}{6}\varphi(e,e) - \frac{1}{2}\varphi(x,e) + \frac{1}{4}[-\hat{\psi}(x,x) - \hat{\psi}(x,x^{-1}) + 2\hat{\psi}(x^2,e)], \\
&\quad k(x) - k(e) - Q(x) - \frac{T(x)-T'(x)}{2} \\
&= M(x) + \frac{1}{6}\varphi(e,e) - \frac{1}{2}\varphi(e,x) + \frac{1}{4}[-\hat{\psi}(x,x) + \hat{\psi}(x,x^{-1}) + 2\hat{\psi}(e,x^2)]
\end{aligned}$$

*hold for all $x \in G$, where*

$$\begin{aligned}
M(x) &= \frac{1}{2}[-\tilde{\varphi}(x,x) + 2\tilde{\varphi}(e,x) + 2\tilde{\varphi}(x,e) - \tilde{\varphi}(x,x^{-1})], \\
\varphi(x,y) &= (\varphi_1(x,y) + \varphi_2(x,y))/2, \\
\psi(x,y) &= (\varphi_1(x,y) - \varphi_2(x,y))/2.
\end{aligned}$$

*The function $Q$ is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{g(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{h(x^{2^n})}{4^n} = \lim_{n\to\infty} \frac{k(x^{2^n})}{4^n}$$

*for all $x \in G$.*

*Proof.* Let $f_e, g_e, h_e, k_e$ be even parts and $f_o, g_o, h_o, k_o$ be odd parts of $f, g, h, k$, respectively. From (3.29) and (3.30), we get

$$\begin{aligned} & f_e(x*y) + g_e(x*y^{-1}) - 2h_e(x) - 2k_e(y) \\ = & \frac{1}{2}[f(x*y) + g(x*y^{-1}) - 2h(x) - 2k(y)] \\ & + \frac{1}{2}[f(y^{-1}*x^{-1}) + g(y*x^{-1}) - 2h(x^{-1}) - 2k(y^{-1})] \\ = & \frac{\varphi_1(x,y) + \varphi_2(x,y)}{2} = \varphi(x,y) \end{aligned}$$

for all $x, y \in G$. By Theorem 3.1, there exists exactly one quadratic function $Q : G \to X$ such that

$$f_e(x^2) - f(e) - Q(x) = M(x^2) + \frac{1}{2}[\varphi(x,x) - \varphi(x,x^{-1})] - \frac{1}{3}\varphi(e,e) \tag{3.32}$$

for all $x \in G$, where

$$Q(x) = \lim_{n\to\infty} [f(x^{2^n}) + f(x^{-2^n})]/(2 \cdot 4^n).$$

From (3.29) and (3.30), we get

$$f_o(x*y) + g_o(x*y^{-1}) - 2h_o(x) - 2k_o(y) = \frac{\varphi_1(x,y) - \varphi_2(x,y)}{2} = \psi(x,y)$$

for all $x \in G$. By Theorem 3.2, the limits $T(x) = \lim_{n\to\infty}[f(x^{2^n}) - f(x^{-2^n})]/(2 \cdot 2^n)$, $T'(x) = \lim_{n\to\infty}[g(x^{2^n}) - g(x^{-2^n})]/(2 \cdot 2^n)$ exist for all $x \in G$. And the two functions $T, T' : G \to X$ satisfy (3.15) and

$$f_o(x^2) - 2T(x) = \frac{1}{2}[-\hat{\psi}(x^2,x^2) + \hat{\psi}(x^2,e) + \hat{\psi}(e,x^2)] \tag{3.33}$$

for all $x \in G \setminus \{e\}$. From (3.32), (3.33) and the equation

$$f(x^2) - f(e) - 4Q(x) - 2T(x) = f_e(x^2) - f(e) - Q(x^2) + f_o(x^2) - T(x^2),$$

we get (3.31). The equation

$$\begin{aligned} \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n} &= \lim_{n\to\infty} \frac{f(x^{2^n}) + f(x^{-2^n})}{2 \cdot 4^n} + \lim_{n\to\infty} \frac{f(x^{2^n}) - f(x^{-2^n})}{2 \cdot 4^n} \\ &= \lim_{n\to\infty} \frac{f(x^{2^n}) + f(x^{-2^n})}{2 \cdot 4^n} + \lim_{n\to\infty} \frac{1}{2^n} \lim_{n\to\infty} \frac{f(x^{2^n}) - f(x^{-2^n})}{2 \cdot 2^n} \\ &= Q(x) + 0 \cdot T(x) \end{aligned}$$

holds for all $x \in G$. By the similar method, we obtain the remaining results.

**Corollary 3.3.** *Let $\varphi : V \times V \to X$ be a mapping satisfying the conditions in Corollary 3.1 and Corollary 3.2. Suppose that the functions $f, g, h, k : V \to X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \le \varphi(x,y) \quad \textit{for all} \quad x, y \in V. \tag{3.34}$$

*Then there exist exactly one quadratic function $Q : V \to X$ and two unique additive functions $T, T' : V \to X$ such that*

$$\begin{aligned}
&\|f(x) - f(0) - Q(x) - T(x)\| \\
\le\ & M(x) + \frac{1}{2}[\psi(\frac{x}{2}, \frac{x}{2}) + \psi(\frac{x}{2}, -\frac{x}{2})] + \frac{1}{2}[\hat{\psi}(x, x) + \hat{\psi}(x, 0) + \hat{\psi}(0, x)] + \frac{1}{3}\psi(0, 0), \\
&\|g(x) - g(0) - Q(x) - T'(x)| \\
\le\ & M(x) + \frac{1}{2}[\psi(\frac{x}{2}, \frac{x}{2}) + \psi(\frac{x}{2}, -\frac{x}{2})] + \frac{1}{2}[\hat{\psi}(x, -x) + \hat{\psi}(x, 0) + \hat{\psi}(0, x)] + \frac{1}{3}\psi(0, 0), \\
&\|h(x) - h(0) - Q(x) - \frac{1}{2}(T(x) + T'(x))\| \\
\le\ & M(x) + \frac{1}{2}\psi(x, 0) + \frac{1}{6}\psi(0, 0) + \frac{1}{4}[\hat{\psi}(x, x) + \hat{\psi}(x, -x) + 2\hat{\psi}(2x, 0)], \quad \textit{and} \\
&\|k(x) - k(0) - Q(x) - \frac{1}{2}(T(x) - T'(x))\| \\
\le\ & M(x) + \frac{1}{2}\psi(0, x) + \frac{1}{6}\psi(0, 0) + \frac{1}{4}[\hat{\psi}(x, x) + \hat{\psi}(x, -x) + 2\hat{\psi}(0, 2x)]
\end{aligned}$$

*for all $x \in V$, where $\psi(x, y) = (\varphi(x, y) + \varphi(-x, -y))/2$ and*

$$M(x) = \frac{1}{2}[\tilde{\psi}(x, x) + 2\tilde{\psi}(0, x) + 2\tilde{\psi}(x, 0) + \tilde{\psi}(x, -x)].$$

*The function $Q$ is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n} = \lim_{n\to\infty} \frac{g(2^n x)}{4^n} = \lim_{n\to\infty} \frac{h(2^n x)}{4^n} = \lim_{n\to\infty} \frac{k(2^n x)}{4^n}$$

*and the functions $T, T'$ are given by*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}, T'(x) = \lim_{n\to\infty} \frac{g(2^n x) - g(-2^n x)}{2^{n+1}}.$$

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