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Stability of Iterative Sequences Approximating Common Fixed Point for a System of Asymptotically Quasi-nonexpansive Type Mappings

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Abstract. In this paper, we introduce the concept of a system of asymptotically quasi-nonexpansive type mappings. Furthermore, we define a k-step iterative sequence approximating common fixed point for a system of asymptotically quasi-nonexpansive type mappings and study its stability in real Banach spaces.

1. Introduction

Let X be an arbitrary real Banach space and C be a nonempty close and convex subset of X. Let $T : C \to C$ be a mapping. Suppose that, for any $x_0 \in X$,

$$x_{n+1} = f(T, x_n) \tag{1.1}$$

yields a sequence of points $\{x_n\}$ in C, where f denotes the iterative process involving T and x_n. Suppose that $F(T) = \{x \in C : Tx = x\} \neq \emptyset$ and $\{x_n\}$ converges strongly to $x^* \in F(T)$. Let $\{y_n\}$ be a sequence in C and $\{\epsilon_n\}$ be a sequence in $[0, \infty)$ defined by

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|.$$

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If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iterative process defined by (1.1) is said to be T-stable or stable with respect to T (see, for example, [6]-[8], [14], [15], [17], [18] and the references therein).

We say that the iterative process $\{x_n\}$ defined by (1.1) is almost T-stable or almost stable with respect to T if $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = x^*$ (see [15]). It is easy to see that an iterative process $\{x_n\}$ which is T-stable is almost T-stable. The example in [15] showed that an iterative process which is almost T-stable need not be T-stable.

Stability results for several iterative processes for some kinds of nonlinear mappings have been shown in recent papers by many authors (see, for example, [1], [2], [6]-[8], [14], [15], [17], [18] and the references therein). Harder and Hicks [8] showed how such sequences $\{y_n\}$ could arise in practice and demonstrated the importance of investigating the stability of various iterative processes for various kinds of nonlinear mappings.

The concepts of quasi-nonexpansive mapping was initated by Tricomi in 1941 for real functions. The concepts of asymptotically nonexpansive mapping and the asymptotically nonexpansive type mapping were introduced by Goebel and Kirk [5] and Kirk [10], respectively, which are closely related to the theory of fixed points in Banach spaces. Recently, the iterative approximating problem of fixed points for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings has been studied by many authors (see, for example, [3], [4], [9], [11]-[13], [16], [19]-[22] and the references therein).

In this paper, we introduce the concept of a system of asymptotically quasi-nonexpansive type mappings. Furthermore, we define a k-step iterative sequence approximating common fixed point for a system of asymptotically quasi-nonexpansive type mappings and study its stability in real Banach spaces. Our results extend, improve and unify the corresponding results of [3], [4], [9], [11]-[13], [16] and [19]-[22].

2. Preliminaries

Definition 2.1. Let X be a real Banach space and C be a nonempty close and convex subset of X. Let $T : C \to C$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Fx = x\}$.

(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C;$$

(2) T is said to be quasi-nonexpansive if, $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \le \|x - x^*\|, \quad \forall x \in C,\ x^* \in F(T);$$

(3) T is said to be asymptotically nonexpansive [5] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C,\ n \ge 0;$$

(4) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - x^*\| \le k_n \|x - x^*\|, \quad \forall x \in C,\ x^* \in F(T),\ n \ge 0;$$

(5) T is said to be asymptotically nonexpansive type [10] if

$$\limsup_{n\to\infty} \sup_{x\in C}\{\|T^n x - T^n y\| - \|x-y\|\} \le 0, \quad \forall y \in C;$$

(6) T is said to be asymptotically quasi-nonexpansive type if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{x\in C}\{\|T^n x - x^*\| - \|x-x^*\|\} \le 0, \quad \forall x^* \in F(T).$$

Remark 2.1. It is easy to see that the following relations hold:

$$\begin{array}{ccc}
(1) & \overset{F(T)\neq\emptyset}{\Longrightarrow} & (2) \\
\Downarrow & & \Downarrow \\
(3) & \overset{F(T)\neq\emptyset}{\Longrightarrow} & (4) \\
\Downarrow & & \Downarrow \\
(5) & \overset{F(T)\neq\emptyset}{\Longrightarrow} & (6).
\end{array}$$

Throughout this paper, let X be a real Banach space, C a nonempty close and convex subset of X. Let $T_1, T_2, \cdots, T_k : C \to C$ be mappings, $F(T_i)$ the set of fixed points of T_i, where k is a given positive integer. Let m and n be the nonnegative integers.

Definition 2.2. Let X be a real Banach space, C a nonempty close and convex subset of X. $T_1, T_2, \cdots, T_k : C \to C$ are said to be a system asymptotically quasi-nonexpansive type mappings if $S = \cap_{i=1}^k F(T_i) \neq \emptyset$ and, for each $i \in \{1, 2, \cdots, k\}$,

$$\limsup_{n\to\infty} \sup_{x\in C}\{\|T_i^n x - x^*\| - \|x-x^*\|\} \le 0, \quad \forall x^* \in S.$$

Remark 2.2. It is easy to see that the concept of a system asymptotically quasi-nonexpansive type mappings defined by Definition 2.2 reduces to that of asymptotically quasi-nonexpansive type mappings defined by Definition 2.1 (6) when $T_1 = T_2 = \cdots = T_k$.

In our main results, we need the following lemma.

Lemma 2.1 ([21]). *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge n_0,$$

where $\sum_{n=0}^\infty b_n < \infty$ and n_0 is some positive integer. Then the $\lim_{n\to\infty} a_n$ exists.

3. The main results

Theorem 3.1. *Let X be a real Banach space and C be a nonempty close and convex subset of X. Let $T_1, T_2, \cdots, T_k : C \to C$ be a system of asymptotically quasi-nonexpansive type mappings defined by Definition 2.2. Assume that, for each $i \in \{1, 2, \cdots, k\}$, there exist constants L_i and $\alpha_i' > 0$ such that*

$$\|T_i x - y^*\| \le L_i \|x - y^*\|^{\alpha_i'}, \quad \forall x \in C,\ y^* \in S = \cap_{i=1}^k F(T_i). \tag{3.1}$$

For any given $x_0 \in C$, Define the k-step iterative sequence $\{x_n\}$ by

$$
(3.2)\qquad \begin{cases} z_{k-1,n} = (1-\alpha_{k,n})x_n + \alpha_{k,n}T_k^n x_n, \quad n \ge 0, \\ z_{k-2,n} = (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n}T_{k-1}^n z_{k-1,n}, \quad n \ge 0, \\ \cdots\cdots\cdots \\ z_{1,n} = (1-\alpha_{2,n})x_n + \alpha_{2,n}T_2^n z_{2,n}, \quad n \ge 0, \\ x_{n+1} = (1-\alpha_{1,n})x_n + \alpha_{1,n}T_1^n z_{1,n}, \quad n \ge 0, \end{cases}
$$

where $\{\alpha_{i,n}\}$ is a sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_{1,n} < \infty$ for each $i \in \{1,2,\cdots,k\}$. Suppose that $\{y_n\}$ is a sequence in C and define a sequence $\{\epsilon_n\}$ of positive real numbers by

$$
(3.3)\qquad \begin{cases} w_{k-1,n} = (1-\alpha_{k,n})y_n + \alpha_{k,n}T_k^n y_n, \quad n \ge 0, \\ w_{k-2,n} = (1-\alpha_{k-1,n})y_n + \alpha_{k-1,n}T_{k-1}^n w_{k-1,n}, \quad n \ge 0, \\ \cdots\cdots\cdots \\ w_{1,n} = (1-\alpha_{2,n})y_n + \alpha_{2,n}T_2^n w_{2,n}, \quad n \ge 0, \\ \epsilon_n = \|y_{n+1} - (1-\alpha_{1,n})y_n - \alpha_{1,n}T_1^n w_{1,n}\|, \quad n \ge 0. \end{cases}
$$

Then we have the following:

(i) $\liminf_{n\to\infty} d(x_n, S) = 0$ *if and only if $\{x_n\}$ converges strongly to a common fixed point x^* of $T_1, T_2, \cdots, T_k$ in C, where $S = \cap_{i=1}^k F(T_i)$ and $d(x_n, S)$ denotes the distance from x_n to the set S, i.e., $d(x_n, S) = \inf_{y^* \in S} \|x_n - y^*\|$.*

(ii) $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *and* $\liminf_{n\to\infty} d(y_n, S) = 0$ *imply that $\{y_n\}$ converges strongly to a common fixed point x^* of $T_1, T_2, \cdots, T_k$ in C.*

(iii) *If $\{y_n\}$ converges strongly to a common fixed point x^* of $T_1, T_2, \cdots, T_k$ in C, then* $\lim_{n\to\infty} \epsilon_n = 0$.

In order to prove Theorem 3.1, we first give the following proposition:

Proposition 3.1. *Assume that all the assumptions in Theorem 3.1 hold and $\sum_{n=0}^{\infty} \epsilon_n < \infty$, then, for any given $\epsilon > 0$, there exists a positive integer n_0 such that*

(1) $\|y_{n+1} - y^*\| \le \|y_n - y^*\| + \epsilon\gamma_{k,n} + \epsilon_n, \quad \forall y^* \in S,\ n \ge n_0,$ *where* $\gamma_{k,n} = \alpha_{1,n} + \alpha_{1,n}\alpha_{2,n} + \cdots + \alpha_{1,n}\alpha_{2,n}\cdots\alpha_{k,n}$;

(2) $\|y_m - y^*\| \le \|y_n - y^*\| + \epsilon\sum_{j=n}^{m-1}\gamma_{k,j} + \sum_{j=n}^{m-1}\epsilon_j, \quad \forall y^* \in S,\ n \ge n_0,\ m > n$;

(3) $\lim_{n\to\infty} d(y_n, S)$ *exists.*

Proof. Take any $y^* \in S$, it follows from (3.3) that

$$
(3.4)\qquad \begin{aligned} & \|y_{n+1} - y^*\| \\ \le\ & \epsilon_n + \|(1-\alpha_{1,n})(y_n - y^*) + \alpha_{1,n}(T_1^n w_{1,n} - y^*)\| \\ \le\ & (1-\alpha_{1,n})\|y_n - y^*\| + \alpha_{1,n}(\|T_1^n w_{1,n} - y^*\| - \|w_{1,n} - y^*\|) \\ & +\alpha_{1,n}\|w_{1,n} - y^*\| + \epsilon_n \end{aligned}
$$

and

$$
\begin{aligned} & \|w_{1,n} - y^*\| \\ =\ & \|(1-\alpha_{2,n})(y_n - y^*) + \alpha_{2,n}(T_2^n w_{2,n} - y^*)\| \\ \le\ & (1-\alpha_{2,n})\|y_n - y^*\| + \alpha_{2,n}(\|T_2^n w_{2,n} - y^*\| - \|w_{2,n} - y^*\|) + \alpha_{2,n}\|w_{2,n} - y^*\|. \end{aligned}
$$

Continuing in this way, we can deduce that

$$
\begin{aligned}
(3.5) \qquad & \|w_{i,n} - y^*\| \\
= \;& \|(1-\alpha_{i+1,n})(y_n - y^*) + \alpha_{i+1,n}(T_{i+1}^n w_{i+1,n} - y^*)\| \\
\le \;& (1-\alpha_{i+1,n})\|y_n - y^*\| + \alpha_{i+1,n}(\|T_{i+1}^n w_{i+1,n} - y^*\| - \|w_{i+1,n} - y^*\|) \\
& + \alpha_{i+1,n}\|w_{i+1,n} - y^*\|, \quad 1 \le i \le k-2,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6) \qquad & \|w_{k-1,n} - y^*\| \\
= \;& \|(1-\alpha_{k,n})(y_n - y^*) + \alpha_{k,n}(T_k^n y_n - y^*)\| \\
\le \;& (1-\alpha_{k,n})\|y_n - y^*\| + \alpha_{k,n}(\|T_k^n y_n - y^*\| - \|y_n - y^*\|) + \alpha_{k,n}\|y_n - y^*\|.
\end{aligned}
$$

Since $T_1, T_2, \cdots, T_k : C \to C$ are a system of asymptotically quasi-nonexpansive type mappings, from definition Definition 2.2, we obtain, for each $i \in \{1, 2, \cdots, k\}$,

$$\limsup_{n\to\infty} \sup_{x\in C}\{\|T_i^n x - y^*\| - \|x - y^*\|\} \le 0,$$

which implies that, for any given $\epsilon > 0$, there exists a positive integer $n_{i,0}$ such that

$$\sup_{x\in C}\{\|T_i^n x - y^*\| - \|x - y^*\|\} < \epsilon, \quad \forall n \ge n_{i,0}.$$

Set $n_0 = \max_{1\le i\le k}\{n_{i,0}\}$. Then it follows that, for any $i \in \{1, 2, \cdots, k\}$,

$$(3.7) \qquad \sup_{x\in C}\{\|T_i^n x - y^*\| - \|x - y^*\|\} < \epsilon, \quad \forall n \ge n_0.$$

Since $\{w_{i,n}\} \subset C$, it follows from (3.7) that

$$(3.8) \qquad \|T_i^n w_{i,n} - y^*\| - \|w_{i,n} - y^*\| < \epsilon$$

for all $n \ge n_0$ and $i \in \{1, 2, \cdots, k-1\}$. Again, since $\{y_n\} \subset C$, (3.7) implies that

$$(3.9) \qquad \|T_k^n y_n - y^*\| - \|y_n - y^*\| < \epsilon, \quad \forall n \ge n_0.$$

Substituting (3.5), (3.6), (3.8) and (3.9) into (3.4), for any $y^* \in S$ and $n \ge n_0$, we have

$$(3.10) \qquad \|y_{n+1} - y^*\| \le \|y_n - y^*\| + \epsilon_n + \epsilon\{\alpha_{1,n} + \alpha_{1,n}\alpha_{2,n} + \cdots + \alpha_{1,n}\alpha_{2,n}\cdots\alpha_{k,n}\}.$$

Set $\gamma_{k,n} = \alpha_{1,n} + \alpha_{1,n}\alpha_{2,n} + \cdots + \alpha_{1,n}\alpha_{2,n}\cdots\alpha_{k,n}$. It follows from (3.10) that

$$\|y_{n+1} - y^*\| \le \|y_n - y^*\| + \epsilon\gamma_{k,n} + \epsilon_n, \quad \forall y^* \in S,\ n \ge n_0.$$

Hence the conclusion (1) holds. From the conclusion (1), we have

$$
\begin{aligned}
\|y_m - y^*\| \;&\le\; \|y_{m-1} - y^*\| + \epsilon\gamma_{k,m-1} + \epsilon_{m-1} \\
&\le\; \|y_{m-2} - y^*\| + \epsilon\gamma_{k,m-2} + \epsilon\gamma_{k,m-1} + \epsilon_{m-2} + \epsilon_{m-1} \\
&\le\; \cdots \\
&\le\; \|y_n - y^*\| + \epsilon\sum_{j=n}^{m-1}\gamma_{k,j} + \sum_{j=n}^{m-1}\epsilon_j, \quad \forall y^* \in S,\ m > n,\ n \ge n_0.
\end{aligned}
$$

Thus the conclusion (2) holds. Again, it follows from the conclusion (1) that

$$d(y_{n+1}, S) \le d(y_n, S) + \epsilon\gamma_{k,n} + \epsilon_n, \quad \forall n \ge n_0.$$

Since

$$\sum_{n=0}^{\infty} \alpha_{1,n} < \infty, \quad \sum_{n=0}^{\infty} \epsilon_n < \infty,$$

we have

$$\sum_{n=0}^{\infty} (\epsilon\gamma_{k,n} + \epsilon_n) \le k\epsilon \sum_{n=0}^{\infty} \alpha_{1,n} + \sum_{n=0}^{\infty} \epsilon_n < \infty.$$

Therefore, Lemma 2.1 implies that the conclusion (3) holds. This completes the proof.

The proof of Theorem 3.1. It is easy to see that the sufficiency of the conclusion (i) is obvious and the necessity follows from the conclusion (ii) by setting $\epsilon_n = 0$ in (3.3) for $n \ge 0$ and considering (3.2).

Now, we prove the conclusion (ii) holds. It follows from Proposition 3.1 (3) that $\lim_{n\to\infty} d(y_n, S)$ exists. Since $\liminf_{n\to\infty} d(y_n, S) = 0$, we have

$$\lim_{n\to\infty} d(y_n, S) = 0. \tag{3.11}$$

First, we prove that $\{y_n\}$ is a Cauchy sequence in C. In fact, from (3.11), the assumptions $\sum_{n=0}^{\infty} \alpha_{1,n} < \infty$ and $\sum_{n=0}^{\infty} \epsilon_n < \infty$ imply that, for any given $\epsilon > 0$, there exists a positive integer $n_1 \ge n_0$ (where n_0 is the positive integer appeared in Proposition 3.1) such that

$$d(y_n, S) < \epsilon, \quad \forall n \ge n_1, \tag{3.12}$$

$$\sum_{n=n_1}^{\infty} \gamma_{k,n} \le k \sum_{n=n_1}^{\infty} \alpha_{1,n} < \epsilon \tag{3.13}$$

and

$$\sum_{n=n_1}^{\infty} \epsilon_n < \epsilon, \tag{3.14}$$

where $\gamma_{k,n}$ is the same as in Proposition 3.1. By the definition of infimum, it follows from (3.12) that, for any given $n \ge n_1$, there exists $y^*(n) \in S$ such that

$$\|y_n - y^*(n)\| < 2\epsilon. \tag{3.15}$$

On the other hand, for any $m, n \ge n_1$ and $m > n$, it follows from Proposition 3.1 (2) that

$$\begin{aligned} \|y_m - y_n\| &\le \|y_m - y^*(n)\| + \|y_n - y^*(n)\| \\ &\le 2\|y_n - y^*(n)\| + \epsilon \sum_{j=n}^{m-1} \gamma_{k,j} + \sum_{j=n}^{m-1} \epsilon_j. \end{aligned} \tag{3.16}$$

From (3.13)-(3.16), for any $m, n \ge n_1$ and $m > n$, we have

$$\|y_m - y_n\| \le 4\epsilon + \epsilon^2 + \epsilon = \epsilon(5 + \epsilon),$$

which implies that $\{y_n\}$ is a Cauchy sequence in C. Since X is complete and C is closed, C is complete. Then there exists $x^* \in C$ such that $y_n \to x^*$ as $n \to \infty$.

Now, we prove that x^* is a common fixed point of $T_1, T_2, \cdots, T_k$ in C. Since $y_n \to x^*$ and $d(y_n, S) \to 0$ as $n \to \infty$, then, for any given $\epsilon > 0$, there exists a positive integer $n_2 \geq n_1 \geq n_0$ such that

$$\|y_n - x^*\| < \epsilon, \quad d(y_n, S) < \epsilon, \quad \forall n \geq n_2. \tag{3.17}$$

The second inequality in (3.17) implies that there exists $y_1^* \in S$ such that

$$\|y_{n_2} - y_1^*\| < 2\epsilon. \tag{3.18}$$

Moreover, it follows from (3.7) that, for any $i \in \{1, 2, \cdots, k\}$,

$$\|T_i^n x^* - y_1^*\| - \|x^* - y_1^*\| < \epsilon, \quad \forall n \geq n_2. \tag{3.19}$$

Thus, from (3.17)-(3.19), for any $i \in \{1, 2, \cdots, k\}$ and $n \geq n_2$, we have

$$\begin{aligned}
\|T_i^n x^* - x^*\| &\leq \{\|T_i^n x^* - y_1^*\| - \|x^* - y_1^*\|\} + 2\|x^* - y_1^*\| \\
&\leq \epsilon + 2\{\|x^* - y_{n_2}\| + \|y_1^* - y_{n_2}\|\} \\
&\leq \epsilon + 2(\epsilon + 2\epsilon) = 7\epsilon,
\end{aligned}$$

which implies that $T_i^n x^* \to x^*$ as $n \to \infty$. Again, since

$$\|T_i^n x^* - T_i x^*\| \leq \{\|T_i^n x^* - y_1^*\| - \|x^* - y_1^*\|\} + \|x^* - y_1^*\| + \|T_i x^* - y_1^*\|$$

for all $1 \leq i \leq k$ and $n \geq n_2$, (3.1) and (3.17)-(3.19) imply

$$\begin{aligned}
&\|T_i^n x^* - T_i x^*\| \\
\leq\ & \epsilon + \|x^* - y_1^*\| + L_i \|x^* - y_1^*\|^{\alpha_i'} \\
\leq\ & \epsilon + \|x^* - y_{n_2}\| + \|y_1^* - y_{n_2}\| + L_i\{\|x^* - y_{n_2}\| + \|y_1^* - y_{n_2}\|\}^{\alpha_i'} \\
<\ & \epsilon + 3\epsilon + L_i(3\epsilon)^{\alpha_i'},
\end{aligned}$$

which shows that $T_i^n x^* \to T_i x^*$ as $n \to \infty$. By the uniqueness of the limit, we have $T_i x^* = x^*$, that is, x^* is a fixed point of $T_1, T_2, \cdots, T_k$ in C. Therefore, the conclusion (ii) holds.

From (3.4)-(3.6), (3.8) and (3.9), we have, for any given $\epsilon > 0$,

$$\begin{aligned}
\epsilon_n &\leq \|y_{n+1} - x^*\| + \|(1 - \alpha_{1,n})(y_n - x^*) + \alpha_{1,n}(T_1^n w_{1,n} - x^*)\| \\
&\leq \|y_{n+1} - x^*\| + (1 - \alpha_{1,n})\|y_n - x^*\| + \alpha_{1,n}(\|T_1^n w_{1,n} - x^*\| - \|w_{1,n} - x^*\|) \\
&\quad + \alpha_{1,n}\|w_{1,n} - x^*\| \\
&\leq \cdots \\
&\leq \|y_{n+1} - x^*\| + \|y_n - x^*\| + \epsilon\{\alpha_{1,n} + \alpha_{1,n}\alpha_{2,n} + \cdots + \alpha_{1,n}\alpha_{2,n} \cdots \alpha_{k,n}\} \\
&= \|y_{n+1} - x^*\| + \|y_n - x^*\| + \epsilon\gamma_{k,n}, \quad \forall n \geq n_0.
\end{aligned}$$

Since $y_n \to x^*$ and $\sum_{n=0}^{\infty} \gamma_{k,n} \leq k \sum_{n=0}^{\infty} \alpha_{1,n} < \infty$, it follows that $\lim_{n\to\infty} \epsilon_n = 0$. Thus the conclusion (iii) holds. This completes the proof.

If take $T_1 = T_2 = \cdots = T_k$ in Theorem 3.1, then we obtain the following conclusion:

Theorem 3.2. *Let X be a real Banach space and C be a nonempty close and convex subset of X. Let $T : C \to C$ be an asymptotically quasi-nonexpansive type mapping defined by Definition 2.1* (6)*. Assume that there exist constants L and $\alpha' > 0$ such that*

$$\|Tx - y^*\| \le L\|x - y^*\|^{\alpha'}, \quad \forall x \in C,\ y^* \in F(T).$$

For any given $x_0 \in C$, define the k-step iterative sequence $\{x_n\}$ by

$$\begin{cases} z_{k-1,n} = (1-\alpha_{k,n})x_n + \alpha_{k,n}T^n x_n, \quad n \ge 0, \\ z_{k-2,n} = (1-\alpha_{k-1,n})x_n + \alpha_{k-1,n}T^n z_{k-1,n}, \quad n \ge 0, \\ \cdots\cdots\cdots \\ z_{1,n} = (1-\alpha_{2,n})x_n + \alpha_{2,n}T^n z_{2,n}, \quad n \ge 0, \\ x_{n+1} = (1-\alpha_{1,n})x_n + \alpha_{1,n}T^n z_{1,n}, \quad n \ge 0, \end{cases}$$

where $\{\alpha_{i,n}\}$ is sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_{1,n} < \infty$ for each $i \in \{1, 2, \cdots, k\}$. Suppose that $\{y_n\}$ is a sequence in C and define a sequence $\{\epsilon_n\}$ of positive real numbers by

$$\begin{cases} w_{k-1,n} = (1-\alpha_{k,n})y_n + \alpha_{k,n}T^n y_n, \quad n \ge 0, \\ w_{k-2,n} = (1-\alpha_{k-1,n})y_n + \alpha_{k-1,n}T^n w_{k-1,n}, \quad n \ge 0, \\ \cdots\cdots\cdots \\ w_{1,n} = (1-\alpha_{2,n})y_n + \alpha_{2,n}T^n w_{2,n}, \qquad n \ge 0, \\ \epsilon_n = \|y_{n+1} - (1-\alpha_{1,n})y_n - \alpha_{1,n}T^n w_{1,n}\|, \quad n \ge 0. \end{cases}$$

Then we have the following:

(i) $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ *if and only if $\{x_n\}$ converges strongly to a fixed point x^* of T in C, where $d(x_n, F(T))$ denotes the distance from x_n to the set $F(T)$, i.e.,* $d(x_n, F(T)) = \inf_{y^* \in F(T)} \|x_n - y^*\|$.

(ii) $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *and* $\liminf_{n\to\infty} d(y_n, F(T)) = 0$ *imply that $\{y_n\}$ converges strongly to a fixed point x^* of T in C.*

(iii) *If $\{y_n\}$ converges strongly to a fixed point x^* of T in C, then* $\lim_{n\to\infty} \epsilon_n = 0$.

Remark 3.1. Theorems 3.1 and 3.2 extend, improve and unify the corresponding results of [3], [4], [9], [11]-[13], [16] and [19]-[22].

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