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A Note on the Spectral Mapping Theorem

Dedicated to the memory of our good friend and colleague professor Allen Shields

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Abstract. In this note we point out how a theorem of Gamelin and Garnett from function theory can be used to establish a spectral mapping theorem for an arbitrary contraction and an associated class of H^∞-functions.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For T in $\mathcal{L}(\mathcal{H})$, we write, as usual, $\sigma(T)$ for the spectrum of T. Recall first that if T satisfies $\|T\| \leq 1$, then T is called a *contraction* (*operator*). If T is a contraction with the property that in its canonical decomposition $T = T_1 \oplus T_2$, where T_1 is completely nonunitary (c.n.u.) and T_2 is a unitary operator (cf. [11]), it happens that T_2 is an absolutely continuous unitary operator, then T is called an *absolutely continuous contraction* (a.c.c.). We will denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$, and by $H^\infty = H^\infty(\mathbb{D})$ the algebra of all bounded analytic functions on $\mathbb{D}$ with the supremum norm $\| \ \|_\infty$.

If f is any function in H^∞ and T is an a.c.c., then the operator $f(T)$ is defined by the Nagy-Foias functional calculus (cf. [2, p.34]), and the relationship between $\sigma(f(T))$ and $f(\sigma(T))$ is sometimes quite complicated. This relationship has been studied fairly extensively; for example, in [5], [8], [6], [7], and [1], where perhaps the best results are to be found. On the other hand, if U is a singular unitary operator, then $f(U)$ cannot even be defined for an arbitrary f in H^∞ because the associated boundary function $\widehat{f} : \partial\mathbb{D} \to \mathbb{C}$ of f is only defined almost everywhere on $\partial\mathbb{D}$ (with respect to arclength measure) in general.

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2. A spectral mapping theorem

In this note we show that, associated with an arbitrary, given contraction $T \in \mathcal{L}(\mathcal{H})$, there is a subspace of H^∞ for which $f(T)$ can be defined, and, moreover, the equation $\sigma(f(T)) = f(\sigma(T))$ is valid for such f. The argument uses a theorem of Gamelin-Garnett [9] from function theory. If E is a nonempty closed subset of $\partial\mathbb{D}$, then H_E^∞ will denote the set of those f in H^∞ that have a continuous extension $\widehat{f}$ to at least $\mathbb{D} \cup E$; it is well known that H_E^∞ is a (closed) subspace of H^∞.

The result of Gamelin-Garnett that we shall need is the following.

Theorem 2.1 ([9]). *If $f \in H_E^\infty$, then there is a sequence $\{f_n\} \subset H^\infty$ such that*

a) $\|f_n - f\|_\infty \to 0$, *and*

b) *each f_n has a holomorphic extension to an open set $\mathcal{U}_n$ containing $\mathbb{D} \cup E$.*

Definition 2.2. Let T be an arbitrary contraction in $\mathcal{L}(\mathcal{H})$, with canonical decomposition $T = T_1 \oplus T_2$ as mentioned above, where T_1 is c.n.u. and T_2 is a unitary operator. (Of course, either T_1 or T_2 may act on the space (0)). Let also $E = E_T := \sigma(T) \cap \partial\mathbb{D}$, and let H_E^∞ be as defined above. Then for $f \in H_E^\infty$, we define $f(T) := f(T_1) \oplus \widehat{f}(T_2)$ where $f(T_1)$ is given by the (Sz.-Nagy-Foias) H^∞-functional calculus mentioned above, and $\widehat{f}(T_2) = (\widehat{f}|_E)(T_2)$ is defined by the usual functional calculus for continuous functions of a normal operator.

Our result is the following.

Theorem 2.3. *Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$ and let $E = \sigma(T) \cap \partial\mathbb{D}$. Then for every $f \in H_E^\infty$, we have $\sigma(f(T)) = f(\sigma(T))$.*

We shall need the following result of Newburgh (which is a consequence of the Gelfand theory of commutative Banach algebras).

Lemma 2.4 ([10]). *If $\{A_n\}$ is an abelian family of operators on a Banach space, and if $\|A_n - A_0\| \to 0$, then $\sigma(A_n) \to \sigma(A_0)$ (where the convergence is with respect to the Hausdorff metric).*

Actually Newburgh's result pertains to general Banach algebras. To apply it to prove the lemma one forms the Banach algebra generated by the set of all operators of the form $r(A_n)$, where $n \in \mathbb{N}$ and r is a rational function with no poles on $\sigma(A_n)$. Then the spectrum of A_n with respect to this algebra is the same as $\sigma(A_n)$. (For more results on the continuity properties of spectra see [4].)

Proof of Theorem 2.3. Let $f \in H_E^\infty$ with continuous extension $\widehat{f}$ to $\mathbb{D} \cup E$. By Theorem 2.1 there is a sequence $\{f_n\} \subset H^\infty$ such that $\|f_n - f\|_\infty \to 0$ and each f_n has an analytic extension to a neighborhood of $\mathbb{D} \cup E$. Since $E = \partial\mathbb{D} \cap \sigma(T)$, it follows easily that each f_n is analytic on a neighborhood of $\sigma(T)$, and $f_n(T)$, defined by the Riesz-Dunford functional calculus (see, for example, [3, p.387-394]), coincides with $f_n(T)$ as given by Definition 2.2. From the well known theory of this functional calculus one has

$$\sigma(f_n(T)) = f_n(\sigma(T)), \qquad n \in \mathbb{N}. \tag{1}$$

Also, we assert that

$$\|f_n(T) - f(T)\| \to 0 \qquad (n \to \infty).$$

To see this, we use the decomposition $f(T) = f(T_1) \oplus f(T_2)$ provided by Definition 2.2. Then

$$\|f_n(T_1) - f(T_1)\| \to 0 \qquad (n \to \infty)$$

via properties of the Sz. Nagy-Foias functional calculus, and

$$\|f_n(T_2) - \widehat{f}(T_2)\| \to 0 \qquad (n \to \infty)$$

via the spectral theorem for normal operators. This establishes the above assertion. The proof of the theorem is now completed by applying Lemma 2.4 to (1). $\square$

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