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On the Stability of Orthogonally Cubic Functional Equations

Choonkil Baak
Department of Mathematics, Chungnam National University, Daejeon 305–764, Korea
e-mail : cgpark@cnu.ac.kr

Mohammad Sal Moslehian
Department of Mathematics, Ferdowsi University, P. O. Box 1159, Mashhad 91775, Iran
e-mail : moslehian@member.ams.org
Homepage : http://www.um.ac.ir/~moslehian/

Abstract. Let f denote a mapping from an orthogonality space $(\mathcal{X}, \perp)$ into a real Banach space $\mathcal{Y}$. In this paper, we prove the Hyers–Ulam–Rassias stability of the orthogonally cubic functional equations $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$ and $f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)=2f(x+y)+4f(x+z)+4f(x-z)+4f(y+z)+4f(y-z)$, where $x \perp y, y \perp z, x \perp z$.

1. Introduction and preliminaries

The stability problem of functional equations originated from the following question of Ulam [27]: *Under what condition does there is an additive mapping near an approximately additive mapping?* In 1941, Hyers [11] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th. M. Rassias [22] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y)-f(x)-f(y)\| \le \varepsilon(\|x\|^p+\|y\|^p),\ (\epsilon > 0,\ p \in [0,1))$. The result of Rassias has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers–Ulam–Rassias. The reader is referred to [1], [2], [5], [12], [17], [23] and references therein for detailed information on stability of functional equations.

There are several concepts of orthogonality in an arbitrary real Banach space $\mathcal{X}$ which are generalizations of orthogonality in the inner product spaces. These are of intrinsic geometric interest and have been studied by many mathematicians. Among them we recall the following ones:

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(i) *Trivial* $\perp_v$: $x \perp_v 0,\ 0 \perp_v x$ for all $x \in \mathcal{X}$ and for non-zero elements $x, y \in \mathcal{X}$, $x \perp_v y$ if and only if x, y are linearly independent.

(ii) *Birkhoff–James* $\perp_B$: $x \perp_B y$ if $\|x\| \le \|x + \alpha y\|$ for all scalars α; cf. [3], [14].

(iii) *Phythagorean* $\perp_P$: $x \perp_P y$ if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ (see [13])

(vi) *Isosceles* $\perp_I$: $x \perp_I y$ if $\|x+y\| = \|x-y\|$ (see [13])

(v) *Diminnie* $\perp_D$: $x \perp_D y$ if $\sup\{f(x)g(y) - f(y)g(x) : f, g \in S^*\} = \|x\|\|y\|$ where S^* is the unit sphere of the dual space $\mathcal{X}^*$ of $\mathcal{X}$; cf. [6]

(vi) *Carlsson* $\perp_C$: $x \perp_C y$ if either $\sum_{i=1}^m \alpha_i \|\beta_i x + \gamma_i y\|^2 = 0$ where $m \ge 2$ and $\alpha_i \neq 0, \beta_i, \gamma_i$ are fixed real numbers such that $\sum_{i=1}^m \alpha_i \beta_i^2 = \sum_{i=1}^m \alpha_i \gamma_i^2 = 0, \sum_{i=1}^m \alpha_i \beta_i \gamma_i = 1$; cf. [4].

(vii) *T-orthogonality* $\perp_T$: Given a linear mapping $T : \mathcal{X} \to \mathcal{X}^*$, $x \perp_T y$ if $T(x)(y) = 0$; cf. [25]

In 1975, Gudder and Strawther [10] defined an abstract orthogonality relation $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation $f(x+y) = f(x) + f(y),\ \ x \perp y$. In 1985, Rätz [24] introduced a new definition of orthogonality by using more restrictive axioms than those of Gudder and Strawther. A (normed) linear space $\mathcal{X}$ equipped with an orthogonal relation $\perp$ in any appropriate sense containing the conditions $x \perp 0,\ 0 \perp x\ (x \in \mathcal{A})$ is called an *orthogonality* (*normed*) *space*. In 1995, Ger and Sikorska [9] investigated the so-called orthogonal stability of the Cauchy functional equation in the sense of Rätz. This result then generalized by Moslehian [19] in the framework of Banach modules.

The orthogonally quadratic equation $f(x+y)+f(x-y) = 2f(x)+2f(y),\ \ x \perp y$ was first studied by Vajzović [28] when $\mathcal{X}$ is a Hilbert space, $\mathcal{Y}$ is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [7], Fochi [8] and Szabó [26] generalized this result. An investigation of the orthogonal stability of the quadratic equation may be found in [20] (see also [18]). In [15], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

In [16], Jung and Chang considered the following cubic functional equation

$$\begin{aligned} & f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ = \ & 2f(x+y) + 4f(x+z) + 4f(x-z) + 4f(y+z) + 4f(y-z). \end{aligned} \tag{1.2}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equations (1.1) and (1.2), which are called *cubic functional equations* and every solution of the cubic functional equations is said to be a *cubic mapping*.

Let $\mathcal{X}$ be an orthogonality space and $\mathcal{Y}$ a real Banach space. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is called *orthogonally cubic* if it satisfies any one of the so-called orthogonally cubic functional equations

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.3}$$

or

$$\begin{aligned} & f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ = \; & 2f(x+y) + 4f(x+z) + 4f(x-z) + 4f(y+z) + 4f(y-z) \end{aligned} \tag{1.4}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y,\ y \perp z,\ x \perp z$. Putting $y = 0$ in (1.3) and $y = z = 0$ in (1.4) we get $f(2x) = 8f(x)$ whence $f(2^n x) = 8^n f(x) \quad (x \in \mathcal{X}, n \in \mathbb{N})$ for all orthogonally cubic mapping f.

In this paper, we investigate the Hyers–Ulam–Rassias stability of the orthogonally cubic functional equations (1.3) and (1.4).

2. Stability of orthogonally cubic functional equations

Throughout this section, $(\mathcal{X}, \perp)$ denotes an orthogonality normed space with the norm $\|\cdot\|_{\mathcal{X}}$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space. We aim to study the conditional stability problems for

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x),$$

and

$$\begin{aligned} & f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ = \; & 2f(x+y) + 4f(x+z) + 4f(x-z) + 4f(y+z) + 4f(y-z) \end{aligned}$$

where $x, y, z \in \mathcal{X}$ with $x \perp y,\ y \perp z,\ x \perp z$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping.

Theorem 2.1. *Let θ and $p \quad (p < 3)$ be nonnegative real numbers. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ fulfilling*

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\|_{\mathcal{Y}} \\ \leq \; & \theta(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p) \end{aligned} \tag{2.1}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\|_{\mathcal{Y}} \leq \frac{\theta}{16 - 2^{p+1}} \|x\|_{\mathcal{X}}^p \tag{2.2}$$

for all $x \in \mathcal{X}$.

Proof. Putting $y = 0$ in (2.1), we get

$$\|2f(2x) - 16f(x)\|_{\mathcal{Y}} \leq \theta \|x\|_{\mathcal{X}}^p \tag{2.3}$$

for all $x \in \mathcal{X}$, since $x \perp 0$. So

$$\|f(x) - \frac{1}{8}f(2x)\|_{\mathcal{Y}} \le \frac{\theta}{16}\|x\|_{\mathcal{X}}^p$$

for all $x \in \mathcal{X}$. Hence

$$\|\frac{1}{8^n}f(2^n x) - \frac{1}{8^m}f(2^m x)\|_{\mathcal{Y}} \le \frac{\theta}{16}\sum_{k=n}^{m-1}\frac{2^{pk}}{8^k}\|x\|_{\mathcal{X}}^p \tag{2.4}$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{8^n}f(2^n x)\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is complete, there exists a mapping $T : \mathcal{X} \to \mathcal{Y}$ defined by

$$T(x) := \lim_{n\to\infty}\frac{1}{8^n}f(2^n x)$$

for all $x \in \mathcal{X}$. Letting $n = 0$ and $m \to \infty$ in (2.4), we get the inequality (2.2). It follows from (2.1) that

$$\begin{aligned}
& \|T(2x+y) + T(2x-y) - 2T(x+y) - 2T(x-y) - 12T(x)\|_{\mathcal{Y}} \\
= & \lim_{n\to\infty}\frac{1}{8^n}\|f(2^n(2x+y)) + f(2^n(2x-y)) - 2f(2^n(x+y)) \\
& - 2f(2^n(x-y)) - 12f(2^n x)\|_{\mathcal{Y}} \\
\le & \lim_{n\to\infty}\frac{2^{pn}\theta}{8^n}(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p) = 0
\end{aligned}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. So

$$T(2x+y) + T(2x-y) - 2T(x+y) - 2T(x-y) - 12T(x) = 0$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Hence $T : \mathcal{X} \to \mathcal{Y}$ is an orthogonally cubic mapping. Let $Q : \mathcal{X} \to \mathcal{Y}$ be another orthogonally cubic mapping satisfying (2.2). Then

$$\begin{aligned}
\|T(x) - Q(x)\|_{\mathcal{Y}} &= \frac{1}{8^n}\|T(2^n x) - Q(2^n x)\|_{\mathcal{Y}} \\
&\le \frac{1}{8^n}(\|f(2^n x) - T(2^n x)\|_{\mathcal{Y}} + \|f(2^n x) - Q(2^n x)\|_{\mathcal{Y}}) \\
&\le \frac{\theta}{8 - 2^p}\cdot\frac{2^{pn}}{8^n}\|x\|_{\mathcal{X}}^p,
\end{aligned}$$

which tends to zero for all $x \in \mathcal{X}$. So we have $T(x) = Q(x)$ for all $x \in \mathcal{X}$. This proves the uniqueness of T. □

Theorem 2.2. *Let θ and p $(p > 3)$ be nonnegative real numbers. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ fulfilling*

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\|_{\mathcal{Y}} \le \theta(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p)$$

for all $x, y \in \mathcal{X}$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\|_{\mathcal{Y}} \leq \frac{\theta}{2^{p+1} - 16}\|x\|_{\mathcal{X}}^p \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof. It follows from (2.3) that

$$\|f(x) - 8f(\frac{x}{2})\|_{\mathcal{Y}} \leq \frac{\theta}{2^{p+1}}\|x\|_{\mathcal{X}}^p$$

for all $x \in \mathcal{X}$. So

$$\|8^n f(\frac{x}{2^n}) - 8^m f(\frac{x}{2^m})\|_{\mathcal{Y}} \leq \frac{\theta}{2^{p+1}} \sum_{k=n}^{m-1} \frac{8^k}{2^{pk}}\|x\|_{\mathcal{X}}^p \tag{2.6}$$

for all nonnegative integers n, m with $n < m$. Thus $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is complete, there exists a mapping $T : \mathcal{X} \to \mathcal{Y}$ defined by

$$T(x) := \lim_{n\to\infty} 8^n f(\frac{x}{2^n})$$

for all $x \in \mathcal{X}$. Letting $n = 0$ and $m \to \infty$ in (2.6), we get the inequality (2.5).

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let θ and p $(p < 3)$ be nonnegative real numbers. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ fulfilling*

$$\begin{aligned} &\|f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) - 2f(x+y) \\ &\quad - 4f(x+z) - 4f(x-z) - 4f(y+z) - 4f(y-z)\|_{\mathcal{Y}} \\ \leq\ &\theta(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p + \|z\|_{\mathcal{X}}^p) \end{aligned} \tag{2.7}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y, y \perp z$ and $x \perp z$. Then there exists a unique orthogonally cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\|_{\mathcal{Y}} \leq \frac{\theta}{8 - 2^p}\|x\|_{\mathcal{X}}^p \tag{2.8}$$

for all $x \in \mathcal{X}$.

Proof. Putting $y = z = 0$ in (2.7), we get

$$\|f(2x) - 8f(x)\|_{\mathcal{Y}} \leq \theta\|x\|_{\mathcal{X}}^p \tag{2.9}$$

for all $x \in \mathcal{X}$, since $x \perp 0$ and $0 \perp 0$. So

$$\|f(x) - \frac{1}{8}f(2x)\|_{\mathcal{Y}} \leq \frac{\theta}{8}\|x\|_{\mathcal{X}}^p$$

for all $x \in \mathcal{X}$. Hence

$$\|\frac{1}{8^n}f(2^nx) - \frac{1}{8^m}f(2^mx)\|_{\mathcal{Y}} \leq \frac{\theta}{8}\sum_{k=n}^{m-1}\frac{2^{pk}}{8^k}\|x\|_{\mathcal{X}}^p \tag{2.10}$$

for all nonnegative integers n, m with $n < m$. Thus $\{\frac{1}{8^n}f(2^nx)\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is complete, there exists a mapping $T : \mathcal{X} \to \mathcal{Y}$ defined by

$$T(x) := \lim_{n\to\infty}\frac{1}{8^n}f(2^nx)$$

for all $x \in \mathcal{X}$. Letting $n = 0$ and $m \to \infty$ in (2.10), we get the inequality (2.8). It follows from (2.7) that

$$\begin{aligned}
&\|T(x+y+2z) + T(x+y-2z) + T(2x) + T(2y) - 2T(x+y) \\
&\quad - 4T(x+z) - 4T(x-z) - 4T(y+z) - 4T(y-z)\|_{\mathcal{Y}} \\
= \;& \lim_{n\to\infty}\frac{1}{8^n}\|f(2^n(x+y+2z)) + f(2^n(x+y-2z)) + f(2^{n+1}x) + f(2^{n+1}y) \\
&\quad - 2f(2^n(x+y)) - 4f(2^n(x+z)) - 4f(2^n(x-z)) - 4f(2^n(y+z)) - 4f(2^n(y-z))\|_{\mathcal{Y}} \\
\leq \;& \lim_{n\to\infty}\frac{2^{pn}\theta}{8^n}(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p + \|z\|_{\mathcal{X}}^p) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y$, $y \perp z$ and $x \perp z$. So

$$\begin{aligned}
&T(x+y+2z) + T(x+y-2z) + T(2x) + T(2y) - 2T(x+y) \\
&\quad - 4T(x+z) - 4T(x-z) - 4T(y+z) - 4T(y-z) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y$, $y \perp z$ and $x \perp z$. Hence $T : \mathcal{X} \to \mathcal{Y}$ is an orthogonally cubic mapping. Let $Q : \mathcal{X} \to \mathcal{Y}$ be another orthogonally cubic mapping satisfying (2.8). Then

$$\begin{aligned}
\|T(x) - Q(x)\|_{\mathcal{Y}} &= \frac{1}{8^n}\|T(2^nx) - Q(2^nx)\|_{\mathcal{Y}} \\
&\leq \frac{1}{8^n}(\|f(2^nx) - T(2^nx)\|_{\mathcal{Y}} + \|f(2^nx) - Q(2^nx)\|_{\mathcal{Y}}) \\
&\leq \frac{2\theta}{8-2^p}\cdot\frac{2^{pn}}{8^n}\|x\|_{\mathcal{X}}^p,
\end{aligned}$$

which tends to zero for all $x \in \mathcal{X}$. So we have $T(x) = Q(x)$ for all $x \in \mathcal{X}$. This proves the uniqueness of T. □

Theorem 2.4. *Let θ and p $(p > 3)$ be nonnegative real numbers. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ fulfilling*

$$\begin{aligned}
\|f(x+y+2z) &+ f(x+y-2z) + f(2x) + f(2y) - 2f(x+y) \\
&- 4f(x+z) - 4f(x-z) - 4f(y+z) - 4f(y-z)\|_{\mathcal{Y}} \\
&\leq \theta(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p + \|z\|_{\mathcal{X}}^p)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y, y \perp z$ and $x \perp z$. Then there exists a unique orthogonally cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - T(x)\|_{\mathcal{Y}} \leq \frac{\theta}{2^p - 8} \|x\|_{\mathcal{X}}^p \tag{2.11}$$

for all $x \in \mathcal{X}$.

Proof. It follows from (2.9) that

$$\|f(x) - 8f(\frac{x}{2})\|_{\mathcal{Y}} \leq \frac{\theta}{2^p} \|x\|_{\mathcal{X}}^p$$

for all $x \in \mathcal{X}$. So

$$\|8^n f(\frac{x}{2^n}) - 8^m f(\frac{x}{2^m})\|_{\mathcal{Y}} \leq \frac{\theta}{2^p} \sum_{k=n}^{m-1} \frac{8^k}{2^{pk}} \|x\|_{\mathcal{X}}^p \tag{2.12}$$

for all nonnegative integers n, m with $n < m$. Thus $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is complete, there exists a mapping $T : \mathcal{X} \to \mathcal{Y}$ defined by

$$T(x) := \lim_{n\to\infty} 8^n f(\frac{x}{2^n})$$

for all $x \in \mathcal{X}$. Letting $n = 0$ and $m \to \infty$ in (2.12), we get the inequality (2.11). The rest of the proof is similar to the proof of Theorem 2.3. □

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