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Meromorphic Functions with Weighted Sharing of One Set

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ABSTRACT. In this article, we investigate the problem of uniqueness of meromorphic functions sharing one set and having deficient values, and obtain a result which improves some earlier results.

1. Introduction and definitions

Let f be a meromorphic function defined on the complex plane C, and let S be a subset of $\overline{C} = C \bigcup \{\infty\}$. Define

$$\begin{split} E(S,f) &= \bigcup_{a \in S} \{z : f(z) - a = 0, \text{ counting multiplicity} \}, \\ E(S,f) &= \bigcup_{a \in S} \{z : f(z) - a = 0, \text{ ignoring multiplicity} \}. \end{split}$$

Let f and g be two nonconstant meromorphic functions. If E(S, f) = E(S, g), we say that f and g share the set S CM (counting multiplicity); if $\overline{E}(S, f) = \overline{E}(S, g)$, we say that f and g share the set S IM (ignoring multiplicity). Specially, if $S = \{a\}$, where $a \in \overline{C}$, we say that f and g share the value a CM (or IM), if E(S, f) = E(S, g) (or $\overline{E}(S, f) = \overline{E}(S, g)$). It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found, for instance in [5] or [10]. Lahiri and Banerjee relaxed the nature of sharing the sets with the aid of the notion of weighted sharing as introduced in [6].

Definition 1. Let k be a nonnegative integer or infinity. For any $a \in \overline{C}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if m > k. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of f-a with multiplicity $m (\leq k)$ if and only if it is a zero of g-a with multiplicity $m (\leq k)$, and z_0 is a zero of f-a with multiplicity m (> k) if and only if it is a zero of g-a with multiplicity n (> k), where m is not necessarily equal to n.

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We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for all integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share (a, 0) or (a, ∞) respectively.

Definition 2. For $S \subset \overline{C}$, we define $E_f(S,k)$ as $E_f(S,k) = \bigcup_{a \in S} E_k(a,f)$, where k is a nonnegative integer or infinity. Clearly $E(S,f) = E_f(S,\infty)$ and $\overline{E}(S,f) = E_f(S,0)$.

Definition 3. For $a \in \overline{C}$, we define

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)}.$$

In 1977, Gross [4] proved that there exist three finite sets S_j (j = 1, 2, 3), such that any two entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for j = 1, 2, 3 must be identical, and asked: Does there exist a finite set S such that, for any pair of nonconstant entire functions f and g, E(S, f) = E(S, g) implies $f \equiv g$? If the answer of this question is affirmative, what is the smallest cardinal of S?

Yi [12] first proved that such that a set exist. In fact, Yi proved the following:

Theorem A. There exists a set S with seven elements such that E(S, f) = E(S, g) implies $f \equiv g$, for any pair of nonconstant entire functions f and g.

Yi [12], Li and Yang [7], [8], Frank and Rienders [3], Bartels [1] and other authors studied the problem for meromorphic functions sharing one set. Yan [11] proved the following result which is an improvement of the result of Fang and Hua [2].

Theorem B. Let f and g be two nonconstant meromorphic functions. If $\Theta(\infty, g) > 3/4$ and $\Theta(\infty, f) > 3/4$, then there exists a set with seven elements such that E(S, f) = E(S, g)implies $f \equiv g$.

Recently, I. Lahiri and A. Banerjee [6] have weakened the condition " $\Theta(\infty, g), \Theta(\infty, f) > 3/4$ " in Theorem B, but their set consists of nine elements, they proved the following result:

Theorem C. Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n \ge 9$ and a, b are nonzero complex number such that $z^n + az^{n-1} + b = 0$ has no repeated root. If for two nonconstant mero-morphic functions f and g satisfy $E_f(S, 2) = E_g(S, 2)$ and $\Theta(\infty; f) + \Theta(\infty; g) > 4/(n-1)$, then $f \equiv g$.

We note that if $(n-1)^{n-1} \neq b(-(n/a))^n$ and $ab \neq 0$, then $z^n + az^{n-1} + b = 0$ has no repeated root.

The purpose of this article, we treat the conditions in Theorems B and C. In fact, the main idea to proving is due to Lin and Yi [9], they proved the following:

Theorem D. Let $S_1 = \{0\}$, $S_2 = \{\infty\}$ and $S_3 = \{w \in C : aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 = 0\}$, where $n(\geq 4)$ is an integer, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. If f and g are two nonconstant meromorphic functions satisfying $E(S_j, f) = E(S_j, g)$, for j = 1, 2, 3, then $f \equiv g$.

A set S with finite elements is called a unique range set of meromorphic functions provided that E(S, f) = E(S, g) can imply $f \equiv g$ for any two meromorphic functions. Li

58

and Yang [8] gave an unique range set with 15 elements, Frank and Rienders [3] showed an unique range set with 11 elements. The questions related to unique range set have been studied by many mathematicians. However, what is the smallest cardinality of unique range set? It is still an open problem.

In this paper, we prove the following result which is an improvement of Theorems B and C.

Theorem 1. Let $S = \{w \in C : aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2 = 0\}$, where $n(\geq 6)$ is an integer, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. Then $f \equiv g$, if f and g are nonconstant meromorphic functions which satisfy one of the following conditions:

 $\begin{array}{ll} (\text{C-1}) & \frac{16-n}{6} < \Theta_f, & \frac{16-n}{6} < \Theta_g \ and \ E_f(S,0) = E_g(S,0). \\ (\text{C-2}) & \frac{3}{14}(12-n) < \Theta_f, & \frac{3}{14}(12-n) < \Theta_g \ and \ E_f(S,1) = E_g(S,1). \\ (\text{C-3}) & \frac{10-n}{4} < \Theta_f, & \frac{10-n}{4} < \Theta_g \ and \ E_f(S,2) = E_g(S,2). \\ Here \ \Theta_f = \Theta(0,f) + \Theta(\infty,f) + \Theta(b,f) \ and \ \Theta_g \ can \ be \ similarly \ defined. \end{array}$

We see that the condition $\left(\frac{10-n}{4} < \Theta_f, \frac{10-n}{4} < \Theta_g\right)$ of C-3 is better than the condition $\left(\Theta(\infty; f) + \Theta(\infty; g) > 4/(n-1)\right)$ in Theorem C, when n > 10. From Theorem 1, we can easy to deduce the following corollary:

Corollary. Suppose that b is a nonzero complex number and $n \ge 6$ is an integer. There exists a finite set S with n elements and $b \notin S$ such that if for two nonconstant meromorphic functions f and g satisfy one of the conditions (C-1), (C-2) and (C-3) in Theorem 1, then $f \equiv g$.

2. Some lemmas

Let $n(\geq 6)$ be an integer, and let a and b be two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. It is obvious that $n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0$ has two distinct roots, say α_1 and α_2 . Set

(1)
$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}.$$

From (1) we have

(2)
$$R'(w) = \frac{(n-2)aw^{n-1}(w-b)^2}{n(n-1)(w-\alpha_1)^2(w-\alpha_2)^2}$$

We see that w = 0 is one root with multiplicity n of the equation R(w) = 0; and w = b is only one root with multiplicity 3 of the equation R(w) - c = 0, where $c = \frac{ab^{n-2}}{2} \neq 1$. Thus,

(3)
$$R(w) - c = \frac{a(w-b)^3 Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)},$$

where $Q_{n-3}(w)$ is a polynomial of degree n-3 and has no multiple root. Set

(4)
$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}$$

where

(5)
$$P(w) = aw^{n} - n(n-1)w^{2} + 2n(n-2)bw - (n-1)(n-2)b^{2}.$$

We observe that P(w) = 0 has only simple zeros. Let us define F and G by

(6)
$$F = R(f), \qquad G = R(g),$$

and we set

(7)
$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - (\frac{G''}{G'} - 2\frac{G'}{G-1})$$

Lemma 1. Let S be a set which is mentioned in Theorem 1 such that $n \ge 6$. If $E_f(S, 0) = E_g(S, 0)$, then $(n-2)T(r, f) \le nT(r, g) + S(r, f)$ and $(n-2)T(r, g) \le nT(r, f) + S(r, g)$. This shows that S(r, f) = S(r, g) and we denote them by S(r).

Proof. Let z_j , $j = 1, \dots, n$ be the zeros of P(w). From the second fundamental theorem, we have

$$(n-2)T(r,f) \le \sum_{j=1}^{n} \overline{N}(r,\frac{1}{f-z_{j}}) + S(r,f) \le \sum_{j=1}^{n} \overline{N}(r,\frac{1}{g-z_{j}}) + S(r,f) \le nT(r,g) + S(r,f).$$

Similarly, we have $(n-2)T(r,g) \le nT(r,f) + S(r,g)$. This proves Lemma 1.

Lemma 2. $FG \not\equiv 1$, when $n \geq 6$.

Proof. Suppose to the contrary that $FG \equiv 1$. Let z_0 be a pole of f with multiplicity p. Then z_0 is a pole of F with multiplicity (n-2)p. This leads us that z_0 is a zero of g with multiplicity q such that (n-2)p = nq. We see that $2q = (n-2)(p-q) \ge n-2$, so that $p = qn/(n-2) \ge n/2$. Therefore, $\overline{N}(r, f) \le (2/n)N(r, f)$. Let z_0 be a zero of $f - \alpha_1$ with multiplicity p. Then z_0 is a zero of g with multiplicity q such that $p = nq \ge n$. Therefore, $\overline{N}(r, 1/(f - \alpha_1)) \le (1/n)N(r, 1/(f - \alpha_1))$. Similarly, we obtain that

$$\overline{N}(r, 1/(f - \alpha_2)) \le (1/n)N(r, 1/(f - \alpha_2)).$$

According to the second fundamental theorem, we get $T(r, f) \leq (4/n)T(r, f) + S(r, f)$, which is impossible. This proves the lemma.

Lemma 3 [9, Lemma 1]. If F and G share (1,0) and $H \neq 0$, then

$$\overline{N}_{1}(r, \frac{1}{F-1}, \frac{1}{G-1}) \le N(r, H) + S(r),$$

where $\overline{N}_{1}(r, \frac{1}{F-1}, \frac{1}{G-1})$ denotes the counting function of common simple 1-points of F and G, and S(r) is defined as in Lemma 1.

Lemma 4 [9, Lemma 5]. If $Q(w) = (n-1)^2(w^n-1)(w^{n-2}-1) - n(n-2)(w^{n-1}-1)^2$, then $Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6})$, where $\beta_j \setminus \{0, 1\}$ $(j = 1, 2, \cdots, 2n-6)$ are distinct complex numbers.

60

3. Proof of Theorem 1

We first assume that $E_f(S,0) = E_g(S,0)$. It follows from (2), (4) and (6) that F and G are nonconstant meromorphic functions sharing 1 IM. Suppose $H \neq 0$. From (2) and (6) we have

(8)
$$F' = \frac{(n-2)af^{n-1}(f-b)^2 f'}{n(n-1)(f-\alpha_1)^2(f-\alpha_2)^2}, \quad G' = \frac{(n-2)ag^{n-1}(g-b)^2 f'}{n(n-1)(g-\alpha_1)^2(g-\alpha_2)^2}$$

From (7) and (8), we see that

(9)
$$N(r,H) \leq \overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-b}) + N_*(r,\frac{1}{F-1},\frac{1}{G-1}) + N_0(r,\frac{1}{f'}) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{g-b}) + N_0(r,\frac{1}{g'}) + S(r),$$

where we write $N_0(r, \frac{1}{f'})$ for the counting function of the zeros of f' that are not zeros of f(f-b) and F-1, $N_0(r, \frac{1}{g'})$ can be similarly defined, and $N_*(r, \frac{1}{F-1}, \frac{1}{G-1})$ is the counting function of those 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G, each point in these counting functions is counted only once. We observe that

$$\begin{aligned} &(10) \quad \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) \\ &\leq \quad \overline{N}_{1)}(r,\frac{1}{F-1},\frac{1}{G-1}) + \frac{1}{2}\{N(r,\frac{1}{F-1}) + N(r,\frac{1}{G-1})\} + \frac{1}{2}N_{*}(r,\frac{1}{F-1},\frac{1}{G-1}) \\ &\leq \quad \overline{N}_{1)}(r,\frac{1}{F-1},\frac{1}{G-1}) + \frac{n}{2}\{T(r,f) + T(r,g)\} + \frac{1}{2}N_{*}(r,\frac{1}{F-1},\frac{1}{G-1}) + S(r). \end{aligned}$$

By using the second fundamental theorem, we have

$$\begin{aligned} (11) & (n+1)T(r,f) + (n+1)T(r,g) \\ & \leq \quad \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f-b}) + \overline{N}(r,f) \\ & \quad + \overline{N}(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g-b}) + \overline{N}(r,g) - N_0(r,\frac{1}{f'}) - N_0(r,\frac{1}{g'}) + S(r). \end{aligned}$$

From (9) - (11) and by applying Lemma 3, we note that

$$(12) \quad (\frac{n}{2}+1)T(r,f) + (\frac{n}{2}+1)T(r,g) \le 2\overline{N}(r,\frac{1}{f}) + 2\overline{N}(r,\frac{1}{f-b}) \\ + 2\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{g}) + 2\overline{N}(r,\frac{1}{g-b}) + 2\overline{N}(r,g) + \frac{3}{2}N_*(r,\frac{1}{F-1},\frac{1}{G-1}) + S(r).$$

We consider that z_i $(i = 1, \dots, n)$ are the roots of P(w). Then, by the first fundamental theorem

$$\begin{split} & N_*(r, \frac{1}{F-1}, \frac{1}{G-1}) \\ & \leq \quad N(r, \frac{1}{F-1}) - \overline{N}(r, \frac{1}{F-1}) + N(r, \frac{1}{G-1}) - \overline{N}(r, \frac{1}{G-1}) \\ & = \quad \sum_{j=1}^n (N(r, \frac{1}{f-z_j}) - \overline{N}(r, \frac{1}{f-z_j})) + \sum_{j=1}^n (N(r, \frac{1}{g-z_j}) - \overline{N}(r, \frac{1}{g-z_j})) \\ & \leq \quad N(r, \frac{1}{\frac{f}{f}}) + N(r, \frac{1}{\frac{g}{g}}) \\ & \leq \quad \overline{N}(r, \frac{1}{f}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, g) + S(r), \end{split}$$

$$\begin{split} & N_*(r, \frac{1}{F-1}, \frac{1}{G-1}) \\ \leq & \sum_{j=1}^n (N(r, \frac{1}{f-z_j}) - \overline{N}(r, \frac{1}{f-z_j})) + \sum_{j=1}^n (N(r, \frac{1}{g-z_j}) - \overline{N}(r, \frac{1}{g-z_j})) \\ \leq & N(r, \frac{1}{\frac{f}{f-b}}) + N(r, \frac{1}{\frac{g}{g-b}}) \\ \leq & \overline{N}(r, \frac{1}{f-b}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{g-b}) + \overline{N}(r, g) + S(r) \end{split}$$

and

$$\begin{split} &N_*(r, \frac{1}{F-1}, \frac{1}{G-1}) \\ &\leq \quad \sum_{j=1}^n (N(r, \frac{1}{f-z_j}) - \overline{N}(r, \frac{1}{f-z_j})) + \sum_{j=1}^n (N(r, \frac{1}{g-z_j}) - \overline{N}(r, \frac{1}{g-z_j})) \\ &\leq \quad N(r, \frac{1}{\frac{f'}{f(f-b)}}) + N(r, \frac{1}{\frac{g'}{g(g-b)}}) \\ &\leq \quad \overline{N}(r, \frac{1}{f-b}) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g-b}) + \overline{N}(r, \frac{1}{g}) + S(r), \end{split}$$

from these three inequalities and (12), we get the following:

$$\begin{array}{ll} (13) & (\frac{n}{2}+1)T(r,f) + (\frac{n}{2}+1)T(r,g) \\ \leq & \frac{7}{2}\overline{N}(r,\frac{1}{f}) + 2\overline{N}(r,\frac{1}{f-b}) + \frac{7}{2}\overline{N}(r,f) + \frac{7}{2}\overline{N}(r,\frac{1}{g}) + 2\overline{N}(r,\frac{1}{g-b}) + \frac{7}{2}\overline{N}(r,g) + S(r), \end{array}$$

$$\begin{array}{rl} (14) & (\frac{n}{2}+1)T(r,f) + (\frac{n}{2}+1)T(r,g) \\ & \leq & 2\overline{N}(r,\frac{1}{f}) + \frac{7}{2}\overline{N}(r,\frac{1}{f-b}) + \frac{7}{2}\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{g}) + & \frac{7}{2}\overline{N}(r,\frac{1}{g-b}) + \frac{7}{2}\overline{N}(r,g) + S(r) \end{array}$$

and

$$\begin{array}{ll} (15) & (\frac{n}{2}+1)T(r,f) + (\frac{n}{2}+1)T(r,g) \\ \leq & \frac{7}{2}\overline{N}(r,\frac{1}{f}) + \frac{7}{2}\overline{N}(r,\frac{1}{f-b}) + 2\overline{N}(r,f) + \frac{7}{2}\overline{N}(r,\frac{1}{g}) + \frac{7}{2}\overline{N}(r,\frac{1}{g-b}) + 2\overline{N}(r,g) + S(r). \end{array}$$

The inequalities (13) - (15) give us

$$\begin{aligned} &\frac{1}{3}(\frac{n}{2}+1)T(r,f)+\frac{1}{3}(\frac{n}{2}+1)T(r,g)\\ \leq &\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-b})+\overline{N}(r,f)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{g-b})+\overline{N}(r,g)+S(r). \end{aligned}$$

Suppose the condition (C-1) in Theorem 1 occurs. We conclude from the last inequality that for each ϵ such that $0 < 3\epsilon < min\{\frac{n-16}{6} + \Theta_f, \ \frac{n-16}{6} + \Theta_g\}$, we have

$$\left(\frac{n-16}{6} + \Theta_f - 3\epsilon\right)T(r, f) + \left(\frac{n-16}{6} + \Theta_g - 3\epsilon\right)T(r, g) \le S(r).$$

Without loss of generality we may suppose that there exists a set I with infinite measure such that $T(r,g) \leq T(r,f)$, $r \in I$. Then, from the last inequality, we obtain that $\frac{n-16}{3} + \Theta_f + \Theta_g \leq 6\epsilon$, which is a contradiction.

Suppose the condition (C-2) in Theorem 1 occurs. Then F and G share (1,1); moreover

(16)

$$N_*(,\frac{1}{F-1},\frac{1}{G-1})$$

$$\leq \frac{1}{2}\{\overline{N}(r,\frac{f'}{f})+\overline{N}(r,\frac{g'}{g})\}+S(r)$$

$$\leq \frac{1}{2}\{\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})\}+S(r),$$

(17)

$$N_*\left(,\frac{1}{F-1},\frac{1}{G-1}\right)$$

$$\leq \frac{1}{2}\left\{\overline{N}\left(r,\frac{f'}{f-b}\right) + \overline{N}\left(r,\frac{g'}{g-b}\right)\right\} + S(r)$$

$$\leq \frac{1}{2}\left\{\overline{N}\left(r,f\right) + \overline{N}\left(r,\frac{1}{f-b}\right) + \overline{N}\left(r,g\right) + \overline{N}\left(r,\frac{1}{g-b}\right)\right\} + S(r),$$

(18)

$$N_{*}(, \frac{1}{F-1}, \frac{1}{G-1}) \\ \leq \frac{1}{2} \{ \overline{N}(r, \frac{f'}{f(f-b)}) + \overline{N}(r, \frac{g'}{g(g-b)}) \} + S(r) \\ \leq \frac{1}{2} \{ \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-b}) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g-b}) \} + S(r)$$

and

(19)

$$\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - \overline{N}_{1}(r, \frac{1}{F-1})$$

$$\leq \frac{1}{2} \{N(r, \frac{1}{F-1}) + N(r, \frac{1}{G-1})\}$$

$$\leq \frac{n}{2} \{T(r, f) + T(r, g)\}.$$

The inequalities (16) - (19) imply that

(20)
$$N_{*}(r, \frac{1}{F-1}, \frac{1}{G-1}) + \overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - \overline{N}_{1}(r, \frac{1}{F-1}) \\ \leq \frac{n}{2} \{T(r, f) + T(r, g)\} + \frac{1}{3} \{\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-b})\} \\ + \frac{1}{3} \{\overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g-b})\} + S(r).$$

By using Lemma 3, we deduce from the inequalities (9), (11), (20) that

$$(21) \quad (\frac{n}{2}+1)\{T(r,f)+T(r,g)\} \\ \leq \quad \frac{7}{3}\{\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-b})\}+\frac{7}{3}\{\overline{N}(r,g)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,\frac{1}{g-b})\}+S(r).$$

Thus, if ϵ is any positive real number such that $0 < 7\epsilon < \min\{((n-12)/2) + (7/3)\Theta_f, ((n-12)/2) + (7/3)\Theta_g\}$ then, by (21), we obtain

$$\left(\frac{n-12}{2} + \frac{7}{3}\Theta_f - 7\epsilon\right)T(r,f) + \left(\frac{n-12}{2} + \frac{7}{3}\Theta_g - 7\epsilon\right)T(r,g) \le S(r).$$

Without loss of generality we may suppose that there exists a set I with infinite measure such that $T(r,g) \leq T(r,f)$, $r \in I$. The last inequality gives us

$$n - 12 + (7/3)\Theta_f + (7/3)\Theta_g \le 14\epsilon$$

which is a contradiction. Suppose the condition (C-3) in Theorem 1 occurs. Then F and G share (1, 2). We see that

(22)
$$\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) - \overline{N}_{1}(r, \frac{1}{F-1}) + N_*(r, \frac{1}{F-1}, \frac{1}{G-1}) \\ \leq \frac{1}{2} \{N(r, \frac{1}{F-1}) + N(r, \frac{1}{G-1})\} \leq \frac{n}{2} \{T(r, f) + T(r, g)\}.$$

It follows from (9), (11), (22) and by applying Lemma 3, we have

$$\{\frac{n-10}{4} + \Theta_f - 3\epsilon\}T(r, f) + \{\frac{n-10}{4} + \Theta_g - 3\epsilon\}T(r, g) \le S(r, f) + S(r, g)$$

Similar to the arguments in the cases (C-1) and (C-2), we can still get a contradiction. Hence $H \equiv 0$. By integration (7), we find that

(23)
$$G = \frac{(B+1)F + A - B - 1}{BF + A - B},$$

where $A \neq 0$ and B are constants. From (23) and the assumptions of Theorem 1, we see that F and G share 1 CM and T(r, F) = T(r, G) + O(1). Thus,

(24)
$$T(r, f) = T(r, g) + S(r).$$

Consequently, from (3) we obtain

(25)
$$\overline{N}(r,\frac{1}{F-c}) \leq \overline{N}(r,\frac{1}{f-b}) + \overline{N}(r,\frac{1}{Q_{n-3}(f)}) + S(r)$$
$$\leq (n-2)T(r,f) + S(r,f) \leq \frac{n-2}{n}T(r,F) + S(r,f).$$

Now we distinguish three cases.

Case 1. $B \neq 0, -1$. Suppose that $A - B - 1 \neq 0$. From (23), (24) and by the second fundamental theorem

$$\begin{split} nT(r,f) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F+\frac{A-B-1}{B+1}}) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-\alpha_1}) + \overline{N}(r,\frac{1}{f-\alpha_2}) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{g}) + S(r) \\ &\leq 5T(r,f) + S(r), \end{split}$$

which contradicts $n \ge 6$. Therefore A = B + 1, we can write (23) as

$$(26) G = \frac{(B+1)F}{BF+1}.$$

If $c \neq -\frac{1}{B}$ then from (24) – (26) and by using the second fundamental theorem, we observe

$$\begin{array}{ll} 2nT(r,f) &\leq & \overline{N}(r,F) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F+\frac{1}{B}}) + \overline{N}(r,\frac{1}{F-c}) + S(r,F) \\ \\ &\leq & \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-\alpha_1}) + \overline{N}(r,\frac{1}{f-\alpha_2}) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,g) \\ \\ &\quad + \overline{N}(r,\frac{1}{g-\alpha_1}) + \overline{N}(r,\frac{1}{g-\alpha_2}) + (n-2)T(r,f) + S(r) \\ \\ &\leq & (n+5)T(r,f) + S(r), \end{array}$$

which is also a contradiction to $n \ge 6$. Therefore, $c = -\frac{1}{B}$. Then (26) implies that $F = \frac{cG}{G - (1 - 0c)}$. Since $c \ne 1/2$, then we get that $c \ne 1 - c$, from (24), (25) and by using the second fundamental theorem, we see that

$$\begin{array}{ll} 2nT(r,g) &\leq & \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-(1-c)}) + \overline{N}(r,\frac{1}{G-c}) + S(r,G) \\ &\leq & \overline{N}(r,\frac{1}{g-\alpha_1}) + \overline{N}(r,\frac{1}{g-\alpha_2}) + \overline{N}(r,g) + + \overline{N}(r,\frac{1}{g}) \\ &\quad + \overline{N}(r,\frac{1}{f-\alpha_1}) + \overline{N}(r,\frac{1}{f-\alpha_2}) + \overline{N}(r,f) + (n-2)T(r,g) + S(r) \\ &\leq & (n+5)T(r,g), \end{array}$$

which contradicts $n \ge 6$.

Case 2. B = -1. We write (23) as

$$(27) G = \frac{A}{-F+A+1}.$$

By Lemma 2, we get $A + 1 \neq 0$. If $A + 1 \neq c$ then from (24), (25) and by using the second fundamental theorem, we have

$$\begin{aligned} 2nT(r,f) &\leq \overline{N}(r,F) + \overline{N}(r,\frac{1}{F-(A+1)}) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-c}) + S(r,F) \\ &\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-\alpha_1}) + \overline{N}(r,\frac{1}{f-\alpha_2}) + \overline{N}(r,g) \\ &\quad + \overline{N}(r,\frac{1}{g-\alpha_1}) + \overline{N}(r,\frac{1}{g-\alpha_2}) + \overline{N}(r,\frac{1}{f}) + (n-2)T(r,f) + S(r) \\ &\leq (n+5)T(r,f) + S(r), \end{aligned}$$

which contradicts $n \ge 6$. So that A + 1 = c; it follows from (27) that $F = \frac{cG + 1 - c}{G}$. If $\frac{c-1}{c} \ne c$ then from (24), (25) and the second fundamental theorem, we obtain

$$2nT(r,g) \leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{G-c}) + \overline{N}(r,\frac{1}{G+\frac{1-c}{c}}) \leq (n+3)T(r,g) + S(r),$$

which is a contradiction to $n \ge 6$. Thus $\frac{c-1}{c} = c$, and hence, F = c((G-c)/G). Since $Q_{n-3}(w)$ has n-3 distinct root, it follows from the last equation that $(n-4)T(r,g) \le \overline{N}(r,1/(G-c)) + S(r) \le \overline{N}(r,1/f) + S(r) \le T(r,f) + S(r)$; from this and (24), we have a contradiction.

Case 3. B = 0. From (23), we get

$$G = \frac{F+A-1}{A}.$$

Suppose $A \neq 1$. If $1 - A \neq c$ then from (24), (25) and by using the second fundamental theorem, we have

$$\begin{array}{ll} 2nT(r,f) &\leq & \overline{N}(r,\frac{1}{F+A-1}) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{F-c}) + S(r,F) \\ &\leq & \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f-\alpha_1}) \\ & & + \overline{N0}(r,\frac{1}{f-\alpha_2}) + (n-2)T(r,f) + S(r,f) \\ &\leq & (n+3)T(r,f) + S(r,f), \end{array}$$

which is a contradiction to $n \ge 6$. Hence 1 - A = c; by (28), we get $F = (1 - c) \{ G - \frac{c}{c-1} \}$.

If $c \neq 2$ then from (24), (25) and the second fundamental theorem, we obtain

$$\begin{aligned} 2nT(r,g) &\leq \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{G-c}) + \overline{N}(r,\frac{1}{G-\frac{c}{c-1}}) \\ &\leq \overline{N}(r,\frac{1}{g}) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{g-\alpha_1}) + \overline{N}(r,\frac{1}{g-\alpha_2}) \\ &+ (n-2)T(r,g) + \overline{N}(r,\frac{1}{f}) + S(r) \\ &\leq (n+3)T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction to $n \ge 6$. So that c = 2, and hence F = 2 - G. From this equation and (24), we deduce that

$$(n-4)T(r,g) \le \overline{N}(r,1/(G-2)) + S(r) = \overline{N}(r,1/f) + S(r) \le T(r,f) + S(r),$$

which is also a contradiction. Hence A = 1. Then, by (28), we have $F \equiv G$; and from (1) and (6) we obtain that

(29)
$$n(n-1)f^2g^2(f^{n-2}-g^{n-2})-2n(n-2)bfg(f^{n-1}-g^{n-1})+(n-1)(n-2)b^2(f^n-g^n)=0.$$

Letting h = f/g. It follows from (29) that

$$n(n-1)h^2g^2(h^{n-2}-1) - 2n(n-2)bhg(h^{n-1}-1) + (n-1)(n-2)b^2(h^n-1) = 0,$$

which implies

(30)
$$n^{2}(n-1)^{2}h^{2}g^{2}(h^{n-2}-1)^{2} - 2n^{2}(n-1)(n-2)bhg(h^{n-1}-1)(h^{n-2}-1) = -n(n-1)^{2}(n-2)b^{2}(h^{n}-1)(h^{n-2}-1).$$

By using Lemma 4 and (30), we conclude that

(31)
$$\{n(n-1)h(h^{n-2}-1)g - n(n-2)b(h^{n-1}-1)\}^2 = -n(n-2)b^2Q(h),$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n-6})$, and $\beta_j \in C \setminus \{0, 1\}$, $(j = 1, 2, \cdots, 2n-6)$, which are pairwise distinct. If h is not constant then from (31), we know that every zero of $h - \beta_j$ $(j = 1, 2, \cdots, 2n-6)$ is of order at least 2. By the second fundamental theorem, we obtain that

$$(2n-8)T(r,h) \leq \overline{N}(r,\frac{1}{h-\beta_1}) + \dots + \overline{N}(r,\frac{1}{h-\beta_{2n-6}}) + S(r,h) \leq \frac{1}{2}N(r,\frac{1}{h-\beta_1}) + \dots + \frac{1}{2}N(r,\frac{1}{h-\beta_{2n-6}}) + S(r,h) \leq (n-3)T(r,h) + S(r,h),$$

that is $n \leq 5$, which is a contradiction. This shows that h is a constant. It follows from (30) that $h^{n-2} - 1 = 0$ and $h^{n-1} - 1 = 0$; that means h = 1, and hence $f \equiv g$. This completes the proof of Theorem 1.

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