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On Semicommutative Modules and Rings

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Abstract. We say a module M_R a *semicommutative module* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$. This paper gives various properties of reduced, Armendariz, Baer, Quasi-Baer, *p.p.* and *p.q.*-Baer rings to extend to modules. In addition we also prove, for a p.p.-ring R, R is semicommutative iff R is Armendariz. Let R be an abelian ring and M_R be a *p.p.*-module, then M_R is a semicommutative module iff M_R is an Armendariz module. For any ring R, R is semicommutative iff $A(R, \alpha)$ is semicommutative. Let R be a reduced ring, it is shown that for number $n \geq 4$ and $k = [n/2], T_n^k(R)$ is semicommutative ring but $T_n^{k-1}(R)$ is not.

1. Introduction

Throughout this paper all rings R are associative with unity and all modules M are unital right R-modules. For a nonempty subset X of a ring R, we write $r_R(X) = \{r \in R \mid Xr = 0\}$ and $l_R(X) = \{r \in R \mid rX = 0\}$, which are called the *right annihilator of* X in R and the *left annihilator of* X in R, respectively. The notation "$\leq$" will denote a submodule. Recall that a ring R is *reduced* if R has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotents are central). In [5] Kaplansky introduced *Baer* rings as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. A ring R is called *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. These definitions are left-right symmetric. A ring R is called a *right* (resp. *left*) *principally quasi-Baer* (or simply *right* (resp. *left*) *p.q.-Baer*) ring if the right (resp. left) annihilator of a principally right (resp. left) ideal of R is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left *p.q.*-Baer.

Another generalization of Baer rings is a *p.p.*-ring. A ring R is called a *right*

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(resp. *left*) *p.p.*-ring if the right (resp. left) annihilator of an element of R is generated by an idempotent. R is called a *p.p.*-ring if it is both a right and left *p.p.*-ring. A ring R is called *Armendariz* if whenever polynomials $f(x) = \sum a_i x^i \in R[x], g(x) = \sum b_i x^i \in R[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$ for every i and j. A ring R is called *semicommutative* if for every $a \in R$, $r_R(a)$ is an ideal of R. (equivalently, for any $a, b \in R$, $ab = 0$ implies $aRb = 0$). An idempotent $e \in R$ is called *central* if $xe = ex$ for all $x \in R$. An idempotent $e^2 = e \in R$ is called a *left* (resp. *right) semicentral idempotent* if eR (resp. Re) is a two sided ideal of R.

According to Lee-Zhou [6], a module M_R is called *α-reduced* if, for any $m \in M$ and any $a \in R$,

(1) $ma = 0$ implies $mR \cap Ma = 0$

(2) $ma = 0$ iff $m\alpha(a) = 0$,

where $\alpha : R \longrightarrow R$ is a ring homomorphism with $\alpha(1) = 1$. The module M_R is called *reduced* if M_R is 1-reduced. In [8] Lee-Zhou introduced a Baer, quasi-Baer and *p.p.*- module as follows: (a) M_R is called *Baer* if, for any subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$. (b) M_R is called *quasi-Baer* if, for any submodule N of M, $r_R(N) = eR$ where $e^2 = e \in R$. (c) M_R is called *p.p.* if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$. In [2] the module M_R is called *principally quasi-Baer* (*p.q.-Baer* for short) if, for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$. In [3], the module M_R is *semicommutative* module if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$, and the module M_R is called *Armendariz* if whenever polynomials $m(x) = \sum m_i x^i \in M[x], f(x) = \sum a_i x^i \in R[x]$ satisfy $m(x)f(x) = 0$, we have $m_i a_j = 0$ for every i and j.

Let M be a right R-module and $S = End_R(M)$. Then M is a left S-module, right R-module and $S-R$-bimodule. In [11], Rizvi and Roman call M a *Baer module* if the right annihilator in M of any left ideal of S is generated by an idempotent of S(or equivalently, for all R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$); and M is a *quasi-Baer* module if the right annihilator in M of any ideal of S is generated by an idempotent of S(or equivalently, for all fully invariant R-submodules N of M, $l_S(N) = Se$ with $e^2 = e \in S$). Among others they have proved that any direct summand of a Baer (resp. quasi-Baer) module M is again a Baer (respect. quasi-Baer) module, and the endomorphism ring $S = End_R(M)$ of a Baer (resp. quasi-Baer) module M is a Baer (resp.quasi-Baer) ring (see Theorem 4.1 in [11]). They gave several results for a direct sum of Baer (resp. quasi-Baer) modules to be a Baer (resp. quasi-Baer) module.

We shortly summarize the content of the paper. In [1, Proposition 2.7] it is shown that if M_R is a semicommutative module, then M_R is a Baer module if and only if it is a quasi-Baer module, and M_R is a p.p.-module if and only if it is a p.q.-Baer module. In Proposition 2.7 we prove that for an abelian ring R and a p.p.-module M_R, M_R is a semicommutative module if and only if it is an Armendariz module. In Proposition 2.11 for a semicommutative ring R we show that R is a p.p.-ring if and only if $R[x]$ is a p.p.-ring, R is a Baer ring if and only if $R[x]$ is a Baer ring, R is a p.q.-Baer ring if and only if $R[x]$ is a p.q.-Baer ring, and R is a quasi-Baer ring if and only if $R[x]$ is a quasi-Baer ring. In Proposition 2.13 we prove that for any ring R, R is semicommutative if and only if $A(R,\alpha)$ is semicommutative, and in Theorem 2.15 for a reduced ring R and any integer $n \geq 4$ and $k = [n/2]$, we show that $T_n^k(R)$ is semicommutative ring but $T_n^{k-1}(R)$ is not.

Examples.

1. Every reduced module is semicommutative but the inverse is not true. For example, $\mathbb{Z}_n$ is semicommutative for any $n \in \mathbb{N}$, but is reduced for only square-free n.

2. For any commutative ring R, any module M_R is semicommutative.

3. Let D be a division ring, $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$, and $A = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$. Then A_R is a semicommutative module.

From [4, Example 2] and [10, Proposition 4.6] we want to restate next results:

1. If a module M_R is semicommutative then $M[x]_{R[x]}$ need not to be semicommutative.

2. If a semicommutative module M_R is Armendariz, then $M[x]_{R[x]}$ is a semicommutative module.

2. Semicommutative modules

We start with some preliminary results on semicommutative modules and rings. Some of the results are known but we state and give their proofs for the sake of completeness.

Lemma 2.1. *Let M_R be a semicommutative module.*
(1) *If $e^2 = e \in R$ with $r_R(m) = eR$ for some $m \in M$, then e is left semicentral idempotent.*
(2) *For any $e^2 = e \in R$, $mea = mae$ for all $m \in M$ and all $a \in R$.*

Proof. (1) Let $e^2 = e \in R$ with $r_R(m) = eR$ for some $m \in M$. Then $me = 0$. To prove e is left semicentral we show $teR \leq eR$ for any $t \in R$. For any $t \in R$, then $met = 0$. By (1), $mte = 0$. Hence $te \in r_R(m) = eR$, and so $teR \leq eR$ This completes the proof.
(2) See also [1] for a proof. For $e^2 = e \in R$, $e(1-e) = (1-e)e = 0$. Then for all $m \in M$, $me(1-e) = 0$ and $m(1-e)e = 0$. Since M_R is semicommutative, we have $meR(1-e) = 0$ and $m(1-e)Re = 0$. Thus, for all $a \in R$, $mea(1-e) = 0$ and $m(1-e)ae = 0$. So, $mea = meae$ and $mae = meae$. Hence, $mea = mae$ for all $a \in R$. □

Proposition 2.2. *Let M be a semicommutative module. Then the following conditions are equivalent*:
(1) *M_R is a p.q.-Baer module.*
(2) *The right annihilator of every finitely generated submodule is generated* (*as a right ideal*) *by an idempotent.*

Proof. (1)⇒(2) Assume that M_R is *p.q.*-Baer and $N = \sum_{i=1}^{k} n_i R$ is a finitely generated submodule of M_R. Then $r_R(N) = \bigcap_{i=1}^{k} e_i R$ where $r_R(n_i R) = e_i R$ and $e_i^2 = e_i$. By Lemma 2.1 each e_i is a left semicentral idempotent, a routine argument yields an idempotent e such that $\bigcap_{i=1}^{k} e_i R = eR$. Therefore, $r_R(N) = eR$. □

Corollary 2.3 [2, Prop. 1.7]**.** *Following conditions are equivalent for a ring R :*
(1) *R is a right p.q.-Baer ring.*
(2) *The right annihilator of every finitely generated ideal of R is generated* (*as a right ideal*) *by an idempotent.*

Proposition 2.4. *Let M_R be a semicommutative module. Consider the following properties*:
(1) *M_R is a Baer module.*
(2) *M_R is a quasi-Baer module.*
(3) *M_R is a p.p.-module.*
(4) *M_R is a p.q.-Baer module.*
Then (1) ⇔ (2) ⇒ (3) ⇔ (4).

Proof. See [1, Proposition 2.7]. □

Proposition 2.5. *Let R be an abelian ring and M_R be a p.p. module. Then M_R is a p.q.-Baer module.*

Proof. See [1, Proposition 2.15]. □

Corollary 2.6. *Abelian right p.p. rings are right p.q.-Baer.*

Proposition 2.7. *Let R be an abelian ring and M_R be a p.p.-module. Then following conditions are equivalent*:
(1) *M_R is a semicommutative module.*
(2) *M_R is an Armendariz module.*

Proof. (1)⇒(2) Let $m(x) = \sum m_i x^i \in M[x], f(x) = \sum a_j x^j \in R[x]$ satisfy $m(x)f(x) = 0$. We have the following system of equations:

$$\begin{aligned} m_0 a_0 &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (1)\\ m_0 a_1 + m_1 a_0 &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (2)\\ m_0 a_2 + m_1 a_1 + m_2 a_0 &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (3)\\ &\vdots\\ m_t a_s &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (n) \end{aligned}$$

for some s and t.

Let $r(m_0) = e_0 R, r(m_1) = e_1 R, r(m_2) = e_2 R, \cdots, r(m_t) = e_t R$ for some idempotents $e_0, e_1, e_2, \cdots, e_t$ in R. Then $a_0 \in r(m_0) = e_0 R$. Since R is abelian, $a_0 = e_0 a_0 = a_0 e_0$. Multiply (2) from right by e_0 to obtain

$m_0 a_1 e_0 + m_1 a_0 e_0 = 0$

From semicommutativity of M_R, $m_0 e_0 = 0$ implies $m_0 a_1 e_0 = 0$. Then $m_1 a_0 e_0 =$

$m_1a_0 = 0$. So $m_0a_1 = 0$ from (2).
Now multiply (3) from right by e_0:
$m_0a_2e_0 + m_1a_1e_0 + m_2a_0e_0 = 0$
From semicommutativity of M_R, $m_0e_0 = 0$ implies $m_0a_2e_0 = 0$. Then $m_1a_1e_0 + m_2a_0e_0 = m_1a_1 + m_2a_0 = 0$
Hence $m_0a_2 = 0$ by (3). Continuing this process we get:
$m_0a_0 = m_0a_1 = m_0a_2 = \cdots = m_0a_s = 0$.
If we use these equalities, the equations (2), (3), $\cdots$, will be :

$$\begin{aligned} m_1a_0 &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (2^{'}) \\ m_1a_1 + m_2a_0 &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (3^{'}) \\ &\vdots \\ m_ta_s &= 0 \quad \cdots\cdots\cdots\cdots\cdots \quad (n^{'}) \end{aligned}$$

Applying the same method to these equalities we get $m_1a_0 = m_1a_1 = m_1a_2 = \cdots = m_1a_s = 0$.
Continuing this process we will have:
$m_2a_0 = m_2a_1 = m_2a_2 = \cdots = m_2a_s = 0 \cdots m_ta_s = 0$.
So $m_ia_j = 0$ for any i, j.

(2)⇒(1): Let $ma = 0$ for $m \in M$, $a \in R$. For any $r \in R$ take $m(x) = mx+mr \in M[x]$ and $f(x) = -ax+ra \in R[x]$. Then $m(x)f(x) = (mx+mr)(-ax+ra) = mr^2a$. By hypothesis $ma = 0$ implies $a \in r(m) = eR$ for some idempotent $e \in R$. Then $a = ae = ea$. So, $m(x)f(x) = mr^2a = mr^2ea = mer^2a = 0$ since R is abelian. As M_R is Armendariz, we get $mra = 0$ for any $r \in R$. □

Corollary 2.8. *Let R be a p.p.-ring. Then the following are equivalent:*
(1) *R is a semicommutative ring.*
(2) *R is an Armendariz ring.*

Proof. In Lemma 2.1 take $M = R$, then every semicommutative ring is abelian for (1)⇒(2) and from [6, Lemma 7] every Armendariz ring is abelian for (2)⇒(1). □

Proposition 2.9. *Let M_R be a semicommutative module and R is a reduced module. Then M_R is an Armendariz module if and only if its torsion submodule $T(M)$ is Armendariz.*

Proof. Assume that the torsion submodule $T(M)$ of M is Armendariz as a right R-module. Let $m(x) = \sum m_jx^j \in M[x], f(x) = \sum a_ix^i \in R[x]$ satisfy $m(x)f(x) = 0$. We have the following system of equations:

$$\begin{aligned} m_0a_0 &= 0 \\ m_0a_1 + m_1a_0 &= 0 \\ m_0a_2 + m_1a_1 + m_2a_0 &= 0 \\ &\vdots \\ m_ta_s &= 0 \end{aligned}$$

for some s and t.

We may assume $a_0 \neq 0$. Multiplying by a_0, the second of these equations yields $m_1 a_0^2 = 0$. Thus a_0^2 annihilates both m_0 and m_1. The third equation now implies $m_2 a_0^3 = 0$. Continuing we get $m(x) \in T(M)[x]$. Since $T(M)$ is Armendariz as $R-$module, we conclude that $m_j a_i = 0$ for all $i,\ j$. The other implication is trivial. □

Given a ring R, the formal power series ring over R is denoted by $R[[x]]$.

Lemma 2.10. *Let R be a semicommutative ring.*
(1) *Every idempotent of $R[x]$ is in R.*
(2) *Every idempotent of $R[[x]]$ is in R.*

Proof. From [6, Lemma 8]. □

Proposition 2.11. *Let R be a semicommutative ring.*
(1) *R is a p.p.-ring if and only if $R[x]$ is a p.p.-ring.*
(2) *R is a Baer ring if and only if $R[x]$ is a Baer ring.*
(3) *R is a p.q.-Baer ring if and only if $R[x]$ is a p.q.-Baer ring.*
(4) *R is a quasi-Baer ring if and only if $R[x]$ is a quasi-Baer ring.*

Proof. (1) Assume that R is a p.p.-ring. From Corollary 2.16, R is an Armendariz ring. Then by [6, Theorem 9], $R[x]$ is a p.p.-ring.

Conversely, assume that $R[x]$ is a p.p.-ring. Let $a \in R$. By Lemma 2.10 there exists an idempotent $e \in R$ such that $r_{R[x]}(a) = eR[x]$. Hence $r_R(a) = r_{R[x]}(a) \cap R = eR$ and therefore R is a p.p.-ring.

(2) Assume that R is a Baer ring. Then R is a p.p.-ring. By Corollary 2.16 R is an Armendariz ring. From [6, Theorem 10] $R[x]$ is a Baer ring.

Conversely, assume that $R[x]$ is a Baer ring. Let B be a nonempty subset of R. Then $r_{R[x]}(B) = eR[x]$ for some idempotent $e \in R$ by Lemma 2.10. Hence $r_R(B) = eR$ and therefore R is a Baer ring.

(3) Assume that R is a p.q.-Baer ring. Let $t(x) = a_0 + a_1 x + \cdots + a_n x^k \in R[x]$. By assumption $r_R(a_i) = e_i R = r_R(a_i R)$, for all $i = 0, 1, 2, \cdots, n$. By Proposition 2.5 $\cap_{i=0}^{n} r_R(a_i R) = eR, e = e_0 e_1 \cdots e_n$. Let $f(x) \in r_{R[x]}(t(x)R[x])$. Then $t(x)R[x]f(x) = 0$ implies $t(x)Rf(x) = 0$ and $a_j R f(x) = 0$ for all $j = 0, 1, 2, ..., n$. So $a_j R b_i = 0$, hence $b_i \in \cap_{i=0}^{n} r_R(a_j R) = eR$ and $b_i = eb_i$ for all $i,\ j$. Then $ef(x) = f(x)$ implies $f(x) \in eR[x]$.

Conversely, assume that $R[x]$ is a p.q.-Baer. Let $a \in R$. There exists idempotent $e \in R$ such that $r_{R[x]}(aR[x]) = eR[x]$. Then $r_{R[x]}(aR[x]) \cap R = (eR[x]) \cap R = eR$. Since $r_R(aR) = r_{R[x]}(aR[x]) \cap R$, we get $r_R(aR) = eR$.

(4) Assume that R is a quasi-Baer. Let A be an ideal of $R[x]$ and A^* be the set of all coefficients of elements of A. Then A^* is an ideal of R, so $r_R(A^*) = eR$ for some idempotent $e \in R$. Since $e \in r_{R[x]}(A)$, we get $eR[x] \subseteq r_{R[x]}(A)$. Now, let $f = b_0 + b_1 x + \cdots + b_n x^n \in r_{R[x]}(A)$. Then $Af = 0$ implies $Ab_i = 0$, so $A^* b_i = 0$ for all $i = 0, 1, 2, \cdots, n$. Hence $b_i \in r_R(A^*) = eR$ and $b_i = eb_i$ for all i. Consequently, $f \in eR[x]$.

Conversely, assume that $R[x]$ is a quasi-Baer ring and A is an ideal of R. Then $A[x]$ is an ideal of $R[x]$. Hence $r_{R[x]}(A[x]) = eR[x]$. Intersecting both sides with R we get $r_{R[x]}(A[x]) \cap R = eR[x] \cap R = eR$. Since $r_R(A) = r_{R[x]}(A[x]) \cap R$, we have $r_R(A) = eR$. □

Proposition 2.12. *Let M_R be a p.p.-module. Then M_R is a semicommutative module if and only if $mre = mer$, for any $m \in M, r \in R$, and $e^2 = e \in R$.*

Proof. Assume that $mre = mer$, for any $m \in M, r \in R$ and $e^2 = e \in R$. Let $ma = 0$ for $m \in M$, $a \in R$. Then $a \in r(m)$. By hypothesis $r(m) = eR$ for some $e^2 = e \in R$. Hence $me = 0$, $a = ea$, and so $mra = mrea$. By assumption $mrea = mrae = mera = 0$ for any $r \in R$. So $mra = 0$ for any $r \in R$. The rest is clear from Lemma 2.1. □

Now we consider D. A. Jordan's construction of the ring $A(R, \alpha)$ (See [5] for more details). Let $A(R, \alpha)$ or A be the subset $\{x^{-i}rx^i \mid r \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, where $\alpha : R \mapsto R$ is an injective ring endomorphism of a ring R. Elements of $R[x, x^{-1}; \alpha]$ are finite sums of elements of the form $x^{-j}rx^i$ where $r \in R$ and $i,\ j$ are non-negative integers. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$. Note that for each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set $A(R, \alpha)$ of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with

$$\begin{aligned} x^{-i}rx^i + x^{-j}sx^j &= x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)} \\ (x^{-i}rx^i)(x^{-j}sx^j) &= x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)} \end{aligned}$$

for $r,\ s \in R$ and $i,\ j \geq 0$.

Proposition 2.13. *The following are equivalent for a ring R:*
(1) *R is semicommutative.*
(2) *$A(R, \alpha)$ is semicommutative.*

Proof. (1) ⇒ (2). Let $(x^{-i}rx^i)(x^{-j}sx^j) \in A(R, \alpha)$. Suppose that $(x^{-i}rx^i)(x^{-j}sx^j) = 0$. Then $x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)} = 0$ and so $\alpha^j(r)\alpha^i(s) = 0$. Hence $\alpha^k(\alpha^j(r)\alpha^i(s)) = \alpha^{k+j}(r)\alpha^{k+i}(s) = 0$, and $\alpha^{j+k}(r)\alpha^{j+i}(t)\alpha^{i+k}(s) = 0$ by (1). For any $x^{-k}tx^k \in A(R, \alpha)$

$$\begin{aligned} (x^{-i}rx^i)(x^{-k}tx^k)(x^{-j}sx^j) &= x^{-(i+k)}(\alpha^k(r)\alpha^i(t))x^{(i+k)}(x^{-j}sx^j) \\ &= x^{-(i+k+j)}\alpha^j(\alpha^k(r)\alpha^i(t))\alpha^{i+k}(s)x^{(i+k+j)} \\ &= x^{-(i+k+j)}\alpha^{j+k}(r)\alpha^{j+i}(t)\alpha^{i+k}(s)x^{(i+k+j)}. \end{aligned}$$

(2) ⇒ (1) From the fact that $R \leq A(R, \alpha)$, R is semicommutative. □

Proposition 2.14. *Assume that the ring $S = R[x]/(x^n)$ is a semicommutative ring for any $n = 2, 3, \cdots$. Then R is a semicommutative ring.*

Proof. Let $ab = 0$. Take $f(x) = a + (x^n)$, $f(x) = b + (x^n) \in R[x]/(x^n) = S$. Then

$f(x)g(x) = 0_S$. By assumption $(a + (x^n))((r + (x^n))(b + (x^n)) = 0_S$. So $arb \in (x^n)$ implies $arb = 0$ for any $r \in R$. $\square$

For a reduced ring R, it is interesting to find which subrings of $T_n(R)$ are semicommutative. For this purpose, we introduce some notation. For number $n \geq 4$ and any m from set $\{1, \cdots, n\}$, we let

$$T_n^m(R) = \{\sum_{i=j}^{n}\sum_{j=1}^{m} a_j E_{(i-j+1)i} + \sum_{i=j}^{n-m}\sum_{j=1}^{n-m} r_{ij} E_{j(m+i)} : a_j, r_{ij} \in R\},$$

where $\{E_{i,j} : 1 \leq i, j \leq n\}$ are the matrix units. Then each element of $T_n^m(R)$ has the matrix form

$$\begin{bmatrix} a_1 & a_2 & ... & a_m & a_{1(m+1)} & ... & a_{1n} \\ 0 & a_1 & ... & a_{m-1} & a_m & ... & a_{2n} \\ 0 & 0 & a_1 & ... & & & a_{3n} \\ & & & ... & & & \\ & & & & & & a_1 \end{bmatrix},$$

where $a_1, \cdots, a_m, a_{1(m+1)}, \cdots, a_{(n-m)n} \in R$.

Theorem 2.15. *Let R be a reduced ring. Then for number $n \geq 4$ and $k = [n/2]$, $T_n^k(R)$ is semicommutative ring but $T_n^{k-1}(R)$ is not.*

Proof. Let $A = \sum_{i=1}^{n-k} E_{i(i+k-1)}$, $B = E_{(n-k+1)n} \in T_n^{k-1}(R)$. Then $AB = 0$. But for $C = \sum_{j=i}^{n}\sum_{i=1}^{n} E_{ij} \in T_n^{k-1}(R)$, $ACB \neq 0$. So $T_n^{k-1}(R)$ is not semicommutative. To complete the proof that $T_n^k(R)$ is semicommutative ring for $n \geq 4$ and $k = [n/2]$, it is enough to consider the case $n = 5$. The same proof will work for any $n \geq 4$ and $k = [n/2]$. Let $n = 5$. Then $k = 2$. Let

$$A = \begin{bmatrix} a_1 & a_2 & a_{13} & a_{14} & a_{15} \\ 0 & a_1 & a_2 & a_{24} & a_{25} \\ 0 & 0 & a_1 & a_2 & a_{35} \\ 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & b_{13} & b_{14} & b_{15} \\ 0 & b_1 & b_2 & b_{24} & b_{25} \\ 0 & 0 & b_1 & b_2 & b_{35} \\ 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & b_1 \end{bmatrix}$$

be elements of $T_5^2(R)$ and $AB = 0$. We show that each term in the following system of equations obtained from $AB = 0$ is zero:

$$A_1B^1 : a_1b_1 = 0 \quad \cdots\cdots\cdots \quad (1)$$
$$A_1B^2 : a_1b_2 + a_2b_1 = 0 \quad \cdots\cdots\cdots \quad (2)$$
$$A_1B^3 : a_1b_{13} + a_2b_2 + a_{13}b_1 = 0 \quad \cdots\cdots\cdots \quad (3)$$
$$A_1B^4 : a_1b_{14} + a_2b_{24} + a_{13}b_2 + a_{14}b_1 = 0 \quad \cdots\cdots\cdots \quad (4)$$
$$A_1B^5 : a_1b_{15} + a_2b_{25} + a_{13}b_{35} + a_{14}b_2 + a_{15}b_1 = 0 \quad \cdots\cdots\cdots \quad (5)$$
$$A_2B^4 : a_1b_{24} + a_2b_2 + a_{24}b_1 = 0 \quad \cdots\cdots\cdots \quad (6)$$

$$A_2B^5 : a_1b_{25} + a_2b_{35} + a_{24}b_2 + a_{25}b_1 = 0 \quad \cdots\cdots\cdots \quad (7)$$
$$A_3B^5 : a_1b_{35} + a_2b_2 + a_{35}b_1 = 0 \quad \cdots\cdots\cdots \quad (8)$$

To prove each term in these equations($n = 5$ or any other n) is zero, we will proceed as follows: For $1 \leq j \leq k$ we show all terms of A_1B^j are zero. Next for $0 \leq i \leq n-k-2$ we prove each term in the equations $A_{n-k-i}B^{n-i}, A_{n-k-i}B^{n-i+1} ..., A_{n-k-i}B^n$ is zero. By using preceeding results finally we show each term of the equations $A_1B^{k+1}, \cdots, A_1B^n$ is zero.

Note that by hypothesis from $rs = 0$ for any r and s in R we get $sr = 0$ and $rRs = 0$. Also from $r^2s = 0$ we have $(rs)^2 = 0$ and so $Rs = sr = 0$. We make use these implications without referring to the hypothesis. Now multiply (2) from left by a_1, we have $a_1^2b_2 + a_1a_2b_1 = 0$. By (1) and hypothesis, $a_1a_2b_1 = 0$. So $a_1^2b_2 = 0$ and then $a_1b_2 = 0$. From (2), $a_2b_1 = 0$. Left multiplying (3) by a_1, we have $a_1^2b_{13} = 0$. Hence $a_1b_{13} = 0$. Then (3) becomes $a_2b_2 + a_3b_1 = 0$. Left multiplying this equation by a_2, we have $a_2^2b_2 = 0 = a_2b_2$. Hence $a_{13}b_1 = 0$ from (3). Hence each term in the equations (1), (2) and (3) are zero.
Now we left multiply (8) by a_1 and obtain $a_1^2b_{35} = 0$ since $a_1b_2 = 0$ and $a_1b_1 = 0$ imply $a_1a_2b_2 + a_1a_{35}b_1 = 0$. From (8) $a_2b_2 + a_{35}b_1 = 0$. Left multiply the latter by a_2 and use $a_2b_1 = 0$ we get $a_2^2b_2 = 0$. Hence $a_2b_2 = 0$. By (8) $a_{35}b_1 = 0$.
Left multiply (6) by a_1 and use $a_1b_2 = 0$ and $a_1b_1 = 0$ to obtain $a_1b_{24} = 0$. From (6) $a_{24}b_1 = 0$ since $a_2b_2 = 0$.
Left multiply (7) by a_1 and use $a_1b_{35} = 0$, $a_1b_2 = 0$ and $a_1b_1 = 0$ to obtain $a_1b_{25} = 0$. (7) induces to $a_2b_{35} + a_{24}b_2 + a_{25}b_1 = 0$. We left multiply the latter by a_2 to obtain $a_2b_{35} = 0$. From (7) we have $a_{24}b_2 + a_{25}b_1 = 0$. Left multiply this by b_1 and use $a_{24}b_1 = 0$ to obtain $b_1^2a_{25} = 0$. Hence $b_1a_{25} = 0$. Now we go to the equation (5) to left multiply it by a_1 and use $a_1b_{35} = 0$, $a_1b_{25} = 0$, $a_1b_2 = 0$, $a_1b_1 = 0$ to get $a_1^2b_{15} = 0$. Hence $a_1b_{15} = 0$. From (5) we have $a_2b_{25} + a_{13}b_{35} + a_{14}b_2 + a_{15}b_1 = 0$. Similarly this procedure continues to obtain each term in the latter equation is zero: $a_2b_{25} = 0$, $a_{13}b_{35} = 0$, $a_{14}b_2 = 0$, $a_{15}b_1 = 0$. As for (4), left multiply it by a_1 to get $a_1^2b_{14} = 0$ since $a_1b_{24} = 0$, $a_1b_2 = 0$ and $a_1b_1 = 0$. So $a_1b_{14} = 0$. From (4) $a_2b_{24} + a_{13}b_2 + a_{14}b_1 = 0$. Left multiply it by a_2 and use $a_2b_2 = 0$ and $a_2b_1 = 0$ to obtain similarly $a_2b_{24} = 0$. We are left with $a_{13}b_2 + a_{14}b_1 = 0$. Left multiply it by b_2 and use $a_{14}b_2 = 0$ to obtain $b_2^2a_{13} = 0$. Hence $b_2a_{13} = 0$. Thus $a_{14}b_1 = 0$. Since R is semicommutative, the rest of the proof is clear. □

Corollary 2.16. *Let R be a prime ring. Then $R[x]/(x^n)$ is Armendariz if and only if $R[x]/(x^n)$ is semicommutative.*

Proof. Clear from [9, Corollary 1.5] and Theorem 2.23. □

Corollary 2.17. *If R is a reduced ring then $R[x]/(x^n)$ is semicommutative.*

Corollary 2.18. *Let R be an Armendariz ring. Then R is semicommutative if and only if $R[x]/(x^n)$ is semicommutative.*

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