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Oscillation and Nonoscillation of Nonlinear Neutral Delay Differential Equations with Several Positive and Negative Coefficients

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Abstract. In this paper, oscillation and nonoscillation criteria are established for nonlinear neutral delay differential equations with several positive and negative coefficients

$$[x(t) - R(t)x(t-r)]' + \sum_{i=1}^{m} P_i(t) H_i(x(t-\tau_i)) - \sum_{j=1}^{n} Q_j(t) H_j(x(t-\sigma_j)) = 0.$$

Our criteria improve and extend many results known in the literature. In addition we prove that under appropriate hypotheses, if every solution of the associated linear equation with constant coefficients,

$$y'(t) + \sum_{i=1}^{m}(p_i - \sum_{k\in J_i} q_k) y(t-\tau_i) = 0,$$

oscillates, then every solution of the nonlinear equation also oscillates.

1. Introduction

In recent years there has been much research activity concerning the linearized oscillation theory of the nonlinear delay differential equations, which in some sense parallels the so called linearized stability theory for differential equations are important in applications in Physics, Ecology, Biology and many applications in epidemics and infectious diseases. For some contributions to this topic we refer to the articles by Kulenovic, Ladas and Meimardou [12], Kulenovic and Ladas ([13], [14], [15]), Gopalsamy, Kulenovic and Ladas [7], Ladas and Qian([16], [17]), Rodica [27], Wei Jun Jie [17], Gyori and Ladas [8], Hiroshi [10], Hua and Joinshe [11], Xiping, Jun and Sui Sun [28], Qirui [21] and Norio [20]. They are considered the case with several coefficients and several delays but with positive coefficients, the oscillation of linear neutral delay differential equation with positive and negative coefficients has been investigate by many authors. See, for example, [2], [5], [6], [8], [23], [24], [26], [32] and the references cited therein and Elabbasy and Saker [4] studied the oscillation of nonlinear delay differential equation.

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Our aim in this paper is to investigate some oscillation results for the nonlinear, nonautonomous neutral delay differential equation with several positive and negative coefficients. our results partially solve an open problem posed by Gyori and Ladas [8] in Chapter 4.

Consider the nonlinear neutral delay differential equation with several positive and negative coefficients

(1)
$$[x(t)-R(t)x(t-r)]'+\sum_{i=1}^{m}P_i(t)H_i(x(t-\tau_i))-\sum_{j=1}^{n}Q_j(t)H_j(x(t-\sigma_j))=0,$$

where we will assume that the following hypotheses are satisfied:
(A_1) $P_i, Q_j, R\in C([t_0,\infty),R^+)$, $H_i, H_j\in C[R,R]$, $r\in(0,\infty)$ and $\tau_i,\sigma_j\in R^+$ for $i=1,2,\cdots,m$ and $j=1,2,\cdots,n$.
(A_2) There exists a positive number $p\le m$ and a partion of the set $(1,2,\cdots,n)$ into p disjoint subsets $J_1,J_2,\cdots,J_p$ such that $k\in J_i$ implies that

$$uH_k(u)>0 \quad \text{for } u\ne 0, \qquad \lim_{u\to 0}\frac{H_i(u)}{u}=1, \qquad H_i\ge\sum_{k\in J_i}H_k,\ \sigma_k\le\tau_i$$

and there exist a positive constant δ such that either

$$H_i(u)\le u \quad \text{for } u\in[0,\delta]$$

or

$$H_i(u)\ge u \quad \text{for } u\in[-\delta,0]$$

(A_3)

$$\begin{aligned} F_i(t) &:= P_i(t)-\sum_{k\in J_i}Q_k(t-\tau_i+\sigma_k)\ge 0\ (\ne 0)\ \text{ for } i=1,2,\cdots,p,\\ F_i(t) &:= P_i(t)\ \text{ for } i=p+1,\cdots,m; \end{aligned}$$

(A_4)
$$\varrho=\max\{r,\tau_i,\sigma_j:1\le i\le m,\ 1\le j\le n\}$$

and

$$\delta=\min\{r,\tau_i,\sigma_j:1\le i\le m,\ 1\le j\le n\}.$$

(A_5) There exist positive constants M_i such that

$$\frac{H_i(u)}{u}\le M_i \text{ and } 1-\sum_{i=1}^{p}\sum_{k\in J_i}M_iq_k(\sigma_i-\tau_k)>0.$$

A function $x(t)\in C([t_1-\varrho,\infty),R)$ is said to be a solution of equation (1) for some $t_1\ge t_0$ if $x(t)-R(t)x(t-r)$ is continuously differentiable on $[t_1,\infty)$ and satisfies (1) for $t>t_1$.

As is customary, a solution of (1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

For convenience, we will assume that all inequalities concerning the values of functions are satisfied eventually for all large t.

2. Some lemmas

We need the following lemmas for the proofs of our main results.

Lemma 1 ([8]). *Let $a \in (-\infty, 0)$, $\tau \in (0, \infty)$, $t_0 \in R$ and suppose that $x(t) \in C[[t_0, \infty), R]$ satisfies the inequality*

$$x(t) \leq a + \max_{t-\tau \leq s \leq t} x(s) \quad \textit{for } t \geq t_0.$$

Then $x(t)$ cannot be non-negative function.

Lemma 2 ([8]). *Let $v(t)$ be a positive and continuously differentiable function on some interval $[t_0, \infty)$. Assume that there exist positive numbers A and α such that for t sufficiently large $\dot{v}(t) \leq 0$ and $v(t-\tau) < Av(t)$ Set $\Lambda = \{\lambda \geq 0 : \dot{v}(t) + \lambda v(t) \leq 0 \quad \textit{for } t \textit{ sufficiently large}\}$. Then $A > 1$, $\lambda_0 = \frac{\ln(A)}{\alpha} \notin \Lambda$.*

Lemma 3 ([8]). *Let p and τ be positive constants, let $z(t)$ be an eventually positive solution of the delay differential inequality $\dot{z}(t) + pz(t-\tau) \leq 0$. Then for t sufficiently large $z(t-\tau) < \beta z(t)$, where $\beta = \frac{4}{(p\tau)^2}$.*

Lemma 4. *Assume that*

$$R(t) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s) ds \leq 1. \tag{2}$$

Let $x(t)$ be an eventually positive solution of the differential inequality

$$[x(t) - R(t) x(t-r)]' + \sum_{i=1}^{m} P_i(t) H_i(x(t-\tau_i)) - \sum_{j=1}^{n} Q_j(t) H_j(x(t-\sigma_j)) \leq 0, \tag{3}$$

and set

$$z(t) = x(t) - R(t) x(t-r) - \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s) H_k(x(s-\sigma_k)) ds. \tag{4}$$

Then

$$z'(t) \leq 0, \; z(t) > 0 \textit{ and } z'(t) + \sum_{i=1}^{m} F_i(t)(1-\varepsilon_i) z(t-\tau_i) \leq 0, \; \varepsilon_i \in (0,1). \tag{5}$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq.(1). We will assume that $x(t)$ is eventually positive (The case where $x(t)$ is eventually negative is similar and will be omitted). Assume that $t_1 \geq t_0 + \varrho$ is such that $x(t) > 0$ for $t \geq t_1$. Then by (3) and (4), we get

$$\begin{aligned} z'(t) &= (x(t) - R(t)x(t-r))' - \sum_{i=1}^{p} \sum_{k\in J_i} Q_k(t) H_k(x(t-\sigma_k)) \\ &\quad + \sum_{i=1}^{p} \sum_{k\in J_i} Q_k(t-\tau_i+\sigma_k) H_k(x(t-\tau_i)). \end{aligned}$$

Hence

$$\begin{aligned} z'(t) &= (x(t) - R(t)x(t-r))' - \sum_{j=1}^{n} Q_j(t) H_j(x(t-\sigma_j)) \\ &\quad + \sum_{i=1}^{p} \sum_{k\in J_i} Q_k(t-\tau_i+\sigma_k) H_k(x(t-\tau_i)). \end{aligned}$$

From (1) we have

$$\begin{aligned} z'(t) &= -\sum_{i=1}^{p} P_i(t) H_i(x(t-\tau_i)) + \sum_{i=1}^{p} \sum_{k\in J_i} Q_k(t-\tau_i+\sigma_k) H_k(x(t-\tau_i)) \\ &\quad - \sum_{i=p+1}^{m} P_i(t) H_i(x(t-\tau_i)). \end{aligned}$$

Hence from (A_1), (A_2) and (A_3) we fined

$$z'(t) + \sum_{i=1}^{m} F_i(t) H_i(x(t-\tau_i)) \leq 0. \tag{6}$$

Then from (A_2) and (A_3) we have

$$z'(t) \leq 0.$$

Thus by (A_2)

$$\lim_{t\to\infty} \frac{H_i(x(t-\tau_i))}{x(t-\tau_i)} = 1.$$

Let $\varepsilon_i \in (0,1)$ then there exists a T_{ε_i} such that for $t \geq T_{\varepsilon_i}$, $x(t-\tau_i) > 0$ and

$$H_i(x(t-\tau_i)) \geq (1-\varepsilon_i) x(t-\tau_i) \quad \text{for } i = 1, 2, \cdots, m.$$

From (6) we have

$$z'(t) + \sum_{i=1}^{m} F_i(t)(1-\varepsilon_i) x(t-\tau_i) \leq 0. \tag{7}$$

In view of $x(t) \geq z(t)$. Hence

$$z'(t) + \sum_{i=1}^{m} F_i(t)(1-\varepsilon_i) z(t-\tau_i) \leq 0.$$

Now we show that $z(t)$ is positive. For otherwise there exists a $t_2 \geq t_1$ such that $z(t_2) \leq 0$. Because $z'(t) \leq 0$ and so there exists $t_3 \geq t_2$ and $\mu > 0$ such that $z(t) \leq -\mu$ for $t \geq t_3$. Hence

$$\begin{aligned} x(t) &= z(t) + R(t) x(t-r) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s) H_k(x(s-\sigma_k))\, ds \\ &\leq -\mu + R(t) x(t-r) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s) H_k(x(s-\sigma_k))\, ds. \end{aligned}$$

From (A_2) we find

$$x(t) \leq -\mu + R(t) x(t-r) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)(x(s-\sigma_k))\, ds$$

Hence

$$x(t) \leq -\mu + \max_{t-\varrho \leq s \leq t} x(s) \left(R(t) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\, ds \right).$$

From (2) we have

$$x(t) \leq z(t_3) + \max_{t-\varrho \leq s \leq t} x(s), \quad \text{for } t \geq t_3.$$

It follows by Lemma 1 that $x(t)$ cannot be a nonnegative function on $[t_3, \infty)$, which contradicts to $x(t) > 0$. The proof is complete. □

Lemma 5. *Assume that*

$$R(t) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)(1-\varepsilon_k)\, ds \geq 1. \tag{8}$$

Let $x(t)$ be an eventually positive solution of (3) *and let $z(t)$ be defined by* (4). *Then the oscillation of all solutions of the second order ordinary differential equation*

$$y''(t) + \varrho^{-1} \sum_{i=1}^{m} F_i(t) y(t) = 0, \quad t \geq t_0 \tag{9}$$

implies that $z'(t) \leq 0$ and $z(t) < 0$ eventually.

Proof. From (7) we have

$$z'(t) \leq -\sum_{i=1}^{m} F_i(t)(1-\varepsilon_i)\, x(t-\tau_i) \leq 0. \tag{10}$$

Therefore, if $z(t) < 0$ does not hold eventually, then $z(t) > 0$ eventually. Let $t_1 > t_0 + \varrho$ be such that $x(t-\varrho) > 0,\ z(t) > 0$ for $t \geq t_1$. Set

$$N = 2^{-1} \min\{x(t) : t_1 - \varrho \leq t \leq t_1\}.$$

Then $x(t) > N$ for $t_1 - \varrho \leq t \leq t_1$. We claim that

$$x(t) > N, \quad t \geq t_1. \tag{11}$$

If (11) does not hold, then there exists a $t^* > t_1$ such that $x(t) > N$ for $t_1 - \varrho \leq t \leq t^*$ and $x(t^*) = N$. By (4) and (8) we get

$$\begin{aligned} N &= x(t^*) = z(t^*) + R(t)\, x(t-r) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\, H_k(x(s-\sigma_k))\, ds \\ &> \left(R(t) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)(1-\varepsilon_k)\, ds \right) N \geq N. \end{aligned}$$

This is a contradiction and so (11) holds. Let $\lim_{t\to\infty} z(t) = a$. There exist two possible cases:
Case I. $a = 0$. There exists a $T_1 > t_1$ such that $z(t) < \frac{N}{2}$ for $t \geq T_1$. Then for any $\bar{t} > T_1$, we have

$$\frac{1}{\varrho} \int_{\bar{t}}^{t+\varrho} z(s)\, ds \leq N < x(t),\ t \in \left[\bar{t}, \bar{t} + \varrho\right].$$

Case II. $a > 0$. Then $z(t) \geq a$ for $t \geq t_1$. From (4) and (11) we get

$$x(t) = a + R(t)\, x(t-r) + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\, H_k(x(s-\sigma_k))\, ds \geq a + N,\ t \geq t_1.$$

By induction, it is easy to see that $x(t) \geq ka + N$ for $t \geq t_1 + (k-1)\varrho$ and so $\lim_{t\to\infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\frac{1}{\varrho} \int_{T}^{t+\varrho} z(s)\, ds \leq 2z(T) < x(t),\ t \in [T, T+\varrho].$$

Combining the cases I and II we see that

$$x(t) > \frac{1}{\varrho} \int_{T}^{t+\varrho} z(s)\, ds \leq 2z(T),\ t \in [T, T+\varrho].$$

Now we prove that

$$x(t) > \frac{1}{\varrho}\int_{T}^{t+\varrho} z(s)\,ds,\ t \geq T+\varrho. \tag{12}$$

Otherwise, there would exists a $t^* > T+\varrho$ such that

$$x(t^*) = \frac{1}{\varrho}\int_{T}^{t^*+\varrho} z(s)\,ds,$$

$$x(t) > \frac{1}{\varrho}\int_{T}^{t+\varrho} z(s)\,ds \leq 2z(T) \quad \text{for } t \in (T+\varrho, t^*).$$

Then, from (4) and (8), we have

$$\begin{aligned} & \frac{1}{\varrho}\int_{T}^{t^*+\varrho} z(s)\,ds \\ = \quad & z(t^*) + R(t^*)\,x(t^*-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t^*-\tau_i+\sigma_k}^{t^*} Q_k(s)\,H_k(x(s-\sigma_k))\,ds \\ > \quad & \frac{1}{\varrho}\int_{t^*}^{t^*+\varrho} z(s)\,ds + \left(R(t^*) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t^*-\tau_i+\sigma_k}^{t^*} Q_k(s)(1-\varepsilon_k)\,ds\right)\frac{1}{\varrho}\int_{T}^{t^*} z(s)\,ds \\ \geq \quad & \frac{1}{\varrho}\int_{T}^{t^*+\varrho} z(s)\,ds. \end{aligned}$$

This is a contradiction and so (12) holds. Thus, for $t > T+\varrho$, we obtain

$$x(t-\tau_i) > \frac{1}{\varrho}\int_{T}^{t} z(s)\,ds. \tag{13}$$

Substituting (13) into (10) leads to

$$z'(t) + \sum_{i=1}^{m} F_i(t)(1-\varepsilon_i)\left(\frac{1}{\varrho}\int_{T}^{t} z(s)\,ds\right) \leq 0, \quad t > T+\varrho.$$

Set

$$y(t) = \int_{T}^{t} z(s)\,ds,\ t > T+\varrho.$$

Then $y'(t) = z(t)$, $y''(t) = z'(t)$ and

$$y''(t) + \frac{1}{\varrho}\sum_{i=1}^{m} F_i(t)(1-\varepsilon_i)\,y(t) \leq 0, \quad t > T+\varrho.$$

By Lemma 2.4 in [25], Eq. (9) has an eventually positive solution. This is a contradiction and the proof is complete. □

Lemma 6. *Assume that*

$$R(t)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)(1-\varepsilon_k)\,ds\equiv 1. \tag{14}$$

Then the fact that the inequality (3) *has an eventually positive solution* $x(t)$ *implies that Eq.* (1) *has a solution* $\bar{x}(t)$ *which satisfies* $0<\bar{x}(t)\le x(t)$ *eventually.*

Proof. Let $z(t)$ be defined by (4). By Lemma 2.1 there exists a $t_1>t_0$ such that $x(t-\varrho)>0,\ z(t)>0$ and $z'(t)\le 0$ for $t\ge t_1$. Set

$$M=2^{-1}\min\{x(t):t_1-\varrho\le t\le t_1\}.$$

Then $x(t)>M$ for $t\ge t_1-\varrho$. From (4) and (5) we have

$$\begin{aligned} x(t) &\ge R(t)x(t-r)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)H_k(x(s-\sigma_k))\,ds \\ &\quad+\int_t^\infty\sum_{i=1}^{m}F_i(s)H_i(x(s-\tau_i))\,ds,\quad t\ge t_1. \end{aligned} \tag{15}$$

Define a sequence of functions $\{x_v(t)\}$ by $x_0(t)=x(t)$ and for $v=1,2,\cdots$ by

$$\begin{aligned} x_v(t) &= R(t)x_{v-1}(t-r)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)H_k(x_{v-1}(s-\sigma_k))\,ds \\ &\quad+\int_t^\infty\sum_{i=1}^{m}F_i(s)H_i(x_{v-1}(s-\tau_i))\,ds,\quad t\ge t_1+\varrho \end{aligned} \tag{16}$$

and

$$x_v(t)=M+\frac{x_v(t_1+\varrho)-M}{x(t_1+\varrho)-M}(x(t)-M),\quad t_1\le t<t_1+\varrho.$$

Then, from (15) and (16), we have for $t\ge t_1+\varrho$

$$\begin{aligned} x_0(t) &= x(t)\ge x_1(t) \\ &= R(t)x(t-r)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)H_k(x(s-\sigma_k))\,ds \\ &\quad+\int_t^\infty\sum_{i=1}^{m}F_i(s)H_i(x(s-\tau_i))\,ds \\ &\ge\left(R(t)+\sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t}Q_k(s)(1-\varepsilon_k)\,ds\right)M=M. \end{aligned}$$

For $t_1 \le t < t_1 + \varrho$ we have

$$x_0(t) = x(t) \ge M + \frac{x_1(t_1+\varrho) - M}{x(t_1+\varrho) - M}(x(t) - M) = x_1(t) \ge M.$$

Thus, $x_0(t) \ge x_1(t) \ge M$ for $t \ge t_1$. By induction, one can easily prove that

$$x_v(t) \ge x_{v+1}(t) \ge M, \quad t \ge t_1,\ v = 1, 2, \cdots$$

Therefore, $\{x_v(t)\}$ has a positive limit function $\bar{x}(t)$ with

$$0 < M \le \lim_{v\to\infty} x_v(t) = \bar{x}(t) \le x(t) \quad \text{for} \quad t \ge t_1.$$

By the Monotone Convergence Theorem we have

$$\bar{x}(t) = R(t)\bar{x}(t-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s) H_k\left(\bar{x}(s-\sigma_k)\right) ds$$

$$+ \int_t^\infty \sum_{i=1}^{m} F_i(s) H_i\left(\bar{x}(s-\tau_i)\right) ds, \quad t \ge t_1 + \varrho.$$

This implies that for $t \ge t_1 + \varrho$

$$\left[\bar{x}(t) - R(t)\bar{x}(t-r)\right]' + \sum_{i=1}^{m} P_i(t) H_i\left(\bar{x}(t-\tau_i)\right) - \sum_{j=1}^{n} Q_j(t) H_j\left(\bar{x}(t-\sigma_j)\right) = 0.$$

The proof is complete. □

Lemma 7. *Assume that* (14) *holds with* $\delta > 0$. *Then Eq.* (1) *has an eventually positive solution if the second order ordinary differential equation*

$$y''(t) + \delta^{-1}\sum_{i=1}^{m} H_i(t)\, y(t) = 0,\ t \ge t_0 \tag{17}$$

has an eventually positive solution.

Proof. Let $y(t)$ be an eventually positive solution of (17). Then there exists a $t_1 > t_0$ such that $y(t) > 0$, $y''(t) \le 0$ and $y'(t) > 0$ for $t \ge t_1$. Define a function $x(t)$ by

$$x(t) = \delta^{-1} y(t), \quad t_1 \le t \le t_1 + \varrho - \delta,$$

$$x(t) = \delta^{-1}\{y(t) + (t - t_1 - \varrho + \delta)\, y'(t+\varrho)\},\ t_1 + \varrho - \delta \le t \le t_1 + \varrho,$$

and

$$x(t) = y'(t) + R(t)\, x(t-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s) H_k(x(s-\sigma_k))\, ds,$$

$$t_1 + \varrho + l\delta < t \le t_1 + \varrho + (l+1)\delta, \quad l = 0, 1, \cdots.$$

Then $x(t)$ is continuous and positive for $t \geq t_1$, and

(18)
$$y'(t) = x(t) - R(t)\,x(t-r) - \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,H_k(x(s-\sigma_k))\,ds, \quad t \geq t_1.$$

Since $y'(t) > 0$ and $y''(t) \leq 0$, we have for $t_1 + \varrho - \delta \leq t \leq t_1 + \varrho$

$$y(t) - y(t_1) = y'(\zeta)(t-t_1) \geq y'(t_1+\varrho)(t-t_1) \geq (t - t_1 - \varrho + \delta)\,y'(t_1+\varrho),$$

and so

$$x(t) \leq \frac{1}{\delta}y(t), \quad t_1 \leq t \leq t_1 + \varrho.$$

For $t_1 + \varrho \leq t \leq t_1 + \varrho + \delta$, we have

$$\begin{aligned} x(t) &= y'(t) + R(t)\,x(t-r) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)\,H_k(x(s-\sigma_k))\,ds \\ &\leq \frac{1}{\delta}(y(t) - y(t-\delta)) + \left(R(t) + \sum_{i=1}^{p}\sum_{k\in J_i}\int_{t-\tau_i+\sigma_k}^{t} Q_k(s)(1-\varepsilon_k)\,ds\right)\frac{1}{\delta}y(t-\delta) \\ &= \frac{1}{\delta}y(t). \end{aligned}$$

By induction, one can prove in general that for $l = 0, 1, \cdots$

$$x(t) \leq \frac{1}{\delta}y(t), \quad t_1 + \varrho + l\delta < t \leq t_1 + \varrho + (l+1)\,\delta.$$

Therefore

$$x(t) \leq \frac{1}{\delta}y(t), \quad t \geq t_1$$

and so

(19)
$$x(t-\tau_i) \leq \frac{1}{\delta}y(t-\tau_i) < \frac{1}{\delta}y(t), \quad t \geq t_1 + \varrho,\ i = 1, 2, \cdots, m.$$

Substituting (18) and (19) into (17) we obtain

$$[x(t) - R(t)\,x(t-r)]' + \sum_{i=1}^{m} P_i(t)\,H_i(x(t-\tau_i)) - \sum_{j=1}^{n} Q_j(t)\,H_j(x(t-\sigma_j)) \leq 0.$$

By Lemma 6, Eq. (1) has an eventually positive solution. The proof is complete.□

Lemma 8 ([1], [9]). *Consider the ordinary differential equation*

(20)
$$y''(t) + p(t)\,y(t) = 0, \quad t \geq t_0,$$

where $p(t) \in C([t_0, \infty), R^+)$. *Then*

(i) *All solutions of* (20) *oscillate if*

$$\liminf_{t\to\infty} t\int_t^\infty p(s)\,ds > \frac{1}{4}.$$

(ii) *Eq.* (20) *has an eventually positive solution if*

$$t\int_t^\infty p(s)\,ds \le \frac{1}{4} \quad \text{for large } t.$$

3. Oscillation of equation (1)

In the following section we establish some sufficient conditions for all solutions of Eq. (1) to be oscillatory, and give a linearized oscillation results.

Theorem 1. *Assume that* (2) *holds,* $\tau_p = \max\{\tau_1, \tau_2, \cdots, \tau_m\}$ *and*

$$\limsup_{t\to\infty} \int_t^{t+\tau_p} F_p(s)\,ds > 0.$$

If

$$\int_{t_0}^\infty \sum_{i=1}^m (1-\varepsilon_i) F_i(t) \ln \left[\begin{array}{c} e(\sum_{i=1}^m (1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds) \\ +1-\operatorname{sgn}(\sum_{i=1}^m (1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds) \end{array} \right] = \infty, \tag{21}$$

where $\varepsilon_i \in (0,1)$, *then all solutions of* (1) *oscillate.*

Proof. On the contrary, assume that (1) has an eventually positive solution $x(t)$ and let $z(t)$ be defined by (4). It follows from Lemma 4 that (5) holds. From Corollary 3.2.2 in [8], we have that the delay differential equation

$$y'(t) + \sum_{i=1}^m F_i(t)(1-\varepsilon_i)\, y(t-\tau_i) = 0, \tag{22}$$

has an eventually positive solution $y(t)$. Let $\lambda(t) = \frac{-y'(t)}{y(t)}$. Then $\lambda(t) \ge 0$ and it satisfies

$$\lambda(t) = \sum_{i=1}^m F_i(t)(1-\varepsilon_i) \exp\left(\int_{t-\tau_i}^t \lambda(s)\,ds\right) \tag{23}$$

or

$$\begin{aligned} \lambda(t)\sum_{i=1}^m (1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds &= \sum_{i=1}^m (1-\varepsilon_i) F_i(t) \left(\sum_{i=1}^m (1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds\right) \\ &\quad \cdot \exp\left(\int_{t-\tau_i}^t \lambda(s)\,ds\right). \end{aligned}$$

One can easily show that

$$\varphi(u)\, ue^{x} \geq \varphi(u)\, x + \varphi(u) \ln (eu + 1 - \operatorname{sgn} u) \text{ for } u \geq 0 \text{ and } x \in R, \tag{24}$$

where $\varphi(0) = 0$ and $\varphi(u) \geq 0$ for $u > 0$. Employing inequality (24) on the right-hand side of (23) we get

$$\lambda(t) \sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds \geq \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t) \int_{t-\tau_i}^{t} \lambda(s)\, ds + \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t)$$
$$\cdot \ln \left[e \left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds \right) + 1 - \operatorname{sgn} \left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds \right) \right] dt$$

or

$$\begin{aligned} &\lambda(t) \sum_{i=1}^{m} \int_{t}^{t+\tau_i} (1-\varepsilon_i) F_i(s)\, ds - \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t) \int_{t-\tau_i}^{t} \lambda(s)\, ds \\ \geq\ & \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t) \ln \left[\begin{array}{c} e\left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds\right) \\ +1 - \operatorname{sgn}\left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds\right) \end{array} \right] dt. \end{aligned} \tag{25}$$

Then for $N > T$

$$\begin{aligned} &\int_{T}^{N} \lambda(t) \sum_{i=1}^{m} \int_{t}^{t+\tau_i} (1-\varepsilon_i) F_i(s)\, ds dt - \int_{T}^{N} \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t) \int_{t-\tau_i}^{t} \lambda(s)\, ds dt \\ \geq\ & \int_{T}^{N} \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t) \ln \left[\begin{array}{c} e\left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds\right) \\ +1 - \operatorname{sgn}\left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds\right) \end{array} \right] dt. \end{aligned} \tag{26}$$

By interchanging the order of integration, we find that

$$\begin{aligned} \int_{T}^{N} F_i(t) \int_{t-\tau_i}^{t} \lambda(s)\, ds dt &\geq \int_{T}^{N-\tau_i} \int_{s}^{s+\tau_i} F_i(t)\, \lambda(s)\, dt ds \\ &= \int_{T}^{N-\tau_i} \lambda(t) \int_{t}^{t+\tau_i} F_i(s)\, ds dt. \end{aligned} \tag{27}$$

From (26) and (27) it follows that

$$\begin{aligned} &\sum_{i=1}^{m} \int_{N-\tau_i}^{N} \lambda(t) \int_{t}^{t+\tau_i} (1-\varepsilon_i) F_i(s)\, ds dt \\ \geq\ & \int_{T}^{N} \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t) \ln \left[e \left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds \right) \right. \\ & \left. +1 - \operatorname{sgn} \left(\sum_{i=1}^{m} (1-\varepsilon_i) \int_{t}^{t+\tau_i} F_i(s)\, ds \right) \right] dt. \end{aligned} \tag{28}$$

On the other hand, since (22) has an eventually positive solution, by Lemma 2 in [18] we have

$$\int_t^{t+\tau_i} F_i(s)\,ds < \frac{1}{1-\varepsilon_i}, \quad i = 1, 2, \cdots, m, \tag{29}$$

eventually. Then by (28) and (29) we obtain

$$\sum_{i=1}^{m} \int_{N-\tau_i}^{N} \lambda(t)\,dt \geq \int_T^N \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t)$$
$$\ln\left[e\left(\sum_{i=1}^{m}(1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds\right) + 1 - \operatorname{sgn}\left(\sum_{i=1}^{m}(1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds\right)\right] dt$$

or

$$\sum_{i=1}^{m} \ln\frac{y(N-\tau_i)}{y(N)} \geq \int_T^N \sum_{i=1}^{m} (1-\varepsilon_i) F_i(t)$$
$$\ln\left[e\left(\sum_{i=1}^{m}(1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds\right) + 1 - \operatorname{sgn}\left(\sum_{i=1}^{m}(1-\varepsilon_i)\int_t^{t+\tau_i} F_i(s)\,ds\right)\right] dt.$$

By the assumption

$$\lim_{t\to\infty} \prod_{i=1}^{m} \frac{y(t-\tau_i)}{y(t)} = \infty.$$

This implies

$$\lim_{t\to\infty} \frac{y(t-\tau_p)}{y(t)} = \infty. \tag{30}$$

However, by Lemma 1 in [18] we have

$$\lim_{t\to\infty}\inf \frac{y(t-\tau_p)}{y(t)} < \infty.$$

This contradicts (30) and completes the proof. □

Corollary 1. *Assume that* $(A_1)-(A_5)$ *above are satisfied and the delay differential equation*

$$z'(t) + \sum_{i=1}^{m} F_i(t)(1-\varepsilon_i) z(t-\tau_i) = 0,$$

has no eventually positive solutions. Then every solution of equation (1) *oscillates, where* $\varepsilon_i \in (0,1)$.

Corollary 2. *In addition to the assumptions* $(A_1)-(A_5)$ *assume that*

$$\liminf_{t\to\infty}\int_{t-\varrho}^{t}\sum_{i=1}^{m}F_i(s)(1-\varepsilon_i)\,ds>\frac{1}{e}.$$

Then every solution of equation (1) *oscillates.*

The following theorem is the main result for the oscillation of all solutions of the nonlinear delay differential equation (1).

Theorem 2. *Assume that* $(A_1)-(A_4)$ *are satisfied,*

$$\lim_{t\to\infty}P_i(t)=p_i,\ Q_j(t)\le q_j \text{ and } \sum_{k\in J_i}q_k<p_i \text{ for } i=1,2,\cdots,m \text{ and } j=1,2,\cdots,n$$

and every solution of equation

$$y'(t)+\sum_{i=1}^{m}f_i y(t-\tau_i)=0, \tag{31}$$

where $f_i=p_i-\sum_{k\in J_i}q_k$ *oscillates, then every solution of equation* (1) *also oscillates.*

Proof. On the contrary, assume that (1) has an eventually positive solution $x(t)$. Since every solution of equation (31) oscillates, the characteristic equation

$$G(\lambda)=\lambda+\sum_{i=1}^{m}f_i e^{-\lambda\tau_i}=0,$$

has no real roots. As $G(\infty)=\infty$ it follows that $G(\lambda)>0$ for $\lambda\in R$. In particular $G(0)=\sum_{i=1}^{m}f_i>0$. Then $G(-\infty)=\infty$ and so $l=\min_{\lambda\in R}G(\lambda)$ exists and positive. Thus

$$\sum_{i=1}^{m}f_i e^{\lambda\tau_i}\ge\lambda+l,\ \lambda\in R. \tag{32}$$

let $z(t)$ be defined by (4). It follows from Lemma 4 that (5) holds and $x(t)\ge z(t)$, $z(t)$ is eventually positive and decreasing. Then

$$z'(t)+F_{i_0}(t)(1-\varepsilon_{i_0})z(t-\tau_{i_0})\le 0.$$

Where the index i_0 is chosen in such a way that

$$F_{i_0}(t)>0 \text{ and } \tau_{i_0}>0.$$

Clearly for t sufficiently large

$$z'(t)+\frac{1}{2}f_{i_0}(t)(1-\varepsilon_{i_0})z(t-\tau_{i_0})\le 0. \tag{33}$$

Set $L = \frac{1}{2} f_{i_0}(t)(1 - \varepsilon_{i_0})$. Hence

$$z'(t) + Lz(t - \tau_{i_0}) \le 0. \tag{34}$$

Define the set $\Lambda = \{\lambda \ge 0 : z'(t) + \lambda z(t) \le 0 \text{ for } t \text{ sufficiently large}\}$. Clearly from (34) Λ is a non-empty subinterval of R^+. The proof that every solution of equation (1) oscillates will be completed by showing that Λ has the following contradictory properties :

(p_1) Λ is bounded above,

(p_2) $\lambda \in \Lambda \Rightarrow \lambda + \frac{l}{2} \in \Lambda$. where l is positive and satisfy (32).

From (34) since $z'(t) \le 0$ Lemma (3) yields $z(t - \tau_{i_0}) \le \beta z(t)$ with $\beta = \frac{4}{(L\tau_{i_0})^2}$.

So Lemma (2) yields a

$$\lambda_0 = \frac{\ln(\beta)}{\varrho} \notin \Lambda. \tag{35}$$

this completes the proof of (p_1). In order to establish (p_2). Let $\lambda \in \Lambda$ and set

$$\phi(t) = e^{\lambda t} z(t).$$

Then

$$\phi'(t) = e^{\lambda t}(z'(t) + \lambda z(t)) \le 0,$$

which shows that $\phi(t)$ is decreasing. Thus $\phi(t - \tau_i) \ge \phi(t)$ for $i = 1, 2, \cdots, m$. Now choose δ_i and $\varepsilon_i > 0$ such that for t sufficiently large $P_i(t) \ge p_i - \delta_i$ for $i = 1, 2, \cdots, m$ and $|x(t - \tau_i)| < \delta$. Hence

$$H_i(x(t - \tau_i)) \ge (1 - \varepsilon_i) x(t - \tau_i).$$

For t sufficiently large we choose δ_i, τ_i such that

$$\sum_{i=1}^{m} (\delta_i + \sum_{k \in J_i} q_k) e^{\lambda \tau_i}(1 - \varepsilon_i) + \sum_{i=1}^{m} p_i \varepsilon_i e^{\lambda \tau_i} \le \frac{l}{2}.$$

So

$$\begin{aligned}
z'(t) + (\lambda + \frac{l}{2}) z(t) &\le -\sum_{i=1}^{m} F_i(t) H_i(x(t - \tau_i)) + (\lambda + \frac{l}{2}) z(t) \\
&= \sum_{i=1}^{m} (-P_i(t) + \sum_{k \in J_i} Q_k(t - \tau_i + \sigma_k)) H_i(x(t - \tau_i)) \\
&\quad + (\lambda + \frac{l}{2}) z(t) \\
&\le \sum_{i=1}^{m} (-P_i(t) + \sum_{k \in J_i} q_k) H_i(x(t - \tau_i)) + (\lambda + \frac{l}{2}) z(t).
\end{aligned}$$

Where $Q_k(t) \le q_k$ for $k \in J_i$. Hence

$$z'(t) + (\lambda + \frac{l}{2})z(t) \le \sum_{i=1}^{m} -((p_i - \delta_i) - \sum_{k \in J_i} q_k)(1 - \varepsilon_i)\, x(t - \tau_i) + (\lambda + \frac{l}{2})z(t).$$

Hence

$$\begin{aligned} z'(t) + (\lambda + \frac{l}{2})z(t) \quad \le \quad & e^{-\lambda t}[-\sum_{i=1}^{m}(p_i - \sum_{k \in J_i} q_k)e^{\lambda \tau_i}\phi(t - \tau_i) \\ & + \sum_{i=1}^{m}(\delta_i + \sum_{k \in J_i} q_k)(1 - \varepsilon_i)\, e^{\lambda \tau_i}\phi(t - \tau_i) \\ & + \sum_{i=1}^{m} p_i \varepsilon_i e^{\lambda \tau_i}\phi(t - \tau_i) + (\lambda + \frac{l}{2})\phi(t)] \end{aligned}$$

As $\phi(t - \tau_i) \ge \phi(t)$ then we have

$$\begin{aligned} z'(t) + (\lambda + \frac{l}{2})z(t) \quad \le \quad & e^{-\lambda t}[-\sum_{i=1}^{m}(p_i - \sum_{k \in J_i} q_k)e^{\lambda \tau_i} \\ & + \sum_{i=1}^{m}(\delta_i + \sum_{k \in J_i} q_k)(1 - \varepsilon_i)\, e^{\lambda \tau_i} \\ & + \sum_{i=1}^{m} p_i \varepsilon_i e^{\lambda \tau_i}\phi(t - \tau_i) + (\lambda + \frac{l}{2})]\phi(t). \end{aligned}$$

As

$$\sum_{i=1}^{m}(\delta_i + \sum_{k \in J_i} q_k)e^{\lambda \tau_i}(1 - \varepsilon_i) + \sum_{i=1}^{m} p_i \varepsilon_i e^{\lambda \tau_i} \le \frac{l}{2}.$$

Then from (32) we have

$$z'(t) + (\lambda + \frac{l}{2})z(t) \le e^{-\lambda t}[-(\lambda + l) + \frac{l}{2} + (\lambda + \frac{l}{2})]\phi(t) = 0.$$

Hence

$$z'(t) + (\lambda + \frac{l}{2})z(t) \le 0.$$

Then $\lambda + \frac{l}{2} \in \Lambda$. Thus (p_2) is proved. Then every solution of equation (1) oscillates.□

Theorem 3. *Assume that* (14) *holds and that*

$$\liminf_{t \to \infty} t \int_t^{\infty} \sum_{i=1}^{m}(1 - \varepsilon_i)\, F_i(t)\, ds > \frac{\varrho}{4}. \tag{36}$$

Then all solutions of (1) *oscillate.*

Proof. Suppose that Eq. (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (4). Then by Lemma 4 we have $z(t) > 0$ eventually. On the other hand, by Lemma 6, (36) implies that all solutions of Eq. (9) oscillate. By Lemma 5, it follows that $z(t) < 0$. This contradiction completes the proof. □

Theorem 4. *Assume that* (8) *and* (36) *hold and that*

$$R(t-\tau_i) H_i(t) \le h H_i(t-r), \quad i = 1, 2, \cdots, m. \tag{37}$$

Also suppose that $\dfrac{H_i(t)}{Q_j(t-\tau_i+\sigma_j)}$ *is nonincreasing and satisfies*

$$H_i(t) Q_j(t-\tau_i) \le h_j H_i(t-\sigma_j), \quad i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n, \tag{38}$$

where $h, \ h_j \ \ (j = 1, 2, \cdots, n)$ *are nonnegative constants satisfying*

$$h + \sum_{i=1}^{p} \sum_{k \in J_i} h_k (\tau_i - \sigma_k) = 1. \tag{39}$$

Then every solution of (1) *oscillates.*

Proof. Assume the contrary. Eq. (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (4). Then by Lemma 5 we have $z(t) < 0$ eventually. From (10), (37), (38) we have

$$\begin{aligned}
z'(t) &\le -\sum_{i=1}^{m} F_i(t) x(t-\tau_i) \\
&= -\sum_{i=1}^{m} F_i(t) [z(t-\tau_i) + R(t-\tau_i) x(t-r-\tau_i) \\
&\quad + \sum_{i=1}^{p} \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^{t} Q_k(s-\tau_i) x(s-\tau_i-\sigma_k) \, ds] \\
&\ge -\sum_{i=1}^{m} F_i(t) z(t-\tau_i) - h \sum_{i=1}^{m} F_i(t-r) x(t-r-\tau_i) \\
&\quad - \sum_{l=1}^{p} \sum_{k \in J_l} \sum_{i=1}^{m} h_k \frac{H_i(t-\sigma_k)}{Q_k(t-\tau_i)} \int_{t-\tau_l+\sigma_k}^{t} Q_k(s-\tau_i) x(s-\tau_i-\sigma_k) \, ds \\
&\ge -\sum_{i=1}^{m} F_i(t) z(t-\tau_i) + h z'(t-r) \\
&\quad - \sum_{l=1}^{p} \sum_{k \in J_l} h_k \sum_{i=1}^{m} \int_{t-\tau_l+\sigma_k}^{t} F_i(s-\sigma_k) x(s-\tau_i-\sigma_k) \, ds
\end{aligned}$$

$$= \quad -\sum_{i=1}^{m} F_i(t)\, z(t-\tau_i) + hz'(t-r) + \sum_{l=1}^{p} \sum_{k\in J_l} h_k \int_{t-\tau_l+\sigma_k}^{t} z'(s-\sigma_k)\, ds$$
$$= \quad -\sum_{i=1}^{m} F_i(t)\, z(t-\tau_i) + hz'(t-r) + \sum_{i=1}^{n} h_j z(t-\sigma_j) - \sum_{i=1}^{p} \sum_{k\in J_i} h_k z(t-\tau_l)\,.$$

Define $\bar{P}_i(t)$ by

$$\bar{P}_i(t) = F_i(t) + \sum_{k\in J_i} h_k, \quad i = 1, 2, \cdots, p,$$

$$\bar{P}_i(t) = F_i(t)\,, \quad i = p+1, p+2, \cdots, m.$$

We obtain

$$[z(t) - hz(t-r)]' + \sum_{i=1}^{m} \bar{P}_i(t)\, z(t-\tau_i) - \sum_{i=1}^{m} h_j z(t-\sigma_j) \geq 0.$$

This implies that $-z(t)$ is a positive solution solution of the inequality

$$[y(t) - hy(t-r)]' + \sum_{i=1}^{m} \bar{P}_i(t)\, y(t-\tau_i) - \sum_{i=1}^{m} h_j y(t-\sigma_j) \leq 0,$$

which yields a contradiction by Lemma 4 and 5. The proof is complete. $\square$

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