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Sagae-Tanabe Weighted Means and Reverse Inequalities

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Abstract. In this paper we consider weighted arithmetic and geometric means of several positive definite operators proposed by Sagae and Tanabe and we establish a reverse inequality of the arithmetic and geometric means via Specht ratio and the Thompson metric on the convex cone of positive definite operators.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathrm{P}(\mathcal{H})$ be the open convex cone of positive (invertible) operators on $\mathcal{H}$. The Thompson metric on $\mathrm{P}(\mathcal{H})$ is defined by

$$d(A, B) = \max\{\log M(A/B), \log M(B/A)\},$$

where $M(A/B) := \inf\{\lambda > 0 : A \leq \lambda B\}$. A. C. Thompson [15] has shown that $\mathrm{P}(\mathcal{H})$ is a complete metric space with respect to this metric and the corresponding metric topology agrees with the relative norm topology. For $A, B \in \mathrm{P}(\mathcal{H})$, the curve $t \mapsto A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is regarded as a minimal geodesic line passing A and B, and $A\#_{1/2}B$ is known as the geometric mean of A and B. The nonpositive curvature property of the Thompson metric is equivalently stated [4], [9], [10]:

$$d(A\#_t B, C\#_t D) \leq (1-t)d(A, C) + td(B, D), \quad t \in [0, 1]. \tag{1}$$

For positive definite operators A and B, the weighted arithmetic and geometric mean inequality is well-known:

$$A\#_t B \leq (1-t)A + tB, \quad t \in [0, 1].$$

Its reverse inequality via Specht ratio is known as

$$(1-t)A + tB \leq S_h \cdot (A\#_t B), \tag{2}$$

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where $S_h = \frac{(h-1)h^{(h-1)^{-1}}}{e \log h}$ and $h = e^{d(A,B)}$ for the Thompson metric d ([13], [2]).

In [14], Sagae and Tanabe proposed weighted arithmetic and geometric means of severable positive definite operators $A = (A_1, A_2, \cdots, A_m) \in P(\mathcal{H})^m$ from a fixed probability vector $\omega = (w_1, w_2, \cdots, w_m) \in R_+^m$, denoted by $A(\omega : A)$ and $G(\omega : A)$ respectively. The arithmetic-geometric inequality is derived: $G(\omega : A) \le A(\omega : A)$. The main purpose of this paper is to establish a reverse inequality of the weighted arithmetic and geometric means of several positive definite operators via Specht ratio, extending the result (2) of two positive definite operators. A similar reverse inequalities for the higher order (weighted) geometric mean proposed by Ando-Li-Mathias [3], [11], [12] are established in [16], [8], [5].

2. Sagae and Tanabe weighted operator means

For $t \in R, A = (A_1, A_2, \cdots, A_m) \in \mathrm{P}(\mathcal{H})^m$ and invertible operator M on $\mathcal{H}$, we denote

$$A^t = (A_1^t, A_2^t, \cdots, A_m^t), \quad MAM^* = (MA_1M^*, MA_2M^*, \cdots, MA_mM^*).$$

Definition 2.1. Let $\omega = (w_1, w_2, \cdots, w_m) \in R_+^m$ be a probability vector: $w_i > 0$ for all i and $\sum_{k=1}^m w_k = 1$. Set for $k = 1, \cdots, m-2$

$$\omega^{(k)} = \frac{1 - \sum_{i=k+1}^m w_i}{1 - \sum_{i=k+2}^m w_i} = 1 - w_{k+1}\left(\sum_{i=1}^{k+1} w_i\right)^{-1}$$

and $\omega^{(m-1)} = 1 - w_m$.

Remark 2.2.

(i) If $\omega = (1/m, 1/m, \cdots, 1/m) \in R^m$, then $\omega^{(k)} = \dfrac{k}{k+1}$.

(ii) Set $\mu = \dfrac{1}{1-w_m}(w_1, w_2, \cdots, w_{m-1}) \in R^{m-1}$. Then for $1 \le k \le m-2$,

$$\mu^{(k)} = \omega^{(k)} \tag{3}$$

from $\mu^{(k)} = 1 - \frac{w_{k+1}}{1-w_m}\left(\sum_{j=1}^{k+1} \frac{w_j}{1-w_m}\right)^{-1} = 1 - w_{k+1}\left(\sum_{j=1}^{k+1} w_j\right)^{-1} = \omega^{(k)}$.

Definition 2.3. The (Sagae-Tanabe) ω-weighted arithmetic, harmonic and geometric means of positive definite operators $A = (A_1, A_2, \cdots, A_m)$ are defined by

$$\begin{aligned} \mathrm{A}(\omega : A) &:= \sum_{i=1}^{m} w_i A_i, \\ \mathrm{H}(\omega : A) &:= \left[\sum_{i=1}^{m} w_i A_i^{-1}\right]^{-1}, \\ G(\omega : A) &:= A_m \#_{\omega^{(m-1)}} A_{m-1} \#_{\omega^{(m-2)}} \cdots \#_{\omega^{(2)}} A_2 \#_{\omega^{(1)}} A_1, \end{aligned}$$

where we used the notation $A_m \#_{\alpha_{m-1}} A_{m-1} \#_{\alpha_{m-2}} \cdots A_2 \#_{\alpha_1} A_1$ in the usual way:

$$A_m \#_{\alpha_{m-1}} A_{m-1} \#_{\alpha_{m-2}} \cdots \#_{\alpha_2} A_2 \#_{\alpha_1} A_1 = A_m \#_{\alpha_{m-1}} \Big(A_{m-1} \#_{\alpha_{m-2}} \cdots \#_{\alpha_2} A_2 \#_{\alpha_1} A_1 \Big)$$

although the geometric mean operation is not associative.

Proposition 2.4. *We have*

$$\begin{aligned} G(\omega : A)^{-1} &= G(\omega : A^{-1}), \\ G(\omega : MAM^*) &= MG(\omega : A)M^*, \\ G(\omega : A) &\le G(\omega : B) \ \text{ if } A_i \le B_i, (i = 1, 2, \cdots, m), \\ H(\omega : A) &\le G(\omega : A) \le A(\omega : A). \end{aligned}$$

If A_i's are mutually commutative then $G(\omega : A) = A_1^{w_1} A_2^{w_2} \cdots A_m^{w_m}$.

Proof. The invariancy under the inversion and congruence transformations and the monotone property follow from that of the geometric mean of two positive definite operators: $(A \#_\alpha B)^{-1} = A^{-1} \#_\alpha B^{-1}, M(A \#_\alpha B)M^* = (MAM^*) \#_\alpha (MBM^*)$, and $A \#_\alpha B \le C \#_\alpha D$ when $A \le C$ and $B \le D$ (Löwner-Heinz inequality) for $\alpha \in [0, 1]$. The weighted arithmetic-geometric-harmonic mean inequalities appear in [14]. □

3. A reverse inequality

For $h, s \ge 1$, the Specht ratio is defined by

$$S_h(s) := \frac{(h^s - 1) h^{s(h^s - 1)^{-1}}}{e \log h^s}, \quad S_h := S_h(1).$$

The maps $s \mapsto S_h(s)^{\frac{1}{s}}$ and $h \mapsto S_h$ are increasing functions for $s \ge 1$ and $h \ge 1$, respectively ([6], [8]). It then follows that

$$S_{h^\rho} \le S_h^\rho, \quad 0 < \rho \le 1. \tag{4}$$

Proposition 3.1. *Let $B \in \mathrm{P}(\mathcal{H})$ and $A = (A_1, A_2, \cdots, A_m) \in \mathrm{P}(\mathcal{H})^m$. Then*

$$d(B, A_m \#_{\alpha_{m-1}} A_{m-1} \#_{\alpha_{m-1}} \cdots \#_{\alpha_2} A_2 \#_{\alpha_1} A_1) \le \Delta(A_1, A_2, \cdots, A_m, B) \tag{5}$$

for any $\alpha_i \in [0,1]$, *where* $\Delta(A_1, A_2, \cdots, A_m) := \max\{d(A_i, A_j) : 1 \leq i, j \leq m\}$ *denotes the diameter of* $\{A_1, A_2, \cdots, A_m\}$ *for the Thompson metric.*

Proof. The proof follows by induction on m. By the nonpositive curvature property of the Thompson metric (1), we have

$$\begin{aligned} & d(B, A_2\#_{\alpha_1}A_1) = d(B\#_{\alpha_1}B, A_2\#_{\alpha_1}A_1) \\ \leq\ & (1-\alpha_1)d(B, A_2) + \alpha_1 d(B, A_1) \leq \Delta(A_1, A_2, B). \end{aligned}$$

Suppose that the inequality (5) holds for $m-1$. That is,

$$d(B, A_{m-1}\#_{\alpha_{m-2}} \cdots \#_{\alpha_2}A_2\#_{\alpha_1}A_1) \leq \Delta(A_1, A_2, \cdots, A_{m-1}, B). \tag{6}$$

Setting $G = A_{m-1}\#_{\alpha_{m-2}} \cdots \#_{\alpha_2}A_2\#_{\alpha_1}A_1$, we have

$$\begin{aligned} & d(B, A_m\#_{\alpha_{m-1}}A_{m-1}\#_{\alpha_{m-2}} \cdots \#_{\alpha_2}A_2\#_{\alpha_1}A_1) \\ =\ & d(B, A_m\#_{\alpha_{m-1}}G) \\ \leq\ & (1-\alpha_{m-1})d(B, A_m) + \alpha_{m-1}d(B, G) \\ \leq\ & \Delta(A_1, A_2, \cdots, A_m, B). \end{aligned}$$

□

Theorem 3.2. *For* $A = (A_1, A_2, \cdots, A_m) \in P(\mathcal{H})^m$ *and* $t \in [0,1]$,

$$A(\omega : A^t) \leq S_h^{(m-1)t} \cdot G(\omega : A^t), \tag{7}$$

where $h = e^{\Delta(\mathbb{A})}$.

Proof. It is enough to show for $t = 1$. Indeed, suppose that the inequality (7) holds true for $t = 1$ and let $s \in [0,1]$. From the non-positive curvature property of the Thompson metric, we have

$$d(A_i^s, A_j^s) = d(I\#_s A_i, I\#_s A_j) \leq (1-s)d(I, I) + sd(A_i, A_j) = sd(A_i, A_j)$$

for all $1 \leq i, j \leq m$. This implies that $h_s := e^{\Delta(A^s)} \leq e^{s\Delta(A)} = h^s$. It then follows from $S_{h_s} \leq S_{h^s} \overset{(4)}{\leq} S_h^s$ that

$$A(\omega : A^s) \leq S_{h_s}^{m-1} \cdot G(\omega : A^s) \leq S_h^{(m-1)s} \cdot G(\omega : A^s).$$

We prove by induction on m. If $m = 2$, then $w_1 = 1 - w_2$ and $\omega^{(1)} = 1 - w_2$ and hence

$$w_1A_1 + w_2A_2 = (1-w_2)A_1 + w_2A_2 \overset{(2)}{\leq} S_h \cdot (A_2\#_{\omega^{(1)}}A_1), \ h = e^{\Delta(A_1, A_2)}. \tag{8}$$

Suppose that the assertion holds true for $m-1$. Let $\omega = (w_1, w_2, \cdots, w_m)$ be a probability vector. Set $\mu = \frac{1}{1-w_m}(w_1, w_2, \cdots, w_{m-1})$. Then by Remark ,

$\mu \in R^{m-1}$ is a probability vector with $\mu^{(j)} = \omega^{(j)}, j = 1, 2, \cdots, m-2$. It follows that by induction

$$\begin{aligned}\frac{1}{1-w_m}\sum_{k=1}^{m-1} w_k A_k &= A(\mu : A_1, A_2, \cdots, A_{m-1}) \\ &\leq S_{h'}^{m-2} \cdot G(\mu : A_1, A_2, \cdots, A_{m-1}), \quad h' = e^{\Delta(A_1, A_2, \cdots, A_{m-1})} \\ &\leq S_h^{m-2} \cdot G(\mu : A_1, A_2, \cdots, A_{m-1}), \quad h = e^{\Delta(A_1, A_2, \cdots, A_m)}\end{aligned}$$

and therefore

$$\begin{aligned}A(\omega : \mathbb{A}) = \sum_{k=1}^{m} w_k A_k &= w_m A_m + (1-w_m)\frac{1}{1-w_m}\sum_{k=1}^{m-1} w_k A_k \\ &\leq w_m A_m + (1-w_m) S_h^{m-2} \cdot G(\mu : A_1, A_2, \ldots, A_{m-1}) \\ &\leq S_h^{m-2} \cdot \Big(w_m A_m + (1-w_m) G(\mu : A_1, A_2, \ldots, A_{m-1})\Big) \\ &\overset{(3)}{\leq} (S_h^{m-2} S_{h''}) \cdot (A_m \#_{\omega^{(m-1)}} G(\mu : A_1, A_2, \ldots, A_{m-1})) \\ &\qquad\qquad h'' := e^{d(A_m, G(\mu : A_1, A_2, \cdots, A_{m-1}))} \\ &= (S_h^{m-2} S_{h''}) \cdot G(\omega : \mathbb{A}) \\ &\overset{(5)}{\leq} (S_h^{m-2} S_h) \cdot G(\mu : \mathbb{A}) = S_h^{m-1} \cdot G(\mu : \mathbb{A}).\end{aligned}$$

□

Corollary 3.3. *Let $\omega = (w_1, w_2, \cdots, w_m)$ be a probability vector and let $A = (A_1, A_2, \cdots, A_m) \in \mathrm{P}(\mathcal{H})^m$. Then*

$$\langle A_1 x, x\rangle^{w_1} \langle A_2 x, x\rangle^{w_2} \cdots \langle A_m x, x\rangle^{w_m} \leq S_h^{m-1} \langle G(\omega : A)x, x\rangle, \qquad h := e^{\Delta(\mathbb{A})}.$$

In particular,

$$(\langle A_1 x, x\rangle \langle A_2 x, x\rangle \cdots \langle A_m x, x\rangle)^{\frac{1}{m}} \leq S_h^{m-1} \langle (A_m \#_{\frac{m-1}{m}} A_{m-1} \#_{\frac{m-2}{m-1}} \cdots \#_{\frac{2}{3}} A_2 \#_{\frac{1}{2}} A_1)x, x\rangle.$$

Proof. It follows from that

$$\begin{aligned}& \langle A_1 x, x\rangle^{w_1} \langle A_2 x, x\rangle^{w_2} \cdots \langle A_m x, x\rangle^{w_m} \\ \leq\ & w_1 \langle A_1 x, x\rangle + w_2 \langle A_2 x, x\rangle \cdots + w_m \langle A_m x, x\rangle \\ =\ & \langle A(\omega : A)x, x\rangle \\ \overset{(7)}{\leq}\ & S_h^{m-1} \langle G(\omega : A)x, x\rangle\end{aligned}$$

for all $x \in \mathcal{H}$. □

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