

A New Approach for the Derivation of a Discrete Approximation Formula on Uniform Grid for Harmonic Functions

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ABSTRACT. The purpose of this article is to find a relation between the finite difference method and the boundary element method, and propose a new approach deriving a discrete approximation formula as like that of the finite difference method for harmonic functions. We develop a discrete approximation formula on a uniform grid based on the boundary integral formulations. We consider three different boundary integral formulations and derive one discrete approximation formula on the uniform grid for the harmonic function. We show that the proposed discrete approximation formula has the same computational molecules with that of the finite difference formula for the Laplace operator ∇^2 .

1. Introduction

The ultimate goal of discrete methods for physical problems governed by partial differential equations is the reduction of continuous systems to ‘equivalent’ discrete systems which are suitable for high-speed computer solution. Once the governing equations are defined in a continuous domain Ω , the basic approximation involves the replacement of a continuous domain Ω by a mesh of discrete points within Ω . Usually, a numerical approach is to find only approximations obtained at the isolated points labelled $P_{i,j}$ instead of developing a solution defined everywhere in Ω .

Since Laplace’s equation, the prototype of elliptic partial differential equations, is one of the basic equations of mathematics, in this article, we are only addressed at the harmonic function u satisfying Laplace’s equation

$$(1.1) \quad \nabla^2 u(A) = 0, \quad A \in \Omega,$$

where Ω is a given domain in \mathbb{R}^2 . Development of discrete approximations can proceed by several avenues, notably finite difference methods(FDM)([1]), and finite

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element methods(FEM)([3], [5], [6]). To obtain the formula for the discrete approximation at the isolated point $P_{i,j}$, FEM starts with a variational formulation for the governing equation (1.1) and then reduce the continuous system to a discrete system by evaluating the variational formula with suitable test functions having supports in each given grid cells. On the other hand, FDM uses the direct approximation formula for the partial derivative ∇^2 obtained from the finite difference formula. For example, if $P_{i,j}$ is an isolated point(mesh point) for Ω with uniform spacing as shown in Figure 1, the central finite difference formula for the Laplace's operator ∇^2 is given by

$$\nabla^2 u(P_{i,j}) = \frac{1}{h^2} \left(\sum_{k=1}^4 u(P_{i,j}^{(k)}) - 4u(P_{i,j}) \right) + O(h^2)$$

and hence the discrete approximation formula for Laplace's equation by centered second differences becomes

$$(1.2) \quad \frac{u(P_{i,j}^{(3)}) - 2u(P_{i,j}) + u(P_{i,j}^{(1)}))}{h^2} + \frac{u(P_{i,j}^{(2)}) - 2u(P_{i,j}) + u(P_{i,j}^{(4)}))}{h^2} = 0.$$

Here, h is the mesh size. Usually, the finite difference formula is derived by Taylor's expansion and hence FDMs have attractions in that they can, in principle, be applied to any system of differential equations. However, unfortunately, it is well known that it is not flexible on the mesh for a domain with irregular boundaries of the majority of practical problems([4]). To overcome such inflexibility about the mesh, we will try to find a relation between the finite difference method and the boundary element method ([2], [4], [7]).

If we observe that the function u is harmonic in the disc $\tilde{\mathcal{D}}(\subset \Omega)$ centered at the isolated point $P_{i,j}$ with the radius h passing through four equidistant points $P_{i,j}^{(k)}$ as shown in Figure 1, then the mean property of the harmonic function shows

$$u(P_{i,j}) = \frac{1}{2\pi h} \int_{\partial \tilde{\mathcal{D}}} u(Q) ds_Q.$$

Hence if we approximate the boundary values $u(Q)$ by the piecewise linear interpolation introduced in (2.4), we can see that the interior value $u(P_{i,j})$ can be approximated by

$$(1.3) \quad u(P_{i,j}) \approx \frac{1}{2\pi h} \int_{\partial \tilde{\mathcal{D}}} \mathcal{P}u(Q) ds_Q = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}),$$

which is similar to the equation (1.2). It is noteworthy that that the discrete formula (1.3) is exactly the mean value of four neighborhood values $u(P_{i,j}^{(k)})$ and hence it has a mean property of the discrete type. Henceforth, we say that a scheme deriving a discrete approximation formula for harmonic functions is *natural* provided the following two conditions are satisfied :

- (i) The discrete approximation formula on the uniform grid stencil has the mean property like (1.3);
- (ii) The approximation procedure is naturally extendable to any types of the grid cell \tilde{D} .

Most recently, in [8], the authors considered new derivation schemes of the five points discrete formula (1.3) for the harmonic function on the disc \tilde{D} . The schemes are based on three different boundary integral formulations, where two of them are the indirect boundary integral equations (IBIE) for the harmonic function based on the single-layer potential and double-layer potential, respectively, and the other one is the direct boundary integral equation (DBIE) based on the Green's representation formula for the harmonic function. The collocation methods with the piecewise linear interpolation for the density functions were introduced for solving the boundary integral equations.

In [8], the authors assert that the derivation schemes of the discrete approximation formula can be naturally extended to an arbitrary shape of the grid cell \tilde{D} and hence the scheme is natural, but any information of the types of the discrete approximation formula are not discussed. The aim of this article is to develop a new approach for deriving a discrete approximation formula having the mean property of the discrete type for the harmonic function on the equilateral rectangle domain as shown in Figure 1. For the derivation of the approximation formula, we follow the fundamental idea of [8] based on the boundary integral formulations for the harmonic function. We show that two discrete approximation formulas derived from IBIE based on the double-layer potential and DBIE are same. The discrete approximation formula derived from IBIE with the single-layer potential also converges to the same formula obtained by the other approaches when the mesh size goes to zero. Also, we show that a slight modification for the latter method gives the exactly same discrete approximation formula with that of the former two methods.

The manuscript is organized as follows. Section 2 begins with the description of potentials and then propose three different boundary integral formulations for the harmonic function. Then, we introduce the discrete approximation formula after approximating the boundary integral equations using the collocation method. Section 3 is devoted to the calculation of the potentials and the proof of the main theorems stated in Section 2. We show that the discrete approximation formula can be constructed without any techniques for solving the boundary integral equations. Finally, we close the article with some comments in Section 4.

2. General frameworks and main theorems

Throughout this paper, we assume that the domain Ω is partitioned with uniform rectangle grid cells and each isolated grid point $P_{i,j}$ is related with four different grid points $P_{i,j}^{(k)}$, as shown in Figure 1, satisfying

$$(2.1) \quad \chi(P_{i,j}^{(k)}) = \frac{\pi}{2}, \quad |\overrightarrow{P_{i,j}P_{i,j}^{(k)}}| = h, \quad |\overrightarrow{P_{i,j}^{(k)}P_{i,j}^{(k+1)}}| = \sqrt{2}h,$$

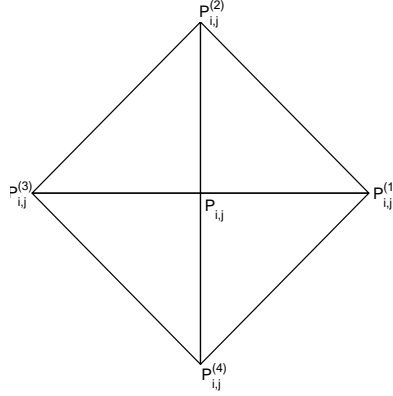


Figure 1: Shape of equilateral rectangle domain.

where h is a fixed constant mesh size and $\chi(P)$ denotes the interior angle to $\partial\mathcal{D}$ at the point P , where we assume that the domain $\mathcal{D} = \mathcal{D}_{i,j}$ is a symmetric equilateral polygon centered at the grid point $P_{i,j}$ with four boundary vertices $P_{i,j}^{(1)} = P_{i,j}^{(5)}, P_{i,j}^{(2)}, P_{i,j}^{(3)}, P_{i,j}^{(4)} = P_{i,j}^{(0)}$ counting with counterclockwise orientation, and $\partial\mathcal{D}$ denotes the boundary of the domain \mathcal{D} .

The aim of this section is to propose a new approach for developing discrete approximation formulas for the harmonic function u on the grid cell \mathcal{D} based on the single and double layer potentials defined by

$$(2.2) \quad \mathcal{S}\rho(P) = \int_{\partial\mathcal{D}} \rho(Q) \log \frac{1}{|P-Q|} ds_Q, \quad \mathcal{K}\rho(P) = \int_{\partial\mathcal{D}} \rho(Q) \frac{\partial}{\partial n_Q} \left[\log \frac{1}{|P-Q|} \right] ds_Q,$$

where n_Q denotes the unit outward normal vector to $\partial\mathcal{D}$ at the point Q . Let $\psi^{(k)}$ be the Lagrange basis functions for a piecewise linear interpolation defined on $\partial\mathcal{D}$ satisfying the interpolation condition

$$(2.3) \quad \psi^{(k)}(P_{i,j}^{(l)}) = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

With these Lagrange basis function $\psi^{(k)}$, we introduce the piecewise linear interpolation operator \mathcal{P} defined by

$$(2.4) \quad \mathcal{P}\rho(Q) = \sum_{k=1}^4 \rho(P_{i,j}^{(k)}) \psi^{(k)}(Q), \quad Q \in \partial\mathcal{D}$$

for a function ρ defined on the boundary $\partial\mathcal{D}$.

2.1. IDBIE based on the single-layer potential

In this subsection, we will consider a derivation of the discrete approximation formula for $u(P_{i,j})$ based on the single layer potential $\mathcal{S}\rho(P)$. Since u is a harmonic function in \mathcal{D} , we can represent it as a single-layer potential given by

$$(2.5) \quad u(A) = \mathcal{S}z^S(A), \quad A \in D,$$

with an unknown density function z^S to be determined. Now recall the well known jump relation for the single layer potential as follows([2]). Letting $A(\in \mathcal{D}) \rightarrow P \in \partial\mathcal{D}$, it is well known that

$$(2.6) \quad \lim_{A \rightarrow P} \mathcal{S}\rho(A) = \mathcal{S}\rho(P), \quad P \in \partial\mathcal{D}.$$

Applying this jump relation into the equation (2.5), we can obtain the boundary integral equation of the first kind

$$(2.7) \quad u(P) = \mathcal{S}z^S(P), \quad P \in \partial\mathcal{D}.$$

Here we remark that the interior value $u_{i,j} = u(P_{i,j})$ in the grid cell \mathcal{D} can be expressed with the boundary values of u by the equation (2.5) after representing the unknown density function z^S with the boundary values of u on $\partial\mathcal{D}$ from the boundary integral equation (2.7).

Now we introduce the collocation method for solving the boundary integral equation (2.7). We first substitute the piecewise linear interpolation $\mathcal{P}z^S$ into the equation (2.7) instead of z^S , and then consider the residual $r_S(P)$ defined by

$$(2.8) \quad r_S(P) = \mathcal{S}\mathcal{P}z^S(P) - u(P) = \sum_{k=1}^4 z^S(P_{i,j}^{(k)})\mathcal{S}\psi^{(k)}(P) - u(P), \quad P \in \partial\mathcal{D}.$$

To determine the coefficients $\{z^S(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$, we impose the collocation condition on the residual $r_S(P)$:

$$(2.9) \quad r_S(P_{i,j}^{(l)}) = 0, \quad l = 1, 2, 3, 4.$$

This collocation condition leads to the linear system for $\{z^S(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$:

$$(2.10) \quad u(P_{i,j}^{(l)}) = \sum_{k=1}^4 z^S(P_{i,j}^{(k)})\mathcal{S}\psi^{(k)}(P_{i,j}^{(l)}), \quad l = 1, 2, 3, 4.$$

With the solution $\{z^S(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$ of the linear system (2.10) and its piecewise linear interpolation polynomial $\mathcal{P}z^S$, we now define a new discrete approximation $U_{i,j}^S$ for $u_{i,j} = u(P_{i,j})$ by approximating the integral (2.5) as follows :

$$(2.11) \quad U_{i,j}^S = \mathcal{S}\mathcal{P}z^S(P_{i,j}) = \sum_{k=1}^4 z^S(P_{i,j}^{(k)})\mathcal{S}\psi^{(k)}(P_{i,j}).$$

Then we claim the following.

Theorem 2.1. *For the solution $\{z^S(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$ of the linear system (2.10), the discrete approximation $U_{i,j}^S$ for $u_{i,j} = u(P_{i,j})$ defined in (2.11) is exactly given by*

$$(2.12) \quad U_{i,j}^S = \lambda(h) \sum_{k=1}^4 u(P_{i,j}^{(k)}), \quad \lambda(h) = \frac{2 \log h - 2 + \frac{\pi}{2}}{4 \log(2h^2) + \log(4) + \pi - 8}.$$

Remark 2.2.

- (i) Since $\lambda'(x) < 0$ and $\lim_{x \rightarrow 0} \lambda(x) = \frac{1}{4}$, $\lambda(x)$ is always positive. Also, the discrete approximation $U_{i,j}^S$ converges to the mean value of the four boundary values $\{u(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$.
- (ii) It is noteworthy that the computational molecules of the discrete approximation formula (2.12) converges to that of the formula (1.2).
- (iii) Since the domain \mathcal{D} is symmetric, the integral values $\mathcal{S}\psi^{(k)}(P_{i,j})$ will be constant independent of k , and so is the values $\sum_{l=1}^4 \mathcal{S}\psi^{(k)}(P_{i,j}^{(l)})$. In fact, we will show these properties in Lemma 3.7 and Corollary 3.9, respectively. In later, we will show that these facts prove Theorem 2.1 without any techniques for solving the linear system (2.11).
- (iv) In the next section, using the inverse system of (2.10), we will show that the discrete approximation $U_{i,j}^S$ can be modified naturally as

$$\tilde{U}_{i,j}^S = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}),$$

which is exactly the mean value of the four boundary values $u(P_{i,j}^{(k)})$ and has the same computational molecules with that of the formula (1.2).

2.2. IDBIE based on the double-layer potential

In this subsection, we will consider a derivation of the discrete approximation formula for $u(P_{i,j})$ based on the double layer potential $\mathcal{K}\rho(P)$. It is well-known that the harmonic function in \mathcal{D} can be represented by the double-layer potential

$$(2.13) \quad u(A) = \mathcal{K}z^D(A), \quad A \in \mathcal{D},$$

with an unknown density function z^D to be determined. Now we recall the well known jump relation for the double-layer potential as follows([2]). Letting $A(\in \mathcal{D}) \rightarrow P \in \partial\mathcal{D}$, it is well known that

$$(2.14) \quad \lim_{A \rightarrow P} \mathcal{K}\rho(A) = (\chi(P) - 2\pi)\rho(P) + \mathcal{K}\rho(P), \quad P \in \partial\mathcal{D}.$$

Applying this jump relation into the equation (2.13), we can get the boundary integral equation of the second kind

$$u(P) = (\chi(P) - 2\pi)z^D(P) + \mathcal{K}z^D(P), \quad P \in \partial\mathcal{D}.$$

It is well known that this equation is not for practical use since the integral operator $\mathcal{K}\rho(P)$, $P \in \partial\mathcal{D}$ has a singularity, which follows from the non-smoothness of the boundary $\partial\mathcal{D}$. So we introduce the following modified boundary integral equation

$$(2.15) \quad u(P) = -2\pi z^D(P) + \mathcal{K}_1 z^D(P), \quad P \in \partial\mathcal{D},$$

where

$$(2.16) \quad \mathcal{K}_1 \rho(P) = \int_{\partial\mathcal{D}} (\rho(Q) - \rho(P)) \frac{\partial}{\partial n_Q} \left[\log \frac{1}{|P - Q|} \right] ds_Q.$$

Here, we used the well known identity for the double-layer potential \mathcal{K} given by

$$(2.17) \quad -\chi(P) = \int_{\partial\mathcal{D}} \frac{\partial}{\partial n_Q} \left[\log \frac{1}{|P - Q|} \right] ds_Q, \quad P \in \partial\mathcal{D},$$

which can be proved by the Green's representation formula introduced in next subsection.

Here we remark that the interior value $u_{i,j} = u(P_{i,j})$ in the grid cell \mathcal{D} can be expressed with the boundary values of u by the equation (2.13) after representing the unknown density function z^D with the boundary values of u on $\partial\mathcal{D}$ from the boundary integral equation (2.15).

Now we consider the collocation method for solving the boundary integral equation (2.15) based on the piecewise linear interpolation $\mathcal{P}z^D$. We first substitute the piecewise linear interpolation $\mathcal{P}z^D$ into the equation (2.15) instead of z^D , and consider the residual $r_D(P)$ defined by

$$(2.18) \quad \begin{aligned} r_D(P) &= -2\pi \mathcal{P}z^D(P) + \mathcal{K}_1 \mathcal{P}z^D(P) - u(P) \\ &= \sum_{k=1}^4 z^D(P_{i,j}^{(k)}) (-2\pi \psi^{(k)}(P) + \mathcal{K}_1 \psi^{(k)}(P)) - u(P), \quad P \in \partial\mathcal{D}. \end{aligned}$$

To determine the coefficients $\{z^D(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$, we impose the collocation condition on the residual $r_D(P)$:

$$(2.19) \quad r_D(P_{i,j}^{(l)}) = 0, \quad l = 1, 2, 3, 4.$$

This collocation condition leads to the linear system for $\{z^D(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$:

$$(2.20) \quad u(P_{i,j}^{(l)}) = -2\pi z^D(P_{i,j}^{(l)}) + \sum_{k=1}^4 z^D(P_{i,j}^{(k)}) \mathcal{K}_1 \psi^{(k)}(P_{i,j}^{(l)}), \quad l = 1, 2, 3, 4.$$

With the solution $\{z^D(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$ of the linear system (2.20) and its piecewise linear interpolation polynomial $\mathcal{P}z^D$, we define a discrete approximation $U_{i,j}^D$ for $u_{i,j} = u(P_{i,j})$ by approximating the integral (2.13) as follows :

$$(2.21) \quad U_{i,j}^D = \mathcal{K} \mathcal{P} z^D(P_{i,j}) = \sum_{k=1}^4 z^D(P_{i,j}^{(k)}) \mathcal{K} \psi^{(k)}(P_{i,j}).$$

Then we claim the following.

Theorem 2.3. *For the solution $\{z^D(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$ of the linear system (2.20), the discrete approximation $U_{i,j}^D$ for $u_{i,j} = u(P_{i,j})$ defined in (2.21) is exactly given by*

$$(2.22) \quad U_{i,j}^D = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}).$$

Remark 2.4.

- (i) The formula (2.22) shows that the discrete approximation $U_{i,j}^D$ is exactly the mean value of the four boundary values $\{u(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$.
- (ii) It is noteworthy that the computational molecules of the discrete approximation formula (2.22) is exactly same with that of the formula (1.2).
- (iii) Since the domain \mathcal{D} is symmetric, the integral values $\mathcal{K} \psi^{(k)}(P_{i,j})$ will be constant independent of k , and so is the values $\sum_{l=1}^4 \mathcal{K}_1 \psi^{(k)}(P_{i,j}^{(l)})$. In fact, we will prove these properties in Lemma 3.7 and Corollary 3.11, respectively. In later, we will prove Theorem 2.3 with only these facts without any technique for solving the linear system (2.21).

2.3. DBIE based on the Green's representation formula

First recall the Green's second identity:

$$(2.23) \quad \int_{\mathcal{D}} [u \nabla^2 w - w \nabla^2 u] dx = \int_{\partial \mathcal{D}} \left[w \frac{\partial u}{\partial n} - u \frac{\partial w}{\partial n} \right] ds.$$

From this identity (2.23), if we take $w = \log \frac{1}{|A-Q|}$, then the harmonic function u in \mathcal{D} has the Green's representation formula :

$$(2.24) \quad 2\pi u(A) = \mathcal{S}z^G(A) - \mathcal{K}u(A), \quad A \in \mathcal{D},$$

where $z^G(Q) = \frac{\partial u}{\partial n_Q}$.

Applying the jump relations in (2.6) and (2.14) into the above equation (2.24), we can have the following boundary integral equation of the first kind with respect to the unknown density $z^G(P)$

$$0 = \mathcal{S}z^G(P) - \chi(P)u(P) - \mathcal{K}u(P), \quad P \in \partial\mathcal{D},$$

or equivalently

$$(2.25) \quad \mathcal{S}z^G(P) = \mathcal{K}_1 u(P), \quad P \in \partial\mathcal{D}$$

where we used the identity (2.17).

Here we remark that the interior value $u_{i,j} = u(P_{i,j})$ in the grid cell \mathcal{D} can be expressed with the boundary values of u by the equation (2.24) after representing the unknown density function z^G with the boundary values of u on $\partial\mathcal{D}$ from the boundary integral equation (2.25). With this notice, we now consider the another derivation for a discrete approximation formula for the value $u_{i,j} = u(P_{i,j})$. As the previous two procedures, we consider the collocation method for the integral equation (2.25) based on the piecewise linear interpolation $\mathcal{P}z^G(P)$.

We first substitute the piecewise linear interpolations $\mathcal{P}z^G$ and $\mathcal{P}u$ into the equation (2.25) instead of z^G and u , respectively, and introduce the residual $r_G(P)$ defined by

$$(2.26) \quad \begin{aligned} r_G(P) &= \mathcal{S}\mathcal{P}z^G(P) - \mathcal{K}_1\mathcal{P}u(P) \\ &= \sum_{k=1}^4 z^G(P_{i,j}^{(k)})\mathcal{S}\psi^{(k)}(P) - \sum_{k=1}^4 u(P_{i,j}^{(k)})\mathcal{K}_1\psi^{(k)}(P), \quad P \in \partial\mathcal{D}. \end{aligned}$$

To determine the coefficients $\{z^G(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$, we impose the collocation condition on the residual $r_G(P)$:

$$(2.27) \quad r_G(P_{i,j}^{(l)}) = 0, \quad l = 1, 2, 3, 4.$$

This collocation condition leads to the linear system for $\{z^G(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$:

$$(2.28) \quad \sum_{k=1}^4 u(P_{i,j}^{(k)})\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(l)}) = \sum_{k=1}^4 z^G(P_{i,j}^{(k)})\mathcal{S}\psi^{(k)}(P_{i,j}^{(l)}), \quad l = 1, 2, 3, 4.$$

With the solution $\{z^G(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$ of the linear system (2.28) and its piecewise linear interpolation polynomial $\mathcal{P}z^G$, we define a discrete approximation

$U_{i,j}^G$ for $u_{i,j} = u(P_{i,j})$ by approximating the integral (2.24) as follows :

$$(2.29) \quad \begin{aligned} U_{i,j}^G &= \frac{1}{2\pi} \left(\mathcal{SP}z^G(P_{i,j}) - \mathcal{KP}u(P_{i,j}) \right) \\ &= \frac{1}{2\pi} \sum_{k=1}^4 \left[z^G(P_{i,j}^{(k)}) \mathcal{S}\psi^{(k)}(P_{i,j}) - u(P_{i,j}^{(k)}) \mathcal{K}\psi^{(k)}(P_{i,j}) \right]. \end{aligned}$$

Then we claim the following.

Theorem 2.5. *For the solution $\{z^G(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$ of the linear system (2.28), the discrete approximation $U_{i,j}^G$ for $u_{i,j} = u(P_{i,j})$ defined in (2.29) is exactly given by*

$$(2.30) \quad U_{i,j}^G = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}).$$

Remark 2.6.

- (1) The discrete approximation formula (2.30) is exactly the mean value of the four neighborhood values $\{u(P_{i,j}^{(k)}) : k = 1, 2, 3, 4\}$.
- (2) It is noteworthy that the computational molecules of the discrete approximation formula (2.30) is exactly same with that of the formula (1.2).

3. Proof of theorems

Throughout this section, the notation $\Gamma^{(k)}$ will be employed for the sub-boundary of $\partial\mathcal{D}$ connecting three consecutive vertices $P_{i,j}^{(k-1)}$, $P_{i,j}^{(k)}$ and $P_{i,j}^{(k+1)}$. Then we can observe that the Lagrange basis function $\psi^{(k)}(Q)$ has only the support on the sub-boundary $\Gamma^{(k)}$.

Since each edge $\overline{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$ is a straight line, we can see that the unit outward normal vector n_Q on each edge $\overline{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$ is a constant vector and it can be represented by

$$(3.1) \quad n_Q = \frac{1}{\sqrt{2}h} \left(\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k+1)}} \right), \quad Q \in \overline{P_{i,j}^{(k)} P_{i,j}^{(k+1)}},$$

where we used the facts in (2.1). Hence a careful calculation gives the following.

Lemma 3.7. *For the piecewise linear function $\psi^{(k)}(P)$, the integral values of $\mathcal{S}\psi^{(k)}(P_{i,j})$ and $\mathcal{K}\psi^{(k)}(P_{i,j})$ can be exactly calculated by*

$$(3.2) \quad \mathcal{S}\psi^{(k)}(P_{i,j}) = -\sqrt{2}h \left(\log h - 1 + \frac{\pi}{4} \right), \quad \mathcal{K}\psi^{(k)}(P_{i,j}) = -\frac{\pi}{2}, \quad \forall k.$$

Proof. For any indices $k = 1, \dots, 4$, the assumptions in (2.1) give

$$(3.3) \quad \begin{aligned} \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \cdot \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} &= |\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}|^2 \cos \frac{\pi}{4} = h^2, \\ \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \cdot \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} &= 0. \end{aligned}$$

Since the unit outward normal vector n_Q on each edge $\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$ is orthogonal to the vector $\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$, the equation (3.1) shows

$$(3.4) \quad \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \cdot (\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j}^{(k+1)} P_{i,j}^{(k)}}) = 0.$$

Now for the edge $\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$, if we consider the following parametrization

$$(3.5) \quad Q = Q(t) = P_{i,j}^{(k)} + t(P_{i,j}^{(k+1)} - P_{i,j}^{(k)}), \quad t \in [0, 1],$$

then the assumptions in (2.1) and (3.3) give

$$\begin{aligned} |Q - P_{i,j}|^2 &= |\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} + t\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}|^2 \\ &= \left| \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} \right|^2 + 2t\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} \cdot \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} + t^2 \left| \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \right|^2 \\ &= h^2(2t^2 - 2t + 1), \quad Q \in \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}. \end{aligned}$$

Also, three equations in (3.1), (3.3) and (3.4) give

$$\begin{aligned} n_Q \cdot (Q - P_{i,j}) &= \frac{1}{\sqrt{2}h} \left(\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j}^{(k+1)} P_{i,j}^{(k)}} \right) \cdot \left(\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} + t\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \right) \\ &= \frac{1}{\sqrt{2}h} \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} \cdot \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k)}} = \frac{h}{\sqrt{2}}, \quad Q \in \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}. \end{aligned}$$

Since the piecewise linear function $\psi^{(k)}(P)$ has support in $\Gamma^{(k)}$, the integrals $\mathcal{S}\psi^{(k)}(P_{i,j})$ and $\mathcal{K}\psi^{(k)}(P_{i,j})$ become

$$\begin{aligned} \mathcal{S}\psi^{(k)}(P_{i,j}) &= \int_{\Gamma^{(k)}} \psi^{(k)}(Q) \log \frac{1}{|Q - P_{i,j}|} ds_Q, \\ \mathcal{K}\psi^{(k)}(P_{i,j}) &= \int_{\Gamma^{(k)}} \psi^{(k)}(Q) \frac{\partial}{\partial n_Q} \left[\log \frac{1}{|Q - P_{i,j}|} \right] ds_Q. \end{aligned}$$

Hence if we use the parametrization given in (3.5) for the point Q on each edge

$\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$, the integral $\mathcal{S}\psi^{(k)}(P_{i,j})$ can be easily calculated as

$$\begin{aligned} \mathcal{S}\psi^{(k)}(P_{i,j}) &= -\sqrt{2}h \int_0^1 \left[t \log \left| \overrightarrow{P_{i,j} P_{i,j}^{(k-1)}} + t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}} \right| \right. \\ &\quad \left. + (1-t) \log \left| \overrightarrow{P_{i,j} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \right| \right] dt \\ &= -\frac{h}{\sqrt{2}} \int_0^1 \log(h^2(2t^2 - 2t + 1)) dt = -\sqrt{2}h \left(\log h - 1 + \frac{\pi}{4} \right). \end{aligned}$$

Also the integral $\mathcal{K}\psi^{(k)}(P_{i,j})$ can be evaluated as

$$\begin{aligned} \mathcal{K}\psi^{(k)}(P_{i,j}) &= -\sqrt{2}h \int_0^1 \left[t \frac{(\overrightarrow{P_{i,j} P_{i,j}^{(k-1)}} + t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}}) \cdot (\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k+1)}})}{\left| \overrightarrow{P_{i,j} P_{i,j}^{(k-1)}} + t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}} \right|^2} \right. \\ &\quad \left. + (1-t) \frac{(\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}) \cdot (\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k+1)}})}{\left| \overrightarrow{P_{i,j} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \right|^2} \right] dt \\ &= -\int_0^1 \frac{1}{2t^2 - 2t + 1} dt = -\frac{\pi}{2}. \end{aligned}$$

Hence we complete the proof. \square

Lemma 3.8. *For the piecewise linear function $\psi^{(k)}(P)$, the values of $\mathcal{S}\psi^{(k)}(P_{i,j}^{(l)})$ can be exactly calculated by*

$$(3.6) \quad \mathcal{S}\psi^{(k)}(P_{i,j}^{(l)}) = \frac{h}{\sqrt{2}} \begin{cases} 3 - \log(2h^2), & k = l, \\ 2 - \frac{\pi}{2} - \log(2h^2), & |k - l| = 1, 3, \\ 1 - \log(8h^2), & |k - l| = 2. \end{cases}$$

Proof. By the conditions of $P_{i,j}^{(k)}$ given in (2.1), the vertices $P_{i,j}^{(k)}$ may be represented as

$$(3.7) \quad P_{i,j}^{(k)} = P_{i,j} + h \left(\cos \left(\theta_0 + \frac{(k-1)\pi}{2} \right), \sin \left(\theta_0 + \frac{(k-1)\pi}{2} \right) \right)$$

for a fixed constant value θ_0 . Hence, a careful calculation gives

$$\begin{aligned} \left| \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(l)}} \right|^2 &= (P_{i,j}^{(l)} - P_{i,j}^{(k)}) \cdot (P_{i,j}^{(l)} - P_{i,j}^{(k)}) \\ &= h^2 \left(\left(\cos \left(\theta_0 + \frac{l-1}{2} \pi \right) - \cos \left(\theta_0 + \frac{k-1}{2} \pi \right) \right)^2 \right. \\ &\quad \left. + \left(\sin \left(\theta_0 + \frac{l-1}{2} \pi \right) - \sin \left(\theta_0 + \frac{k-1}{2} \pi \right) \right)^2 \right) \\ &= h^2 \left(2 - 2 \cos \left(\frac{l-k}{2} \pi \right) \right) = 4h^2 \sin^2 \frac{(l-k)\pi}{4}, \end{aligned}$$

and similarly,

$$\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(l)}} \cdot \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} = h^2 \left(1 - \sqrt{2} \sin \frac{(2k - 2l + 1)\pi}{4} \right).$$

These formulas and the assumptions in (2.1) give

$$(3.8) \quad \left| \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(l)}} - t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \right|^2 = \left| \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(l)}} \right|^2 + 2h^2 t^2 - 2t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(l)}} \cdot \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \\ = 2h^2 a_l^{(k)}(t),$$

where

$$a_l^{(k)}(t) = t^2 - t \left(1 - \sqrt{2} \sin \frac{(2k - 2l + 1)\pi}{4} \right) + 2 \sin^2 \frac{(k - l)\pi}{4}.$$

Since the piecewise linear function $\psi^{(k)}$ has support in $\Gamma^{(k)}$, the parametrization (3.5) gives

$$\begin{aligned} \mathcal{S}\psi^{(k)}(P_{i,j}^{(l)}) &= - \int_{\Gamma^{(k)}} \log |Q - P_{i,j}^{(l)}| ds_Q \\ &= - \frac{h}{\sqrt{2}} \int_0^1 \left(t \log |\overrightarrow{P_{i,j}^{(l)} P_{i,j}^{(k-1)}} + t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}}|^2 \right. \\ &\quad \left. + (1 - t) \log |\overrightarrow{P_{i,j}^{(l)} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}|^2 \right) dt \\ &= - \frac{h}{\sqrt{2}} \int_0^1 \left(t \log(2h^2 a_l^{(k-1)}(t)) + (1 - t) \log(2h^2 a_l^{(k)}(t)) \right) dt \\ &= - \frac{h}{\sqrt{2}} \left(\log(2h^2) + A_{k,l} \right), \end{aligned}$$

where

$$A_{k,l} = \int_0^1 \left(t \log(a_l^{(k-1)}(t)) + (1 - t) \log(a_l^{(k)}(t)) \right) dt.$$

Then, direct calculations give

$$A_{k,k} = \int_0^1 \left(t \log(1 - t)^2 + (1 - t) \log t^2 \right) dt = -3,$$

and if $|k - l| = 1, 3$,

$$A_{k,l} = \int_0^1 \left(t \log t^2 + (1 - t) \log(1 + t^2) \right) dt = \frac{1}{2}(-4 + \pi).$$

Finally, if $|k - l| = 2$, then

$$A_{k,l} = \int_0^1 \left(t \log(t^2 + 1) + (1 - t) \log(t^2 - 2t + 2) \right) dt = -1 + \log 4.$$

Hence we can complete the proof. \square

Corollary 3.9. *For the piecewise linear function $\psi^{(k)}(P)$, we have*

$$\sum_{l=1}^4 \mathcal{S}\psi^{(k)}(P_{i,j}^{(l)}) = \frac{h}{\sqrt{2}} \left(8 - \pi - \log 4 - 4 \log(2h^2) \right), \quad \forall k.$$

Lemma 3.10. *For the piecewise linear function $\psi^{(k)}(P)$, the values of $\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(l)})$ can be exactly calculated by*

$$(3.9) \quad \mathcal{K}_1\psi^{(k)}(P_{i,j}^{(l)}) = \begin{cases} \frac{\pi}{2}, & k = l, \\ \frac{1}{4}(-\pi + \log 4), & |k - l| = 1, 3, \\ -\log 2, & |k - l| = 2. \end{cases}$$

Proof. We first consider the calculation of $\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(k)})$, $k = 1, 2, 3, 4$. Since the square \mathcal{D} is symmetric, we can see that

$$\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(k)}) = \mathcal{K}_1\psi^{(j)}(P_{i,j}^{(j)}), \quad \forall k, j.$$

So we consider only the integral $\mathcal{K}_1\psi^{(1)}(P_{i,j}^{(1)})$. From two facts (3.1), (3.4), and the parametrization (3.5), we can see that for each $k = 1, 2, 3, 4$,

$$\begin{aligned} n_Q \cdot (Q - P_{i,j}^{(1)}) &= n_Q \cdot \left(\overrightarrow{P_{i,j}^{(1)} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}} \right) \\ &= \frac{1}{\sqrt{2}h} \left(\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k+1)}} \right) \cdot \overrightarrow{P_{i,j}^{(1)} P_{i,j}^{(k)}}, \quad Q \in \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}. \end{aligned}$$

Now if we use the representation (3.7), we can get

$$n_Q \cdot (Q - P_{i,j}^{(1)}) = \frac{h}{\sqrt{2}} \left(1 - \cos \frac{k\pi}{2} - \sin \frac{k\pi}{2} \right), \quad Q \in \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}.$$

This shows that

$$n_Q \cdot (Q - P_{i,j}^{(1)}) = \begin{cases} 0, & Q \in \Gamma^{(1)}, \\ \sqrt{2}h, & \text{otherwise.} \end{cases}$$

Also, from the parametrization (3.5) and the equation (3.8), we get

$$\begin{aligned} |Q - P_{i,j}^{(1)}|^2 &= |\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(1)}} - t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}|^2 \\ &= 2h^2 \left(t^2 - t \left(1 - \sqrt{2} \sin \frac{(2k-1)\pi}{4} \right) + 2 \sin^2 \frac{(k-1)\pi}{4} \right), \end{aligned}$$

for any $Q \in \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}$. Hence

$$|Q - P_{i,j}^{(1)}|^2 = 2h^2 \begin{cases} t^2 + 1, & Q \in \overrightarrow{P_{i,j}^{(2)} P_{i,j}^{(3)}}, \\ t^2 - 2t + 2, & Q \in \overrightarrow{P_{i,j}^{(3)} P_{i,j}^{(4)}}. \end{cases}$$

Hence, the interpolation condition (2.3) and the parametrization (3.5) show

$$\begin{aligned}\mathcal{K}_1\psi^{(1)}(P_{i,j}^{(1)}) &= - \int_{\partial\mathcal{D}} (\psi^{(1)}(Q) - 1) \frac{n_Q \cdot (Q - P_{i,j}^{(1)})}{|Q - P_{i,j}^{(1)}|^2} ds_Q \\ &= \int_0^1 \left(\frac{1}{t^2 + 1} + \frac{1}{t^2 - 2t + 2} \right) dt = \frac{\pi}{2}.\end{aligned}$$

Now for two distinct indices k and l , we consider the calculation of $\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(l)})$. Since the support of the Lagrange basis function $\psi^{(k)}(Q)$ is $\Gamma^{(k)}$ and $\psi^{(k)}(P_{i,j}^{(l)}) = 0$ if $k \neq l$, the representation (3.1) and the parametrization (3.5) give

$$\begin{aligned}\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(l)}) &= - \int_0^1 \left[t \frac{(\overrightarrow{P_{i,j}^{(l)} P_{i,j}^{(k-1)}} + t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}}) \cdot (\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k-1)}})}{|\overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(l)}} - t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}}|^2} \right. \\ &\quad \left. + (1-t) \frac{(\overrightarrow{P_{i,j}^{(l)} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}) \cdot (\overrightarrow{P_{i,j} P_{i,j}^{(k+1)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k)}})}{|\overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(l)}} - t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}|^2} \right] dt,\end{aligned}$$

for $k \neq l$. Now if we use the representation (3.7) for $P_{i,j}^{(k)}$, we can see that

$$\begin{aligned}&(\overrightarrow{P_{i,j}^{(l)} P_{i,j}^{(k-1)}} + t \overrightarrow{P_{i,j}^{(k-1)} P_{i,j}^{(k)}}) \cdot (\overrightarrow{P_{i,j} P_{i,j}^{(k)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k-1)}}) \\ &= 2\sqrt{2}h^2 \sin \frac{(k-l)\pi}{4} \sin \frac{(k-l-1)\pi}{4}, \\ &(\overrightarrow{P_{i,j}^{(l)} P_{i,j}^{(k)}} + t \overrightarrow{P_{i,j}^{(k)} P_{i,j}^{(k+1)}}) \cdot (\overrightarrow{P_{i,j} P_{i,j}^{(k+1)}} + \overrightarrow{P_{i,j} P_{i,j}^{(k)}}) \\ &= 2\sqrt{2}h^2 \sin \frac{(k-l)\pi}{4} \sin \frac{(k-l+1)\pi}{4}.\end{aligned}$$

Thus the formula (3.8) shows that the integrand function of (3.10) becomes

$$\begin{aligned}&-\sqrt{2} \sin \frac{(k-l)\pi}{4} \left(\frac{t \sin \frac{(k-l-1)\pi}{4}}{t^2 - t(1 - \sqrt{2} \sin \frac{(2k-2l-1)\pi}{4}) + 2 \sin^2 \frac{(k-l)\pi}{4}} \right. \\ &\quad \left. \frac{(1-t) \sin \frac{(k-l+1)\pi}{4}}{t^2 - t(1 - \sqrt{2} \sin \frac{(2k-2l+1)\pi}{4}) + 2 \sin^2 \frac{(k-l)\pi}{4}} \right).\end{aligned}$$

This implies

$$\mathcal{K}_1\psi^{(k)}(P_{i,j}^{(l)}) = \begin{cases} \int_0^1 \frac{t-1}{1+t^2} dt, & |k-l| = 1, 3, \\ \int_0^1 \frac{-1 + (-1+t)t}{(2+(t-2)t)(1+t^2)} dt, & |k-l| = 2. \end{cases}$$

Hence a direct calculation shows

$$\mathcal{K}_1 \psi^{(k)}(P_{i,j}^{(l)}) = \begin{cases} \frac{1}{4}(-\pi + \log 4), & |k - l| = 1, 3, \\ -\log 2, & |k - l| = 2. \end{cases}$$

Thus we can complete the proof. \square

Corollary 3.11. *For the piecewise linear function $\psi^{(k)}(P)$, we have*

$$\sum_{l=1}^4 \mathcal{K}_1 \psi^{(k)}(P_{i,j}^{(l)}) = 0, \quad \forall k.$$

(Proof of the theorems) First, Lemma 3.7 shows that the discrete approximations $U_{i,j}^S$, $U_{i,j}^D$ and $U_{i,j}^G$ defined in (2.11), (2.21) and (2.29), respectively, can be simplified as follows.

$$\begin{aligned} U_{i,j}^S &= -\sqrt{2}h \left(\log h - 1 + \frac{\pi}{4} \right) \sum_{k=1}^4 z^S(P_{i,j}^{(k)}), \\ (3.11) \quad U_{i,j}^D &= -\frac{\pi}{2} \sum_{k=1}^4 z^D(P_{i,j}^{(k)}), \\ U_{i,j}^G &= \frac{1}{2\pi} \left[-\sqrt{2}h \left(\log h - 1 + \frac{\pi}{4} \right) \sum_{k=1}^4 z^G(P_{i,j}^{(k)}) + \frac{\pi}{2} \sum_{k=1}^4 u(P_{i,j}^{(k)}) \right], \end{aligned}$$

where z^S , z^D and z^G are the solutions of the linear systems (2.10), (2.20) and (2.28), respectively. By Corollary 3.9 and the linear system (2.10), we see that

$$\begin{aligned} \sum_{l=1}^4 u(P_{i,j}^{(l)}) &= \sum_{k=1}^4 \left(z^S(P_{i,j}^{(k)}) \sum_{l=1}^4 \mathcal{S} \psi^{(k)}(P_{i,j}^{(l)}) \right) \\ &= \frac{h}{\sqrt{2}} \left(8 - \pi - \log 4 - 4 \log(2h^2) \right) \sum_{k=1}^4 z^S(P_{i,j}^{(k)}), \end{aligned}$$

or equivalently

$$\sum_{k=1}^4 z^S(P_{i,j}^{(k)}) = \frac{\sqrt{2}}{h(8 - \pi - \log 4 - 4 \log(2h^2))} \sum_{k=1}^4 u(P_{i,j}^{(k)}).$$

Hence the first equation of (3.11) shows that

$$U_{i,j}^S = \frac{2 \log h - 2 + \frac{\pi}{2}}{4 \log(2h^2) + \log(4) + \pi - 8} \sum_{k=1}^4 u(P_{i,j}^{(k)}),$$

which prove Theorem 2.1.

From Corollary 3.11 and the linear system (2.20), we see that

$$\sum_{l=1}^4 u(P_{i,j}^{(l)}) = -2\pi \sum_{l=1}^4 z^D(P_{i,j}^{(l)}).$$

Hence the second equation of (3.11) gives

$$U_{i,j}^D = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}),$$

which prove Theorem 2.3.

Finally, two Corollary 3.9 and 3.11, and the linear system (2.28) show that

$$\sum_{l=1}^4 z^G(P_{i,j}^{(l)}) = 0.$$

Hence the third equation of (3.11) shows

$$U_{i,j}^G = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}),$$

which prove Theorem 2.5. □

For a further discussion, we define

$$\mathcal{M} = \begin{pmatrix} \mathcal{S}\psi^{(1)}(P_{i,j}^{(1)}) & \mathcal{S}\psi^{(2)}(P_{i,j}^{(1)}) & \mathcal{S}\psi^{(3)}(P_{i,j}^{(1)}) & \mathcal{S}\psi^{(4)}(P_{i,j}^{(1)}) \\ \mathcal{S}\psi^{(1)}(P_{i,j}^{(2)}) & \mathcal{S}\psi^{(2)}(P_{i,j}^{(2)}) & \mathcal{S}\psi^{(3)}(P_{i,j}^{(2)}) & \mathcal{S}\psi^{(4)}(P_{i,j}^{(2)}) \\ \mathcal{S}\psi^{(1)}(P_{i,j}^{(3)}) & \mathcal{S}\psi^{(2)}(P_{i,j}^{(3)}) & \mathcal{S}\psi^{(3)}(P_{i,j}^{(3)}) & \mathcal{S}\psi^{(4)}(P_{i,j}^{(3)}) \\ \mathcal{S}\psi^{(1)}(P_{i,j}^{(4)}) & \mathcal{S}\psi^{(2)}(P_{i,j}^{(4)}) & \mathcal{S}\psi^{(3)}(P_{i,j}^{(4)}) & \mathcal{S}\psi^{(4)}(P_{i,j}^{(4)}) \end{pmatrix}.$$

Lemma 3.12. Assume that $Z_0 = (z_0(P_{i,j}^{(1)}), z_0(P_{i,j}^{(2)}), z_0(P_{i,j}^{(3)}), z_0(P_{i,j}^{(4)}))^T$ is the solution of the linear system

$$\mathcal{M}Z_0 = (1, 1, 1, 1)^T.$$

Then the weight $\lambda(h)$ in Theorem 2.1 becomes

$$\lambda(h) = \frac{\mathcal{S}\mathcal{P}z_0(P_{i,j})}{4}.$$

Proof. Lemma 3.8 and a careful calculation show that

$$\begin{aligned} \mathcal{M}^{-1} = & -\frac{\sqrt{2}}{4h(-8 + 4\log(2h^2) + \pi + \log 4)} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ & + \frac{\sqrt{2}}{4h(\pi - \log 4)} \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \\ & + \frac{\sqrt{2}}{4h} \left(\frac{1}{1 + \log 2} - \frac{1}{\pi - \log 4} \right) \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence for each $k = 1, 2, 3, 4$, we have

$$z_0(P_{i,j}^{(k)}) = -\frac{\sqrt{2}}{h(-8 + 4\log(2h^2) + \pi + \log 4)}.$$

Thus, Lemma 3.7 shows that

$$\begin{aligned} \mathcal{SP}z_0(P_{i,j}) &= \sum_{k=1}^4 z_0(P_{i,j}^{(k)}) \mathcal{S}\psi^{(k)}(P_{i,j}) \\ &= \frac{8(\log h - 1 + \frac{\pi}{4})}{-8 + 4\log(2h^2) + \pi + \log 4} = 4\lambda(h), \end{aligned}$$

which complete the proof. \square

From this lemma, we now modify the discrete approximation $U_{i,j}^S$ given in the formula (2.12) as follows

$$(3.12) \quad \tilde{U}_{i,j}^S = \frac{\mathcal{S}z_0(P_{i,j})}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}),$$

where the density function $z_0(P)$ is the solution of the boundary integral equation

$$(3.13) \quad \mathcal{S}z_0(P) = 1, \quad P \in \partial\mathcal{D}.$$

Then we have the following theorem.

Theorem 3.13. *For the solution $z_0(P)$ of the boundary integral equation (3.13), the modified discrete approximation $\tilde{U}_{i,j}^S$ defined in (3.12) is exactly given by*

$$\tilde{U}_{i,j}^S = \frac{1}{4} \sum_{k=1}^4 u(P_{i,j}^{(k)}).$$

Proof. Consider the Dirichlet problem

$$\begin{cases} \nabla^2 v(P) = 0, & P \in \mathcal{D}, \\ v(P) = 1, & P \in \partial\mathcal{D}. \end{cases}$$

Then clearly this problem has the solution $v(P) = 1$ throughout the square \mathcal{D} . On the other hand, the solution v can be represented as a single layer potential

$$1 = v(P) = \int_{\partial\mathcal{D}} z_0(Q) \log \frac{1}{|P - Q|} ds_Q, \quad P \in \mathcal{D},$$

where z_0 is the solution of the integral equation (3.13). So we get

$$S z_0(P_{i,j}) = 1.$$

Hence we can prove the modified discrete approximation formula. \square

4. Conclusion

In this article we propose a new approach deriving a discrete approximation formula on the uniform mesh grid for a two dimensional harmonic function. Three different boundary integral formulations are considered, and the same discrete approximation formula is obtained. Also, we show the discrete approximation formula has the same computational molecules with that of the finite difference formula.

Usually, the finite difference formula is derived by Taylor's expansions and hence it is known that the finite difference method is not flexible on the mesh. However, the proposed discrete approximation formulas are derived from the boundary integral formulations and so it can be extended to an arbitrary shape of the grid cell \mathcal{D} . Hence the proposed technique may be extended the whole domain Ω and one can obtain a discrete approximation formula which is highly fluid to the mesh. In our further study, we will discuss these problems and also treat the truncation error for the discrete approximation formula.

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