KYUNGPOOK Math. J. 47(2007), 495-500

Fractional Derivative Associated with the Multivariable Polynomials

Vinod Bihari Lal Chaurasia *and* Ashok Singh Shekhawat
Department of Mathematics, University of Rajasthan, Jaipur–302004, India
e-mail : csmaths2004@yahoo.com

Abstract. The aim of this paper is to derive a fractional derivative of the multivariable H-function of Srivastava and Panda [7], associated with a general class of multivariable polynomials of Srivastava [4] and the generalized Lauricella functions of Srivastava and Daoust [9]. Certain special cases have also been discussed. The results derived here are of a very general nature and hence encompass several cases of interest hitherto scattered in the literature.

1. Introduction

In this paper the H-function of several complex variables introduced and studied by Srivastava and Panda [7] is an extension of the multivariable G-function and includes Fox's H-function, Meijer's G-function of one and two variables, the generalized Lauricella functions of Srivastava and Daoust [9], Appell functions etc.. In this note we derive a fractional derivative of H-function of several complex variables of Srivastava and Panda [7], associated with a general polynomials (multivariable) of Srivastava [4] and the generalized Lauricella functions of Srivastava and Daoust [9].

2. Definitions and notations

By Oldham and Spanner [2] and Srivastava and Goyal [5], the fractional derivatives of a function $f(t)$ of complex order γ (or alternatively, a γ-th order fractional integral of $f(t)$) by

$$(2.1)\qquad {}_aD_t^{\gamma}\{f(t)\} = \begin{cases} \dfrac{1}{\Gamma(-\gamma)}\displaystyle\int_0^t (t-x)^{-\gamma-1}\, f(x)dx, & \mathrm{Re}(\gamma) < 0, \\[2ex] \dfrac{d^m}{dt^m}\, {}_aD_t^{\gamma-m}\ \{f(t)\}, & 0 \le \mathrm{Re}(\gamma) < m, \end{cases}$$

where m is a positive integer.

Received September 20, 2006.
2000 Mathematics Subject Classification: 44A10, 33C20, 26A33.
Key words and phrases: fractional derivative, multivariable H-function, general class of multivariable polynomials, generalized lauricella functions.

The multivariable H-function is defined by Srivastava and Panda [7] in the following manner

$$(2.2)\qquad H[z_1,\cdots,z_r]\equiv H^{0,\lambda:(u';v');\cdots;(u^{(r)};v^{(r)})}_{A,C:[B';D'];\cdots;[B^{(r)};D^{(r)}]}$$

$$\cdot\left[\begin{array}{c|c} z_1 & [(a):\theta';\cdots;\theta^{(r)}]:[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}] \\ \vdots & \\ z_r & [(c):\psi';\cdots;\psi^{(r)}]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}] \end{array}\right]$$

$$(2.3)\qquad =\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_1}\Psi(\xi_1,\cdots,\xi_r)\ \phi_1(\xi_1)\cdots\phi_r(\xi_r)\ z_1^{\xi_1}\cdots z_r^{\xi_r}d\xi_1\cdots d\xi_r,$$

where $i=\sqrt{-1}$.

The convergence conditions and other details of the H-function of several complex variables $H[z_1,\cdots,z_r]$ are given by Srivastava, Gupta and Goyal [6]. The general polynomials (multivariable) defined by Srivastava [4] represented in the following manner

$$(2.4)\qquad S^{p_1,\cdots,p_s}_{q_1,\cdots,q_s}[x_1,\cdots,x_s] = \sum_{K_1=0}^{[q_1/p_1]}\cdots\sum_{K_s=0}^{[q_s/p_s]}\frac{(-q_1)_{p_1K_1}}{K_1!}\cdots\frac{(-q_s)_{p_sK_s}}{K_s!}$$
$$A[q_1,K_1;\cdots;q_s,K_s]x_1^{K_1}\cdots x_s^{K_s},$$

where $q_j=0,1,2,\cdots$; $p_j\neq 0$ $(j=1,\cdots,s)$ are non-zero arbitrary positive integer. The coefficients $A[q_1,K_1;\cdots;q_s,K_s]$ being arbitrary constants, real or complex.

The following known result of Srivastava and Panda [8].

Lemma. *If* $\lambda(\geq 0)$, $0<x<1$, $Re(1+p)>0$, $Re(q)>-1$, $\lambda_i>0$ *and* $\Delta_i>0$ *or* $\Delta_i=0$ *and* $|z_i|<\sigma$, $i=1,2,\cdots,r$, *then*

$$(2.5)\qquad x^{\lambda}F\begin{pmatrix} z_1x^{\lambda_1} \\ \vdots \\ z_rx^{\lambda_r}\end{pmatrix} = \sum_{M=0}^{\infty}\frac{(1+p+q+2M)(-\lambda)_M\ (1+p)_\lambda}{M!\ (1+p+q+M)_{\lambda+1}}$$
$$\cdot F_M[z_1,\cdots,z_r]\cdot\ {}_2F_1\left[\begin{array}{ll} -M,1+p+q+M & ; \\ & \quad x \\ 1+p & ; \end{array}\right],$$

where

$$(2.6)\qquad F_M[z_1,\cdots,z_r]$$
$$= F^{E+2:U';\cdots;U^{(r)}}_{P+2:V';\cdots;V^{(r)}}\left[\begin{array}{l} [(e):\eta';\cdots;\eta^{(r)}],[1+p+\lambda:\lambda_1;\cdots;\lambda_r], \\ \\ [(g):\xi';\cdots;\xi^{(r)}],[2+p+q+M+\lambda:\lambda_1;\cdots;\lambda_r], \end{array}\right.$$
$$\left.\begin{array}{lll} [\lambda+1;\cdots;\lambda_r] & :[(w'):x'];\cdots;[(w^{(r)}:x^{(r)}] & ; \\ & & z_1,\cdots,z_r \\ [\lambda-M+1;\lambda_1;\cdots;\lambda_r] & :[(V'):t'];\cdots;[(V^{(r)}):t^{(r)}] & ; \end{array}\right],$$

where $M \geq 0$.

In this paper, we also use the shorthand notations as follows:

Following the notations given by Srivastava and Daoust [9]

$$F^{E:U';\cdots;U^{(r)}}_{P:V';\cdots;V^{(r)}}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_r\end{pmatrix} \equiv F\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_r\end{pmatrix}, \tag{2.7}$$

denote the generalized Lauricella function of several complex variables.

The special case of the fractional derivatives of Oldham and Spanier [2] is

$$D_t^{\gamma}(t^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\gamma+1)}\, t^{\mu-\gamma}, \qquad \mathrm{Re}(\mu) > -1. \tag{2.8}$$

3. The main result

Our main result of the present paper is the fractional derivative formula involving the Lauricella functions, generalized polynomials and the multivariable H-function as follows:

$$D_t^{\gamma}\left\{(t-x)^{\sigma}\eta^{\sigma}(y-t)^{\sigma+\rho}\, F\begin{pmatrix}\tau_1\{\eta(y-t)\}^{\sigma_1}\\ \\ \tau_r\{\eta(y-t)\}^{\sigma_r}\end{pmatrix} S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s} \right. \tag{3.1}$$

$$\left. \cdot\begin{bmatrix}(t-x)^{a_1}(y-t)^{b_1}\\ \vdots\\ (t-x)^{a_s}(y-t)^{b_s}\end{bmatrix} \times H\begin{bmatrix}z_1\{t(t-x)\}^{\sigma_1}\{t(y-t)\}^{\rho_1}\\ \vdots\\ z_r\{t(t-x)\}^{\sigma_r}\{t(y-t)\}^{\rho_r}\end{bmatrix}\right\}$$

$$= \sum_{\alpha,\beta=0}^{\infty}\sum_{K,M=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$\cdot(N_1,K_1;\cdots;N_s,K_s)\Delta\cdot\ H^{0,\lambda+3:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A+3,C+3:[B',D'];\cdots;[B^{(r)},D^{(r)}]}$$

$$\cdot\left[\begin{matrix}z_1(-x)^{\sigma_1}y^{\rho_1}t^{\rho_1+\sigma_1}\\ \vdots\\ z_r(-x)^{\sigma_r}y_1^{\rho_r}t^{\rho_r+\sigma_r}\end{matrix}\ \middle|\ \begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r),\\ \\ [(c):\psi';\cdots;\psi^{(r)}],\end{matrix}\right.$$

$$(-\sigma-\sum_{i'=1}^{s}a_{i'},K_{i'}:\sigma_1;\cdots;\sigma_r),(-\rho-K-\sum_{i'=1}^{s}b_{i'}K_{i'}:\rho_1;\cdots;\rho_r),$$

$$(\alpha-\sigma-\sum_{i'=1}^{s}a_{i'}K_{i'}:\sigma_1;\cdots;\sigma_r),(\beta-\rho-K-\sum_{i'=1}^{s}b_{i'}K_{i'}:\rho_1;\cdots;\rho_r),$$

$$\left.\begin{matrix}[(a):\theta';\cdots;\theta^{(r)}] & :[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}]\\ (\gamma-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r) & :[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right],$$

where

$$\Delta = (-1)^{\alpha} \frac{(1+p+q+2M)(1+p+q+M)_k\,(-M)_K\,(-\sigma_k)(1+p)_\sigma}{k!\;M!\;(1+p+q+M)_{\sigma+1}\;(1+p)_k\;\Gamma(\alpha+1)\;\Gamma(\beta+1)}$$
$$\cdot\,\eta^k(-x)^{\sigma-\alpha+\sum_{i'=1}^{s} a_{i'}k_{i'}}\;y^{\rho+k-\beta+\sum_{i'=1}^{s} b_{i'}k_{i'}}\;t^{\alpha+\beta+\gamma}\;F_M[z_1,\cdots,z_r];$$
$$\sigma_i>0,\quad \rho_i>0,\quad i=1,2,\cdots,r\;;$$

and

$$\mathrm{Re}(\sigma)+\sum_{i=1}^{r}\sigma_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1,\qquad \mathrm{Re}(\rho)+\sum_{i=1}^{r}\rho_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1.$$

Proof. In order to prove (3.1), express the Lauricella function by (2.5) and the multivariable H-function in terms of Mellin-Barnes type of contour integrals by (2.3) and generalized polynomials given by (2.4) respectively and collecting the powers of $(t-x)$ and $(y-t)$. Finally making use of the result (2.8), we get (3.1). □

4. Special cases

(I) With $\lambda = A = C = 0$, the multivariable H-function breaks into product of r Fox's H-functions and consequently there holds the following result

$$(4.1)\qquad D_t^{\gamma}\left\{(t-x)^{\sigma}\eta^{\sigma}(y-t)^{\sigma+\rho}\;F\begin{pmatrix}\tau_1\{\eta(y-t)\}^{\sigma_1}\\ \\ \tau_r\{\eta(y-t)\}^{\sigma_r}\end{pmatrix}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\right.$$
$$\cdot\begin{bmatrix}(t-x)^{a_1}(y-t)^{b_1}\\ \vdots\\ (t-x)^{a_s}(y-t)^{b_s}\end{bmatrix}\cdot\prod_{i=1}^{r}H_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}$$
$$\left.\cdot\left[z_i\{t(t-x)\}^{\sigma_i}\{t(y-t)\}^{\rho_i}\;\middle|\;\begin{matrix}[b^{(i)}:\phi^{(i)}]\\ \\ [d^{(i)}:\delta^{(i)}]\end{matrix}\right]\right\}$$
$$=\sum_{\alpha,\beta=0}^{\infty}\sum_{K,M=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$
$$A(N_1,K_1;\cdots;N_s,K_s)\Delta\cdot\;H_{3,3:[B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,3:(u',v');\cdots;(u^{(r)},v^{(r)})}$$
$$\cdot\left[\begin{matrix}z_1(-x)^{\sigma_1}y^{\rho_1}t^{\rho_1+\sigma_1}\\ \vdots\\ z_r(-x)^{\sigma_r}y^{\rho_r}t^{\rho_r+\sigma_r}\end{matrix}\;\middle|\;\begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r),\\ \\ (\alpha-\sigma-\sum_{i'=1}^{s}a_{i'}K_{i'}:\sigma_1;\cdots;\sigma_r),\end{matrix}\right.$$

$$(-\sigma-\sum_{i'=1}^{s}a_{i'},K_{i'}:\sigma_1;\cdots;\sigma_r),(-\rho-K-\sum_{i'=1}^{s}b_{i'}K_{i'}:\rho_1;\cdots;\rho_r):$$

$$(\beta-\rho-K-\sum_{i'=1}^{s}b_{i'}K_{i'}:\rho_1;\cdots;\rho_r),(\gamma-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r):$$

$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}]\\ \\ [(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right],$$

valid under the conditions surrounding (3.1).

(II) If $\phi^{(i)}=\delta^{(i)}=1$, $(i=1,2,\cdots)$ equation (4.1) reduces to

$$D_t^{\gamma}\left\{(t-x)^{\sigma}\eta^{\sigma}(y-t)^{\sigma+\rho}\ F\begin{pmatrix}\tau_1\{\eta(y-t)\}^{\sigma_1}\\ \vdots\\ \tau_r\{\eta(y-t)\}^{\sigma_r}\end{pmatrix}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\right.$$
$$\cdot\begin{bmatrix}(t-x)^{a_1}(y-t)^{b_1}\\ \vdots\\ (t-x)^{a_s}(y-t)^{b_s}\end{bmatrix}\cdot\prod_{i=1}^{r}G_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}$$
$$\left.\cdot\left[z_i\{t(t-x)\}^{\sigma_i}\{t(y-t)\}^{\rho_i}\ \middle|\ \begin{matrix}(b^{(i)})\\ \\ (d^{(i)})\end{matrix}\right]\right\}\tag{4.2}$$

$$=\sum_{\alpha,\beta=0}^{\infty}\sum_{K,M=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$
$$\cdot(N_1,K_1;\cdots;N_s,K_s)\Delta\cdot\ H_{3,3:[B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,3:(u',v');\cdots;(u^{(r)},v^{(r)})}$$

$$\cdot\left[\begin{matrix}z_1(-x)^{\sigma_1}y^{\rho_1}t^{\rho_1+\sigma_1}\\ \vdots\\ z_r(-x)^{\sigma_r}y^{\rho_r}t^{\rho_r+\sigma_r}\end{matrix}\ \middle|\ \begin{matrix}(-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r),\\ \\ (\alpha-\sigma-\sum_{i'=1}^{s}a_{i'}K_{i'}:\sigma_1;\cdots;\sigma_r),\end{matrix}\right.$$

$$(-\sigma-\sum_{i'=1}^{s}a_{i'},K_{i'}:\sigma_1;\cdots;\sigma_r),(-\rho-K-\sum_{i'=1}^{s}b_{i'}K_{i'}:\rho_1;\ldots;\rho_r):$$

$$(\beta-\rho-K-\sum_{i'=1}^{s}b_{i'}K_{i'}:\rho_1;\cdots;\rho_r),(\gamma-\alpha-\beta:\rho_1+\sigma_1;\cdots;\rho_r+\sigma_r):$$

$$\left.\begin{matrix}(b');\cdots;(b^{(r)})\\ \\ (d');\cdots;(d^{(r)})\end{matrix}\right],$$

valid under the conditions as obtainable from (3.1).

(III) Letting $N_i = 0$, $(i = 1, \cdots, s)$, the result in (3.1) reduces to the known result given by Sharma and Singh [3] , after a little simplification.

(IV) Replacing $N_1, \cdots, N_s$ by N in (3.1), we have a known result recently obtained by Chaurasia and Singhal [1].

Acknowledgment. The authors are grateful to Professor H. M. Srivastava, University of Victoria, Canada for his kind help and valuable suggestions in the preparation of this paper.

References

[1] V. B. L. Chaurasia and V. K. Singhal, *Fractional derivative of the multivariable polynomials*, Bull. Malaysian Math. Sc. Soc.(Second Series), **26**(2003), 1-8.

[2] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

[3] C. K. Sharma and Singh Indra Jeet, *Fractional derivatives of the Lauricella functions and the multivariable H-function*, Jñānābha, **1**(1991), 165-170.

[4] H. M. Srivastava, *A multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials*, Pacific J. Math., **117**(1985), 157-191.

[5] H. M. Srivastava and S. P. Goyal, *Fractional derivatives of the H-function of several variables*, J. Math. Anal. Appl., **112**(1985), 641-651.

[6] H. M. Srivastava, K. C. Gupta and S. P. Goyal, The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi-Madras, 1982.

[7] H. M. Srivastava and R. Panda, *Some bilateral generating functions for a class of generalized hypergeometric polynomials*, J. Reine Angew. Math., **283/284**(1976), 265-274.

[8] H. M. Srivastava and R. Panda, *Certain expansion formulas involving the generalized Lauricella functions*, II Comment Math. Univ. St. Paul., **24**(1974), 7-14.

[9] H. M. Srivastava and M. C. Daoust, *Certain generalized Neuman expansions associated with the Kampé de Fériet function*, Nederl. Akad. Wetensch Indag. Math., **31**(1969), 449-457.