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The Cauchy Representation of Integrable and Tempered Boehmians

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Abstract. This paper deals with, by employing the relation between Cauchy representation and the Fourier transform and properties of the former in L_1-space, the investigation of the Cauchy representation of integrable Boehmians as a natural extension of tempered distributions, we have investigated Cauchy representation of tempered Boehmians. An inversion formula is also proved.

1. Preliminaries

A relation between the Cauchy representation of the Fourier transform of the functions in L_2-space and a decomposition of the Fourier transform into two parts, each of which gives an analytic function in a half plane, defines that the decomposed transform is convergent for classes of functions larger than those in L_2-space.

The paper is divided into three sections. In Section 1 we give some definitions and results useful for our investigations. In Section 2 we have investigated the Cauchy representation of integrable Boehmians, by using the relation between the Cauchy representation and the Fourier transform and the properties of the former in L_1-space. Section 3 deals with the investigation of the Cauchy representation of tempered Boehmians, and inversion formulae are proved for both the investigations, in Sections 2 and 3.

In what follows, we mention basic properties of the Cauchy representation, some definitions and few terminologies.

Property 1. *Suppose* $f \in L_2$, $g(\omega) = \mathcal{F}(f, \omega)$ *and* $\hat{g}(z)$ *is the Cauchy representa-*

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tion of $g(z)$, where $z = x + iy$. Then

$$\hat{g}(z) = \begin{cases} \int_0^\infty f(t)e^{itz}dt, & y > 0 \\ -\int_{-\infty}^0 f(t)e^{itz}dt, & y < 0 \end{cases} \tag{1}$$

Equation (1) remains true even if the assumption $f \in L_2$ is replaced by $f, g \in L_1$.

Property 2. *Suppose $g \in L_2$, $f(t) = F^{-1}(g,t)$ and $\hat{f}(z), z = x + iy$ be the Cauchy representation of f. Then*

$$\hat{f}(z) = \begin{cases} (2\pi)^{-1} \int_{-\infty}^0 g(\omega)e^{-i\omega z}d\omega, & y > 0 \\ -(2\pi)^{-1} \int_{-\infty}^0 g(\omega)e^{-i\omega z}d\omega, & y < 0 \end{cases} \tag{2}$$

Definition 1. A function f is called a tempered function (or a function of slow growth or a slowly increasing function) if f is a continuous function and for some α,

$$| f | = O(| t |)^\alpha). \tag{3}$$

Definition 2. The generalized Fourier transform and the generalized inverse Fourier transform of a tempered function f are defined and denoted, respectively, as

$$\hat{\mathcal{F}}(f,z) = \begin{cases} \int_0^\infty f(t)e^{itz}dt, & y > 0 \\ -\int_{-\infty}^0 f(t)e^{itz}dt, & y < 0 \end{cases} \tag{4}$$

and

$$\hat{\mathcal{F}}^{-1}(f,z) = \begin{cases} (2\pi)^{-1} \int_{-\infty}^0 f(t)e^{-itz}dt, & y > 0 \\ -(2\pi)^{-1} \int_0^\infty f(t)e^{-itz}dt, & y < 0 \end{cases} \tag{5}$$

where $z = x + iy$.

Property 3. *$\hat{\mathcal{F}}(f,z)$ and $\hat{\mathcal{F}}^{-1}(f,z)$, defined in equations (4) and (5) are analytic* *functions for* $y \neq 0$. *Following identities hold true between* $\hat{\mathcal{F}}(f,z)$ *and the classical Fourier transform :*

$$\left.\begin{array}{lcll} \hat{\mathcal{F}}(f,x+i\epsilon) & = & \mathcal{F}(f(t)H(t)e^{-\epsilon t},x), & \epsilon>0 \\ \text{and} \quad \hat{\mathcal{F}}(f,x-i\epsilon) & = & -\mathcal{F}(f(t)H(-t)e^{-\epsilon t},x), & \epsilon>0 \end{array}\right\}, \tag{6}$$

where $H(t)$ *is a positive function, which is convenient to consider in the proof of inversion theorem and has a positive Fourier transform whose integral is easily calculated. Therefore, we have*

$$\hat{\mathcal{F}}(f,x+i\epsilon)-\hat{\mathcal{F}}(f,x-i\epsilon)=\mathcal{F}(e^{-\epsilon|t|}f(t),x)=\int_{-\infty}^{\infty}e^{-\epsilon|t|}f(t)e^{itx}dt. \tag{7}$$

Property 4. *Let* f *is a tempered function, then the generalized Fourier transform of* f *has the property*

$$\langle\mathcal{F}(f),\varphi\rangle \;=\; \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(\hat{\mathcal{F}}(f,x+i\epsilon)-\hat{\mathcal{F}}(f,x-i\epsilon))\varphi(x)\,dx, \tag{8}$$

for all $\varphi\in S$ *and* $\epsilon>0$.

Property 5. *For a given tempered function* f, *the inversion formula for the generalized Fourier transform* $\hat{\mathcal{F}}(f,z)$ *has the property*

$$(2\pi)^{-1}\int_{-\infty}^{\infty}(\hat{\mathcal{F}}(f,x+i\epsilon)-\hat{\mathcal{F}}(f,x-i\epsilon))e^{-ixt}\,dt=e^{-\epsilon|t|}f(t),\ \epsilon>0 \tag{9}$$

and

$$\lim_{\epsilon\to 0}\mathcal{F}^{-1}(\hat{\mathcal{F}}(f,x+i\epsilon)-\hat{\mathcal{F}}(f,x-i\epsilon),t) \;=\; f(t),\quad \epsilon>0. \tag{10}$$

Property 6. *Let* T *is a functional for the* m*th derivative of the tempered function defined in* L_1, *then the generalized Fourier transform of* T *is defined as* $\hat{\mathcal{F}}(T,z)=(-iz)^m\hat{\mathcal{F}}(f,z)$.

Property 7. *Let* $T\in S'$ *and* $\hat{\mathcal{F}}(T,z)$ *be a generalized Fourier transform of* T. *Then* $\hat{\mathcal{F}}(t,z)$ *is an analytic (Cauchy) representation of* $\hat{\mathcal{F}}(T)$ *in the sense, that*

$$\lim_{\epsilon\to 0}\langle\hat{\mathcal{F}}(f,x+i\epsilon)-\hat{\mathcal{F}}(f,x-i\epsilon),\varphi(x)\rangle \;=\; \langle\mathcal{F}(T),\varphi\rangle,\quad \varphi\in S, \tag{11}$$

where S' *is the space of linear functional on* S *or the space of tempered distributions.*

Definition 3. The generalized Fourier transform $\mathcal{F}(f,z)$ for a tempered function f, for multi-variables is defined by

$$\int_0^{\infty}\cdots\int_0^{\infty}f(t)e^{i(t,z)}dt,\quad y_1>0,\cdots,y_n>0, \tag{12}$$

and

$$-\int_{-\infty}^{0}\int_{0}^{\infty}\cdots\int_{0}^{\infty} f(t)e^{i(t,z)}dt, \quad y_1 < 0,\ y_2, y_3, \cdots, y_n > 0, \tag{13}$$

where $z_j = x_j + iy_j$, $j = 1, 2, 3, \cdots$.

Equations (12) and (13) can be expressed in the form

$$\hat{\mathcal{F}}(f,z) \ = \ \int_{E^n} f(t)e^{i\langle t,z\rangle} \prod_{j=1}^{n} (\sigma_j H(\sigma_j t))\, dt, \tag{14}$$

where $\sigma_j = \text{sgn}(y_j)$, sgn is the signum function, and $\{z : \sigma_1 y_1, \sigma_2 y_2, \cdots, \sigma_n y_n > 0\}$. E^n is the n-dimensional linear space, over which the integral is defined. Further, equation (14) can be written as

$$\hat{\mathcal{F}}(t,z) \ = \ \int_{E^n} f(t)e^{itz}(\sigma H(\sigma t))\, dt \tag{15}$$

Also,

$$\mathcal{F}^{-1}(H(t)e^{itz}, \omega) \ = \ [2\pi i(\omega - z)]^{-1}, \quad y > 0 \tag{16}$$

$$\mathcal{F}^{-1}(H(-t)e^{itz}, \omega) \ = -[2\pi i(\omega - z)]^{-1}, \quad y < 0 \tag{17}$$

and

$$\mathcal{F}\left(\frac{1}{2\pi i(\omega - z)}, t\right) \ = \ H(t)e^{itz}, \quad a.e. \text{ for } y > 0 \tag{18}$$

$$\mathcal{F}\left(\frac{1}{2\pi i(\omega - z)}, t\right) \ = -H(-t)e^{itz}, \quad a.e. \text{ for } y < 0. \tag{19}$$

Following properties signify the equivalence of generalized Fourier transform and Cauchy (analytic) representation of classical Fourier transform.

Property 8. *Suppose $f \in L_1$, $\mathcal{F}(f) \in L_1$ and $F = \mathcal{F}(f)$, then the Cauchy representation of $\hat{F}(z)$ by means of the Cauchy kernel is given by*

$$\hat{F}(z) \ = \ \hat{\mathcal{F}}(f,z) \tag{20}$$

$$\hat{f}(z) \ = \ \hat{\mathcal{F}}(f,z), \tag{21}$$

where $z = x + iy$.

As a consequence of the Property 8, if $f \in S$ and F denote $\mathcal{F}(f)$, then

$$\hat{F}(z) \ = \ \hat{\mathcal{F}}(f,z), \quad y_1, y_2, \cdots, y_n \neq 0. \tag{22}$$

Property 9. *If $f, g \in L_1$ and $h(x) = (f \star g)(x)$, then*

$$\hat{h}(z) \ = \ \int_{-\infty}^{\infty} f(t)\hat{g}(z - t)\, dt$$

$$= \int_{-\infty}^{\infty} g(t)\hat{f}(z-t)\, dt. \tag{23}$$

This property is called the Cauchy representation of convolutions of functions belonging to L_1-space.

Example 1. Using equations (16)-(19), we have

$$\mathcal{F}\left(\frac{t}{t-z}, \omega\right) \;=\; 2\pi i H(\omega) e^{i\omega z}, \quad y > 0$$

where $H(\omega) \in L_1$ and L_2. Thus,

$$(2\pi i H(\omega) e^{i\omega z}) \star (2\pi i H(\omega) e^{i\omega z}) \;=\; 2\pi \mathcal{F}((t-z)^{-2}),$$

and therefore,

$$\mathcal{F}((t-z)^{-2}, \omega) \;= -2\pi H(\omega) \omega e^{i\omega z}, \quad y > 0.$$

Similarly,

$$\mathcal{F}((t-z)^{-n}, \omega) \;=\; 2\pi (i)^n H(\omega)\, \frac{\omega^{n-1}}{(n-1)!} e^{i\omega z}, \quad y > 0.$$

Definition 4. For any tempered distribution space S, the space of linear functional S' on S, the space $\mathcal{D}$ of all infinitely differentiable functions on R^n with compact support (A set $K \subset X$, a topological space, is called compact if every open cover of K contains a finite subcover) and its dual $\mathcal{D}'$ if $\mathcal{S} \in \mathcal{D}'$ and $T \in \mathcal{E}'$, then the convolution of the distributions is defined by

$$\mathcal{S} \star T \;=\; \mathcal{F}^{-1}(\mathcal{F}(\mathcal{S}) \cdot \mathcal{F}(T)) \;=\; 2\pi \mathcal{F}(\mathcal{F}^{-1}(\mathcal{S}) \cdot \mathcal{F}^{-1}(T)). \tag{24}$$

The space $\mathcal{E}(a, b)$ is the space of smooth functions on (a, b) and $\mathcal{E}'(a, b)$, or simply $\mathcal{E}'$, is the dual of the space $\mathcal{E}$.

As a consequence of equation (24) or the Definition 4, if $\mathcal{S} \in S'$ and $T \in \mathcal{E}'$, then $\mathcal{S} \star T \in S'$. Further, the convolution of the generalized Fourier transform is given by

$$\hat{\mathcal{F}}(f \star g, z) \;=\; \begin{cases} \hat{\mathcal{F}}(f, z)\hat{\mathcal{F}}(g, z), & y > 0 \\ 0, & y < 0 \end{cases} \tag{25}$$

Property 10. *Let $\mathcal{S}, T \in S'$ have support in the half axis $\{t : t > 0\}$. Then,*

$$\mathcal{S} \star T \;=\; \lim_{\epsilon \to 0} \mathcal{F}^{-1}(\hat{F}(\mathcal{S}, x + i\epsilon)\hat{\mathcal{F}}(T, x + i\epsilon)), \tag{26}$$

where $\mathcal{S} \star T \in S'$.

The proofs of all the properties mentioned in this section may be referred to [1].

2. Cauchy representation for integrable Boehmians

In this section we investigate the Cauchy representation of integrable Boehmians by using the relation between the Cauchy representation and the Fourier transform, Properties 8 and 9, and employing the properties of the Cauchy representation for function in L_1-space, enumerated in Section 1. These properties indeed support those of integrable Boehmians.

The Fourier transform for Boehmians has been defined independently by J. Burzyk (through an oral communication) and by Nemzer [6]. Definition of Burzyk is, indeed, general in nature and interestingly, in this definition, the Fourier transform of a Boehmian is not necessarily a function (whereas the Fourier transform of a tempered distribution is always a function). An exhaustive account of Boehmians may be seen in [2].

Considering L_1 as the space of complex valued Lebesgue integrable functions on the real line R, the norm of a function f in L_1 is given by

$$\|f\| = \int_R | f(x) | \, dx.$$

If $f, g \in L_1$, then the convolution $f \star g$ is

$$(f \star g)(x) = \int_R f(u)g(x-u)\, du \tag{27}$$

is an element of the space L_1 and

$$\|f \star g\| \le \|f\| \cdot \|g\|.$$

A sequence of continuous real functions $\delta \in L_1$ is called a delta sequence if

(i) $\displaystyle\int_R \delta_n(x)dx = 1, \forall\, n \in N$

(ii) $\|\delta_n\| < M$, for some $M \in R$, $\forall\, n \in N$.

(iii) $\displaystyle\lim_{n\to\infty} \int_{|x|>\epsilon} | \delta_n(x) | \, dx = 0$, for each $\epsilon > 0$.

If (φ_n) and (ψ_n) are delta sequence, the $(\varphi_n \star \psi_n)$ is also a delta sequence. If $f \in L_1$ and (δ_n) is the delta sequence, then $\|(f \star \delta_n) - f\| \to 0$, as $n \to \infty$.

A pair of sequence (f_n, φ_n) is called a quotient of sequences, and denoted by f_n/φ_n, $f_n \in L_1$ $(n = 1, 2, \cdots)$ (φ_n) is a delta sequence, and $f_m \star \varphi_n = f_n \star \varphi_m$, $\forall\, m, n \in N$. Two quotients of sequence f_n/φ_n and g_n/ψ_n are equivalent if $f_n \star \psi_n = g_n \star \varphi_n$, $\forall\, n \in N$. The equivalence class of a quotient of sequences will be called an integrable Boehmian. The space of all integrable Boehmians will be denoted by B_{L_1}. For convergence of Boehmians one may refer to [3], while the convergence and properties of integrable Boehmians may be found in [4].

Lemma 1. *If* $[f_n/\delta_n] \in B_{L_1}$*, then the sequence*

$$\hat{f}_n(z) \;=\; \int_R f(t)e^{itz}(\sigma H(\sigma t))\, dt$$

converges uniformly on each compact set in R*.*

Proof. If (δ_n) is a delta sequence, then $(\hat{\delta}_n)$ converges on each compact set to the constant function 1. Thus, for each compact set K, $\hat{\delta}_k > 0$ on K, for almost all $k \in K$ and,

$$\hat{f}_n \;=\; \hat{f}_n \frac{\hat{\delta}_k}{\hat{\delta}_k} \;=\; \frac{(f_n \star \delta_k)^\wedge}{\hat{\delta}_k} = \frac{(f_k \star \delta_n)^\wedge}{\hat{\delta}_k} \;=\; \frac{\hat{f}_k}{\hat{\delta}_k}\, \hat{\delta}_n \;\text{ on } K.$$

Indeed, the Cauchy representation of an integrable Boehmians $F = [f_n/\delta_n]$ can be defined as the limit of the sequence $(\hat{f}_n)$ in the space of continuous function on R. This proves that the Cauchy representation of an integrable Boehmian is a continuous function. Thus the lemma is proved. □

Theorem 1. *Let* $F, G \in B_{L_1}$*. Then*

(i) $(\lambda F)^\wedge = \lambda\hat{F}$*, (for any complex number* λ*), and* $(F+G)^\wedge \;=\; \hat{F} + \hat{G}$*.*

(ii) $(F \star G)^\wedge \;=\; \hat{F}\hat{G}$

(iii) $\hat{\mathcal{F}}(f^{(m)}, z) = (-iz)^m \hat{\mathcal{F}}(f, z)$

(iv) *if* $\hat{F} = 0$*, then* $F = 0$

(v) *if* $\Delta - \lim_{n\to\infty} F_n = F$*, then* $\hat{F}_n \to \hat{F}$ *uniformly on each compact set.*

Proof. Appealing to the Properties 8 and 9, the relation between the Cauchy representation and the Fourier transform, the proofs of (i)-(iii) are obvious consequence. Proof of (iv) can be completed by employing uniqueness theorem of the Fourier transform (if $f \in L_1$ and $\hat{f}(t) = 0$ for all $t \in R^1$, then $f(x) = 0$ a.e.) in L_1 (or the Theorem 2, that follows).

Now we prove the part (v):

We have

$$\delta - \lim_{n\to\infty} F_n - F \Rightarrow \hat{F}_n \to \hat{F}$$

uniformly on each compact set.

Let (δ_n) be a delta sequence such that $F_n \star \delta_k$, $F \star \delta_k \in L_1$, $\forall\, n, k \in N$ and

$$\|(F_n - F) \star \delta_k\| \to 0, \qquad \text{as } n \to \infty,$$

for each $k \in N$.

Let K be a compact set in R, then $\hat{\delta}_k > 0$ on K for some $k \in N$. Since $\hat{\delta}_k$ is a continuous function, we observe

$$\hat{F}_n \cdot \hat{\delta}_k \to \hat{F}\hat{\delta}_k$$

uniformly on K. Since,

$$\hat{F}_n \cdot \hat{\delta}_k - \hat{F} \cdot \delta_k \;=\; ((F_n - F) \star \delta_k)^{\wedge}$$

and

$$\|(F_n - F) \star \delta_k\| \to 0, \quad \text{as} \quad n \to \infty.$$

The theorem is thus completely proved. □

Let $f \in L_1$ and from Properties 4,5 and 8 i.e. (equations (8),(9) and(21))

$$f_n(t) \;=\; \mathcal{F}^{-1}(\hat{\mathcal{F}}(f,z)) \;=\; \mathcal{F}^{-1}(\hat{f}(z)) \tag{28}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(z)e^{itz}\, dt. \tag{29}$$

Then the sequence (f_n) converges to f in L_1-norm. The details may be seen in [4].

Theorem 2. *Let $f \in B_{L_1}$ and*

$$f_n(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(z)e^{itz}\, dt. \tag{30}$$

Then $\delta - \lim_{n\to\infty} f_n \;=\; F$, and hence, also $\Delta - \lim\limits_{n\to\infty} f_n \;=\; F$.

Proof. Let $F = [g_n/\delta_n]$, $k \in N$. Then

$$(f_n \star \delta_k)(t) \;=\; \int_R f_n(t-u)\delta_k(u)\, du$$

i.e.

$$= \frac{1}{2\pi}\int_R \int_{-\infty}^{\infty} e^{it(z-u)}\hat{F}(z)\delta_k(u)\, dtdu,$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itz}\hat{F}(z)\hat{\delta}_k(t)\, dt$$

i.e.

$$(f_n \star \delta_k)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itz} F \star \delta_k(t)\, dt.$$

By Lemma 2, $\|f_n \star \delta_k - F \star \delta_k\| \to 0$, as $n \to \infty$. Since k being an arbitrary positive integer, thus, $\delta - \lim\limits_{n\to\infty} f_n = F$. This proves the theorem. □

3. Cauchy representation for tempered Boehmians

Since tempered Boehmians are the natural consequence of tempered distributions, indeed using the relation (15) we in this section, investigate the Cauchy representation of tempered Boehmians, by following the terminologies of that of [5].

Let $\mathcal{J}$ be the space of slowly increasing infinitely differentiable complex-valued functions on R^N (f is called slowly increasing, if there exists a polynomial p, such that $|f(x)| \leq p(x)$, $\forall\ x \in R^N$). $\mathcal{D}$ is the space of all infinitely differentiable complex valued function on R^N with compact support.

A pair of sequence (f_n, φ_n) is called a quotient of sequences, f_n/φ_n, if $f_n \in \mathcal{J}$, $\forall\ n \in N$, and $\{\varphi_n\}$ is a delta sequence. If the function space is of slowly increasing infinitely differentiable complex-valued functions, then the investigated space of Boehmians consists of the tempered Boehmians and will be denoted by $B_{\mathcal{J}}$.

Let $F = [f_n/\varphi_n] \in B_{\mathcal{J}}$. Partial derivatives of F are defined as

$$\frac{\partial F}{\partial x_n} \ = \ \left[\left(\frac{\partial f_n}{\partial x_m} \star \varphi_n\right)/(\varphi_n \star \varphi_n)\right], \tag{31}$$

where $((\partial f_n/\partial x_m) \star \varphi_n)$ is a slowly increasing function for every $n \in N$ and $((\partial f_n/\partial x_m) \star \varphi_n)/(\varphi_n \star \varphi_n)$ is a quotient of sequence. Thus, the partial derivatives of tempered Boehmians are tempered Boehmians.

Let f be an infinitely differentiable complex valued function on R^N. If

$$\sup_{|\alpha|\leq m}\ \sup_{x\in R^N} (1 + x_1^2 + x_2^2 + \cdots + x_N^2) \left|\frac{\partial^{|\alpha|} f(x)}{\partial x^\alpha}\right| < \infty \ \ , \tag{32}$$

for every non-negative m, then f is called a rapidly decreasing function, $\alpha = (\alpha_1, \cdots, \alpha_N)$ is a multi-index, α_N are non negative integers, $\mid \alpha \mid = \mid \alpha_1 \mid + \cdots + \mid \alpha_N \mid$, and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} \ = \ \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \tag{33}$$

Let $S(R^N)$ denote the space of all rapidly decreasing function on R^N. A tempered Boehmian $F = [f_n/\varphi_n]$ is called a rapidly decreasing Boehmian if $f_n \in S$, $\forall\ n \in N$. The space of all rapidly decreasing Boehmian is denoted by B_S. If $F = [f_n/\varphi_n] \in B_{\mathcal{J}}$ and $G = [g_n/\gamma_n] \in B_S$, then the convolution $F \star G$ is defined as

$$F \star G \ = \ [(f_n \star g_n)/(\varphi_n \star \gamma_n)], \quad F \star G \in B_{\mathcal{J}}. \tag{34}$$

Let f/φ is a convolution quotient and $\dfrac{f}{\varphi}$ is the general quotient. Let $f \in \mathcal{J}$. The relation between the Cauchy representation and the Fourier transform, as defined in (15) and (21) and the Cauchy representation for tempered distributions and convolution as defined in Property 9, Example 1, Definition 4 and Property 10 in Section 1, satisfy properties of tempered Boehmians, which have been established in the theorems those follow.

The Cauchy representation of tempered function f is denoted by $\hat{f}(z)$ and the distribution of the tempered function is

$$\langle \hat{f}, \varphi\rangle \ = \ \langle f, \hat{\varphi}\rangle, \quad \varphi \in S, \tag{35}$$

where

$$\hat{\varphi}(z) \;=\; \int \varphi(t) e^{itz}(\sigma H(\sigma(t))\,dt. \tag{36}$$

Theorem 3. *If $[f_n/\varphi_n] \in B_{\mathcal{J}}$, then the sequence $\{\hat{f}_n\}$ converges in $\mathcal{D}'$. Moreover, if $[f_n/\varphi_n] = [g_n/\gamma_n] \in B_{\mathcal{J}}$, then the sequence $\{\hat{f}_n\}$ and $\{\hat{g}_n\}$ converge to the same limit.*

Proof. Let $\varphi \in \mathcal{D}$, $k \in N$ be such that $\hat{\varphi} > 0$ on the support of φ. Since

$$f_n \star \varphi_m \;=\; f_m \star \varphi_n, \quad \forall\; m, n \in N$$

and

$$\hat{f}_n \star \hat{\phi}_m \;=\; \hat{f}_m \star \hat{\phi}_n,$$

we have

$$\begin{aligned}
\hat{f}_n(\varphi) &= \hat{f}_n\left(\frac{\varphi\hat{\varphi}_k}{\hat{\varphi}_k}\right)\\
&= (\hat{f}_n\hat{\varphi}_k)\left(\frac{\varphi}{\hat{\phi}_k}\right)\\
&= (\hat{f}_k\hat{\varphi}_n)\left(\frac{\varphi}{\hat{\varphi}_k}\right)\\
&= (\hat{f}_k)\left(\frac{\varphi\hat{\varphi}_n}{\hat{\varphi}_k}\right).
\end{aligned}$$

Since the sequence $\left\{\dfrac{\varphi\hat{\varphi}_n}{\hat{\varphi}_k}\right\}$ converges to $\dfrac{\varphi}{\hat{\varphi}_k}$ in $\mathcal{D}$, therefore, sequence $\{(\hat{f}_n, \varphi)\}$ converges in $\mathcal{D}$. This proves that $\{\hat{f}_n\}$ converges in $\mathcal{D}'$. Further, assuming that $[f_n/\phi_n] = [g_n/\gamma_n] \in B_{\mathcal{J}}$, and define

$$h_n \;=\; \begin{cases} f_{\frac{n+1}{2}} \star \gamma_{\frac{n+1}{2}} & , \text{ if } n \text{ is odd}\\ g_{\frac{n}{2}} \star \varphi_{\frac{n}{2}} & , \text{ if } n \text{ is even}\end{cases}$$

and

$$\delta_n \;=\; \begin{cases} \varphi_{\frac{n+1}{2}} \star \gamma_{\frac{n+1}{2}} & , \text{ if } n \text{ is odd}\\ \varphi_{\frac{n}{2}} \star \gamma_{\frac{n}{2}} & , \text{ if } n \text{ is even.}\end{cases}$$

Then $[h_n/\delta_n] = [f_n/\phi_n] = [g_n/\gamma_n]$. Therefore, the sequence $\{\hat{h}_n\}$ converges in $\mathcal{D}'$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \hat{h}_{2n-1}(\varphi) &= \lim_{n\to\infty}(f_n \star \gamma_n)^\wedge(\varphi)\\
&= \lim_{n\to\infty}(\hat{f}_n\hat{\gamma}_n)(\varphi) = \lim_{n\to\infty}(\hat{f})(\hat{\gamma}\varphi)\\
&= \lim_{n\to\infty}\hat{f}_n(\varphi).
\end{aligned}$$

Thus, $\{\hat{h}_k\}$ and $\{\hat{f}_n\}$ have the same limit. Similarly, it can be shown that the sequence $\{\hat{h}_n\}$ and $\{\hat{g}_n\}$ will have the same limit. This completes the proof of the theorem. □

Definition 5. The mapping from $B_{\mathcal{J}}$ into the dual space $\mathcal{D}'$ can be defined by a Cauchy representation $\hat{F}$ of F, if $F = [f_n/\varphi_n] \in B_{\mathcal{J}}$ and the limit of sequence $\{\hat{f}_n\}$ is in $\mathcal{D}'$.

Theorem 4. *Let* $F = [f_n/\varphi_n] \in B_{\mathcal{J}}$ *and* $G = [g_n/\gamma_n] \in B_S$. *Then*

(i) $(\partial F/\partial x_m)^\wedge = ix_m\hat{F}$,

(ii) $\hat{G}$ *is an infinitely differentiable function,*

(iii) $(F \star G)^\wedge \;=\; \hat{F}\hat{G}$ *and*

(iv) $\hat{F}\hat{\varphi}_n \;=\; \hat{f}_n$, $\forall\, n \in N$.

Proof. (i) Considering the left hand side, we write

$$\left(\frac{\partial F}{\partial x_m}\right)^\wedge = \left[\left(\frac{\partial f_n}{\partial x_m} \star \varphi_n\right)/(\varphi_n \star \varphi_n)\right]^\wedge$$

$$= \lim_{n\to\infty}\left(\frac{\partial f_n}{\partial x_m} \star \varphi_n\right)^\wedge = \lim_{n\to\infty} ix_m\hat{f}\hat{\varphi}_m = ix_m\hat{F}. \tag{37}$$

The proof of (i) is completed owing to the relation $[f_n/\varphi_n] = [(f_n \star \varphi_n)/(\varphi_n \star \varphi_n)$, and due to Theorem 3.

(ii) Let $G = [g_n/\gamma_n] \in B_S$ and let U be a bounded open subset of R^N. Then there exists $m \in N$ such that $\hat{\gamma}_m > 0$ on U, and

$$\hat{G} = \lim_{n\to\infty}\hat{g}_n = \lim_{n\to\infty}\frac{\hat{g}_n\hat{\gamma}_m}{\hat{\gamma}_m} = \lim_{n\to\infty}\frac{\hat{g}_m\hat{\gamma}_n}{\hat{\gamma}_m}$$

$$= \frac{\hat{g}_m}{\hat{\gamma}_m}\lim_{n\to\infty}\hat{\gamma}_n = \frac{\hat{g}_m}{\hat{\gamma}_m}, \quad \text{on } U. \tag{38}$$

Since $\hat{g}_m, \hat{\gamma}_m \in S$ and $\hat{\gamma}_m > 0$ on U, $\hat{G}$ is an infinitely differentiable function on U. The proof of (ii) is thus completed.

(iii) Let $\varphi \in \mathcal{D}$. Then there exists $m \in N$ such that $\hat{\gamma}_m > 0$ on the support of φ, and we have

$$\begin{aligned}(F\star G)^{\wedge}(\varphi) &= \lim_{n\to\infty}(f_n\star g_n)^{\wedge}(\varphi)\\ &= \lim_{n\to\infty}\hat{f}_n(\hat{g}_n\varphi)\\ &= \lim_{n\to\infty}\left(\frac{\hat{g}_n\hat{\gamma}_m\varphi}{\hat{\gamma}_m}\right)\\ &= \hat{G}\lim_{n\to\infty}(\hat{f}_n\hat{\gamma}_n)(\varphi)\\ &= \hat{G}\lim_{n\to\infty}(f_n\star\hat{\gamma}_n)^{\wedge}(\varphi)\\ &= (\hat{F}\hat{G})(\varphi)\end{aligned}$$

i.e.
$$(F\star G)^{\wedge}(\varphi) = (\hat{F}\hat{G})(\varphi). \tag{39}$$

The proof of (iii) is thus completed owing to reasons given for the proof of part (i).

(iv) Let $\varphi\in\mathcal{D}$, $\forall\, m\in N$

$$\begin{aligned}(\hat{F}\hat{\varphi}_m)(\varphi) &= \hat{F}(\hat{\varphi}_m\varphi)\\ &= \lim_{n\to\infty}\hat{f}_n(\hat{\varphi}_m\varphi)\\ &= \lim_{n\to\infty}(\hat{f}_n\hat{\varphi}_m)(\varphi)\end{aligned}$$

i.e.
$$(\hat{F}\hat{\varphi}_m)(\varphi) = \hat{f}_m(\varphi) = \hat{f}_m. \tag{40}$$

The theorem is, thus, completely proved. □

Theorem 5. *A distribution of the tempered function f is Cauchy representation of a tempered Boehmian if and only if there exists a delta sequence $\{\delta_n\}$ such that $f\delta_n$ is a tempered distribution, for every $n\in N$.*

Proof. If $F=[f_n/\phi_n]\in B_{\mathcal{J}}$ and $f=\hat{F}$, then due to Theorem 4(iv),

$$f\hat{\varphi}_n = \hat{F}\hat{\varphi}_n = \hat{f}_n.$$

Thus, $f\hat{\varphi}_n$ is a tempered distribution.

Let $f\in\mathcal{D}'$ and $\{\delta_n\}$ be a delta sequence such that $f\hat{\delta}_n$ is a tempered distribution for every $n\in N$, we have

$$F = [((f\hat{\delta}_n)^{\vee}\star\delta_n)/(\delta_n\star\delta_n)], \tag{41}$$

where $(f\hat{\delta}_n)^{\vee}$ is the inverse generalized Fourier transform of the Cauchy representation of $f\hat{\delta}_n$, the tempered distribution, so is $(f\hat{\delta}_n)^{\vee}$. Indeed, F is a tempered Boehmian and $\hat{F}=f$. This completes the proof. □

Theorem 6. *Let F be a tempered Boehmian and $\hat{F}=f$. Then*

$$F = [((f\hat{\delta}_n)^{\vee}\star\delta_n)/(\delta_n\star\delta_n)], \tag{42}$$

where $\{\delta\}_n$ *be a delta sequence such that* $f\hat{\delta}_n$ *is a tempered distribution for every* $n \in N$.

Proof. If $F = [f_n/\phi_n] \in B_{\mathcal{J}}$ then appealing to the inverse formula (41). Let $\hat{\delta}_n = \varphi_n$ which reduces equation (42) to $F = [(f\hat{\varphi}_n)^{\vee}/\varphi_n]$. Hence the theorem is proved. □

Remark/Observation. The definition given by Burzyk is very general in nature in the sense that the Fourier transform of a Boehmian is not necessarily a function (whereas the Fourier transform of tempered distribution is a function). The relation between the generalized Fourier transform and Cauchy representation and the properties given in Section 1, establish the fact that, the Cauchy representation of a integrable Boehmians and tempered Boehmian is a distribution.

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