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Generalizations of Dixon's and Whipple's Theorems on the Sum of a ${}_3F_2$

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Abstract. In this paper we consider generalizations of the classical Dixon's theorem and the classical Whipple's theorem on the sum of a ${}_3F_2$. The results are derived with the help of generalized Watson's theorem obtained earlier by Mitra. A large number of results contiguous to Dixon's and Whipple's theorems obtained earlier by Lavoie, Grondin and Rathie, and Lavoie, Grondin, Rathie and Arora follow special cases of our main findings.

1. Introduction and preliminaries

In 1943, Mitra [6] generalized the classical Watson's theorem [1] on the sum of a ${}_3F_2$:

$$(1.1)\qquad {}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\,\middle|\, 1\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}$$

$$(\Re(2c-a-b)>-1)$$

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in the form

$$
\begin{aligned}
(1.2)\qquad & {}_3F_2\left(\begin{matrix} a, \quad b, & c \\ \frac{1}{2}(a+b+1), & \delta \end{matrix}\,\Big|\, 1\right) \\
= \; & \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma(a)\,\Gamma(b)} \\
& \cdot\left\{ \frac{\Gamma\left(c+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}\,\mathcal{A}(a,\,b,\,c,\,\delta) \right. \\
& \left. +\frac{2c-\delta}{\delta}\,\frac{\Gamma\left(c+\frac{3}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\,\Gamma\left(c-\frac{1}{2}a+1\right)\,\Gamma\left(c-\frac{1}{2}b+1\right)}\,\mathcal{B}(a,\,b,\,c,\,\delta) \right\} \\
& (\Re(2\delta-2c-a-b)>-1),
\end{aligned}
$$

where, for convenience,

$$
\begin{aligned}
& \mathcal{A}(a,b,c,\delta) \\
:= \; & {}_7F_6\left(\begin{matrix} c-\frac{1}{2}, \frac{1}{2}c+\frac{3}{4}, c, c-\frac{1}{2}\delta, c-\frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}a, \frac{1}{2}b \\ \frac{1}{2}c-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\delta+\frac{1}{2}, \frac{1}{2}\delta, c-\frac{1}{2}a+\frac{1}{2}, c-\frac{1}{2}b+\frac{1}{2} \end{matrix}\,\Big|\, 1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathcal{B}(a,b,c,\delta) \\
:= \; & {}_7F_6\left(\begin{matrix} c+\frac{1}{2}, \frac{1}{2}c+\frac{5}{4}, c, c-\frac{1}{2}\delta+\frac{1}{2}, c-\frac{1}{2}\delta+1, \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}c+\frac{1}{4}, \frac{3}{2}, \frac{1}{2}\delta+1, \frac{1}{2}\delta+\frac{1}{2}, c-\frac{1}{2}a+1, c-\frac{1}{2}b+1 \end{matrix}\,\Big|\, 1\right).
\end{aligned}
$$

For $\delta = 2c$, we get the Watson's theorem (1.1). For another generalization of the Watson's theorem (1.1), see [3].

We, for easy reference, record here a fundamental transformation formula [1]:

$$
\begin{aligned}
(1.3)\qquad & {}_3F_2\left(\begin{matrix} a, & b, & c \\ e, & f & \end{matrix}\,\Big|\, 1\right) \\
= \; & \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\,{}_3F_2\left(\begin{matrix} e-a, & f-a, & s \\ s+b, & & s+c \end{matrix}\,\Big|\, 1\right) \\
& (s=e+f-a-b-c;\, \Re(s)>0).
\end{aligned}
$$

Recently Choi, Rathie and Malani [2] obtained an interesting result for a special summation formula for ${}_6F_5$, but here we shall use their result in a slightly different equivalent form:

$$(1.4)\quad {}_7F_6\left(\begin{matrix} c-\frac{1}{2}, & \frac{1}{2}c+\frac{3}{4}, & c, & 1, & \frac{3}{2}, & \frac{1}{2}a, & \frac{1}{2}b \\ \frac{1}{2}c-\frac{1}{4}, & \frac{1}{2}, & c-\frac{1}{2}, & c-1, & c-\frac{1}{2}a+\frac{1}{2}, & c-\frac{1}{2}b+\frac{1}{2} \end{matrix}\,\middle|\, 1\right)$$
$$= \frac{\alpha\,(2c-a-1)\,(2c-b-1)}{(2c-a-b-3)\,(2c-a-b-1)\,(2c-1)\,(c-1)},$$

where $\alpha = 2c^2-(a+b+5)c+(a+1)(b+1)+2$.

The aim of the paper is to obtain generalizations of the classical Dixon's and Whipple's theorems.

2. Generalizations of Dixon's theorem

In this section we establish generalizations of Dixon's theorem in the following form:

$$(2.1)\quad {}_3F_2\left(\begin{matrix} A, & B, & C \\ & 1+A-B, & 1+A-C+j \end{matrix}\,\middle|\, 1\right)$$
$$= \frac{2^{2A-2B-4C+2j+1}\,\Gamma(1+A-B)\,\Gamma(1+A-C+j)\,\Gamma(C-j)}{\Gamma(C)\,\Gamma(1+A-2C+j)\,\Gamma(2+2A-2B-2C+j)}$$
$$\cdot\left\{\frac{\Gamma\!\left(\frac{3}{2}+A-B-C\right)\Gamma\!\left(1+\frac{1}{2}A-B-C+\frac{1}{2}j\right)\Gamma\!\left(\frac{1}{2}+\frac{1}{2}A-C+\frac{1}{2}j\right)}{\Gamma\!\left(\frac{1}{2}\right)\Gamma\!\left(\frac{1}{2}+\frac{1}{2}A-\frac{1}{2}j\right)\Gamma\!\left(1+\frac{1}{2}A-B-\frac{1}{2}j\right)}\,\mathcal{C}(A,\,B,\,C,\,j)\right.$$
$$-\frac{j}{2+2A-2B-2C+j}\,\frac{\Gamma\!\left(\frac{5}{2}+A-B-C\right)\Gamma\!\left(\frac{3}{2}+\frac{1}{2}A-B-C+\frac{1}{2}j\right)\Gamma\!\left(1+\frac{1}{2}A-C+\frac{1}{2}j\right)}{\Gamma\!\left(\frac{3}{2}\right)\Gamma\!\left(1+\frac{1}{2}A-\frac{1}{2}j\right)\Gamma\!\left(\frac{3}{2}+\frac{1}{2}A-B-\frac{1}{2}j\right)}$$
$$\left.\cdot\mathcal{D}(A,\,B,\,C,\,j)\right\}\qquad (\Re(A-2B-2C+j)>-2),$$

where, for convenience,

$$\mathcal{C}(A,\,B,\,C,\,j) := {}_7F_6\left(\begin{matrix} A_1,\,A_2,\,A_3,\,A_4,\,A_5,\,A_6,\,A_7 \\ B_1,\,B_2,\,B_3,\,B_4,\,B_5,\,B_6 \end{matrix}\,\middle|\, 1\right),$$

where

$$A_1 = \frac{1}{2}+A-B-C,\quad A_2 = \frac{5}{4}+\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}C,$$
$$A_3 = 1+A-B-C,\quad A_4 = -\frac{1}{2}j,\quad A_5 = -\frac{1}{2}j+\frac{1}{2},$$

$$A_6 = 1 + \frac{1}{2}A - B - C + \frac{1}{2}j, \quad A_7 = \frac{1}{2} + \frac{1}{2}A - C + \frac{1}{2}j,$$

$$B_1 = \frac{1}{4} + \frac{1}{2}A - \frac{1}{2}B - \frac{1}{2}C, \quad B_2 = \frac{1}{2}, \quad B_3 = \frac{3}{2} + A - B - C + \frac{1}{2}j,$$

$$B_4 = 1 + A - B - C + \frac{1}{2}j, \quad B_5 = \frac{1}{2}A - \frac{1}{2}j + \frac{1}{2},$$

$$B_6 = 1 + \frac{1}{2}A - B - \frac{1}{2}j$$

and

$$\mathcal{D}(A,\, B,\, C,\, j) := {}_7F_6 \left(\begin{matrix} C_1,\, C_2,\, C_3,\, C_4,\, C_5,\, C_6,\, C_7 \\ D_1,\, D_2,\, D_3,\, D_4,\, D_5,\, D_6 \end{matrix} \middle| 1 \right),$$

where

$$C_1 = \frac{3}{2} + A - B - C, \quad C_2 = \frac{7}{4} + \frac{1}{2}A - \frac{1}{2}B - \frac{1}{2}C,$$

$$C_3 = 1 + A - B - C, \quad C_4 = -\frac{1}{2}j + \frac{1}{2}, \quad C_5 = -\frac{1}{2}j + 1,$$

$$C_6 = \frac{3}{2} + \frac{1}{2}A - B - C + \frac{1}{2}j, \quad C_7 = 1 + \frac{1}{2}A - C + \frac{1}{2}j,$$

$$D_1 = \frac{3}{4} + \frac{1}{2}A - \frac{1}{2}B - \frac{1}{2}C, \quad D_2 = \frac{3}{2}, \quad D_3 = 2 + A - B - C + \frac{1}{2}j,$$

$$D_4 = \frac{3}{2} + A - B - C + \frac{1}{2}j, \quad D_5 = 1 + \frac{1}{2}A - \frac{1}{2}j,$$

$$D_6 = \frac{3}{2} + \frac{1}{2}A - B - \frac{1}{2}j.$$

To verify (2.1), consider the transformation (1.3) in the form:

$$(2.2) \qquad {}_3F_2 \left(\begin{matrix} e - a, & f - b, & s \\ & s + b, & s + c \end{matrix} \middle| 1 \right) = \frac{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}\; {}_3F_2 \left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix} \middle| 1 \right)$$
$$(s = e + f - a - b - c;\ \Re(s) > 0).$$

If we set $e - a = A$, $f - a = B$, $s = C$, $s + b = 1 + A - C + j$ and $s + c = 1 + A - B$, and can use Mitra's theorem (1.2) to sum the series on the right hand side of the resulting transformation, after a little simplification, we arrive at the desired generalizations of Dixon's theorem (2.1).

If we take $j = 0$ in (2.1), we get, after a little simplification, the theorem due to Dixon [1].

If we take $j = \pm 1,\ \pm 2$ in (2.1), we get the known results due to Lavoie, Grondin, Rathie and Arora [5]. However it should be remarked in passing that in the case of $j = 2$, the resulting involved ${}_7F_6$ has -1 in the numerator, so it gets terminated. Thus, this case of $j = 2$ is also seen to be equivalent to a result given by Lavoie, Grondin, Rathie and Arora [5]. Similarly, there remain ${}_5F_4$ and ${}_7F_6$ in the case of $j=-2$. Applying a known result to ${}_5F_4$ and (1.4) to ${}_7F_6$, after a little simplification, the resulting equation of the case of $j=-2$ is seen to be equivalent to a result given by Lavoie, Grondin, Rathie and Arora [5]. Other results can also be seen to equivalent to their corresponding ones due to Lavoie, Grondin, Rathie and Arora [5].

3. Generalizations of Whipple's theorem

In this section we shall establish the following generalizations of Whipple's Theorem:

$$(3.1)\qquad {}_3F_2\left(\begin{matrix}a, & b, & c\\ & e, & f\end{matrix}\,\middle|\,1\right)=\frac{2^{2c-2a-1}\,\Gamma(e)\,\Gamma(a-j)\,\Gamma(c-j)\,\Gamma(2c-e+1)}{\Gamma(a)\,\Gamma(e-a)\,\Gamma(2c-j)\,\Gamma(2c-e-a+1)}$$
$$\cdot\left\{\frac{\Gamma\left(\frac{1}{2}e-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}e+\frac{1}{2}a-j\right)\Gamma\left(c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}-j\right)}\right.$$
$$\cdot{}_7F_6\left(\begin{matrix}E_1,\,E_2,\,E_3,\,E_4,\,E_5,\,E_6,\,E_7\\ F_1,\,F_2,\,F_3,\,F_4,\,F_5,\,F_6\end{matrix}\,\middle|\,1\right)$$
$$-\frac{j}{2c-j}\,\frac{\Gamma\left(c-j+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}e-\frac{1}{2}a+1\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}-j\right)\Gamma\left(c-\frac{1}{2}e+\frac{1}{2}a+1-j\right)}$$
$$\left.\cdot{}_7F_6\left(\begin{matrix}G_1,\,G_2,\,G_3,\,G_4,\,G_5,\,G_6,\,G_7\\ H_1,\,H_2,\,H_3,\,H_4,\,H_5,\,H_6\end{matrix}\,\middle|\,1\right)\right\}$$
$$(e+f=2c+1;\ a+b=1+j;\ \Re(c-j)>0),$$

where

$$E_1=c-j-\frac{1}{2},\quad E_2=\frac{1}{2}c-\frac{1}{2}j+\frac{3}{4},\quad E_3=c-j,\quad E_4=-\frac{1}{2}j,$$
$$E_5=-\frac{1}{2}j+\frac{1}{2},\quad E_6=\frac{1}{2}e-\frac{1}{2}a,\quad E_7=c-\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2},$$
$$F_1=\frac{1}{2}c-\frac{1}{2}j-\frac{1}{4},\quad F_2=\frac{1}{2},\quad F_3=c-\frac{1}{2}j+\frac{1}{2},\quad F_4=c-\frac{1}{2}j,$$
$$F_5=c-\frac{1}{2}e+\frac{1}{2}a+\frac{1}{2}-j,\quad F_6=\frac{1}{2}e+\frac{1}{2}a-j$$

and

$$G_1=c-j+\frac{1}{2},\quad G_2=\frac{1}{2}c-\frac{1}{2}j+\frac{5}{4},\quad G_3=c-j,\quad G_4=-\frac{1}{2}j+\frac{1}{2},$$
$$G_5=-\frac{1}{2}j+1,\quad G_6=\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2},\quad G_7=c-\frac{1}{2}e-\frac{1}{2}a+1,$$

$$H_1 = \frac{1}{2}c - \frac{1}{2}j + \frac{1}{4}, \quad H_2 = \frac{3}{2}, \quad H_3 = c - \frac{1}{2}j + 1, \quad H_4 = c - \frac{1}{2}j + \frac{1}{2},$$
$$H_5 = c - \frac{1}{2}e + \frac{1}{2}a + 1 - j, \quad H_6 = \frac{1}{2}e + \frac{1}{2}a + \frac{1}{2} - j.$$

If we set $a + b = 1 + j$, $e + f = 2c + 1$ in (1.3), and can use Mitra's theorem (1.2) to sum the series on the right hand side of the resulting transformation, after a little simplification, we arrive at the desired generalizations of Whipple's theorem (3.1).

If we take $j = 0$ in (3.1), we get, after a little simplification, the theorem due to Whipple [1].

If we take $j = \pm 1, \ \pm 2$ in (3.1), we get the known results due to Lavoie, Grondin, Rathie [4]. However it should be remarked in passing that in the case of $j = 2$, the resulting involved ${}_7F_6$ has -1 in the numerator, so it gets terminated. Thus, this case of $j = 2$ is also seen to be equivalent to a result given by Lavoie, Grondin, Rathie [4]. Similarly, there remain ${}_5F_4$ and ${}_7F_6$ in the case of $j = -2$. Applying a known result to ${}_5F_4$ and (1.4) to ${}_7F_6$, after a little simplification, the resulting equation of the case of $j = -2$ is seen to be equivalent to a result given by Lavoie, Grondin, Rathie [4]. Other results can also be seen to be equivalent to their corresponding ones due to Lavoie, Grondin, Rathie [4].

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