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Entire Functions That Share One Value With Their Derivatives

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ABSTRACT. In the paper, we use the theory of normal family to study the problem on entire function that share a finite non-zero value with their derivatives and prove a uniqueness theorem which improve the result of J.P. Wang and H.X. Yi.

1. Introduction and main results

Let f and g be some non-constant meromorphic functions. We say f and g share a value b IM(CM) iff $f-b=0 \Leftrightarrow g-b=0(f-b=0 \rightleftharpoons g-b=0)$, ignoring multiplicities (counting multiplicities). We assume that the reader is familiar with fundamental results and the standard notations of the Nevanlinna theory([5],[9],[10]).

In 1986, Jank, Mues and Volkmann proved the following result.

Theorem A. *Let f be a nonconstant entire function. If f and f' share a finite, nonzero value a IM, and if $f''(z)=a$ whenever $f(z)=a$, then $f\equiv f'$.*

Remark 1. From the hypothesis of Theorem A, it can be easily seen that the value a is shared by f and f' CM. Theorem A suggests the following Question of Yi and Yang.

Question(see [9], [10]). Let f be a nonconstant meromorphic function, let a be a finite, nonzero constant, and let n and $m(n<m)$ be positive integers. If f , $f^{(n)}$ and $f^{(m)}$ share a CM, where n and m are not both even or both odd, must $f\equiv f^{(n)}$?

An example ([7]) given by Yang shows that the answer to the above Question is,

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in general, negative. Recently, related to the Question, Li and Yang ([4]) obtained the following theorem.

Theorem B. *Let f be an entire function, let a be a finite nonzero value, and let $n(\geq 2)$ be a positive integer. If f , f', and $f^{(n)}$ share the value a CM, then f assumes the form $f(z) = be^{cz} + a - \frac{a}{c}$, where b, c are nonzero constants and $c^{n-1} = 1$.*

In 2003, J. P. Wang and H. X. Yi ([6]) proved the next result.

Theorem C. *Let f be a nonconstant entire function, let $a(\neq 0)$be a constant, and $k(\geq 2)$ be a positive integer. If f and f' share a CM, and if $f^{(k)}(z) = a$ whenever $f(z) = a$, then f assumes the form $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where $A(\neq 0)$ and λ are constants satisfying $\lambda^{k-1} = 1$.*

Remark 2. Under the hypothesis of Theorem C, we must have $f' \equiv f^{(k)}$. In Theorem C, if $k = 2$, then we have $\lambda = 1$ which implies $f \equiv f'$. So Theorem C contains Theorem A. Obviously, Theorem C has improved Theorem B.

It is natural to ask the following question: what can we say if CM is replaced by IM in Theorem C? In this paper, we use the theory of normal families to prove the following results.

Theorem 1. *Let f be a nonconstant entire function, let $a(\neq 0)$be a constant, and $k(\geq 2)$ be a positive integer. If f and f' share a IM, and $f^{(k)}(z) = a$ whenever $f(z) = a$, and if there exist $z_0 \in C$ satisfying $f^{(k)}(z_o) = f'(z_o) = b$, where $b \neq a$ is a constant, then f assumes the form $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where $A(\neq 0)$ and λ are constants satisfying $\lambda^{k-1} = 1$.*

Corollary 1. *Let f be a nonconstant entire function, let $a(\neq 0)$ be a constant, let $k \geq 2$ be a positive integer. If f and f' share a IM and $f'(z) = a \to f^{(k)}(z) = a$, then f assumes the form $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where $A(\neq 0)$ and λ are constants satisfying $\lambda^{k-1} = 1$.*

Corollary 2. *Let f be a nonconstant entire function, let $a(\neq 0)$ be a constant, let $k \geq 2$ be a positive integer. If $f(z) = a \rightleftharpoons f'(z) = a \Rightarrow |f^{(k)}(z)| \leq M$, M is a positive number, then $\frac{f' - a}{f - a} = c$, where c is a nonzero constant.*

2. Some lemmas

Lemma 1([1]). *Let ζ be a family of holomorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let α be a function holomorphic in D, such that $\alpha(z) \neq 0$ for $z \in D$. If for every $f \in \zeta$, $f(z) = 0 \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow |f^{(k)}(z)| \leq h$, where h is a positive number, then ζ is normal in D.*

Lemma 2([2]). *Let f be an entire function and M be a positive number. If $f^{\sharp}(z) \leq$*

M for any $z \in C$, then f is of exponential type.

Here, as usual, $f^{\sharp}(z) = \dfrac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative.

Lemma 3([3]). *Let ζ be a family of meromorphic functions in a domain D, then ζ is normal in D if and only if the spherical derivatives of functions $f \in \zeta$ are uniformly bounded on compact subsets of D.*

Lemma 4([8]). *Let $Q(z)$ be a nonconstant polynomial. Then every solution F of the differential equation $F^{(k)} - e^{Q(z)}F = 1$ is an entire function of infinite order.*

Using the same argument as in the proof of Lemma 4, we can prove the following lemma. We omit the details here.

Lemma 5. *Let $P(z)(\not\equiv 0)$ be a polynomial and $Q(z)$be a nonconstant polynomial. Then every solution F of the differential equation $F^{(k)} - P(z)e^{Q(z)}F = 1$ is an entire function of infinite order.*

Lemma 6. *Let f be a transcendental entire function with $\rho(f) \leq 1$. Let $k \geq 2$ be a positive integer. Let h be a positive number and a be a nonzero constant. If $f(z) = 0 \Rightarrow f'(z) = a$, $\quad f'(z) = a \Rightarrow |f^{(k)}(z)| \leq h$ and $N(r, \dfrac{f}{f'-a}) = S(r, f)$, then $\dfrac{f'-a}{f} = c$, where c is a nonzero constant.*

Proof. From $f(z) = 0 \Rightarrow f'(z) = a$, we get $f(z)$ only has simple zeros. Let

$$\mu = \frac{f'-a}{f}, \tag{2.1}$$

then μ is a entire function. Since f is a transcendental function, we get $\mu \not\equiv 0$, then

$$T(r, \mu) = m(r, \mu) \leq m(r, \frac{a}{f}) + S(r, f) \leq T(r, f) + S(r, f).$$

From this we can get $\rho(\mu) \leq \rho(f) \leq 1$, where $\rho(f)$ denote the order of f.

$$N(r, \frac{1}{\mu}) = N(r, \frac{f}{f'-a}) = S(r, f) = O(\log\ r) \quad (r \notin E).$$

Hence μ has finite zeros. We set $\mu = P(z)e^{bz}$, where $P(z)$ is a polynomial and b is a constant. Form (2.1), we have

$$f' - P(z)e^{bz}f = a. \tag{2.2}$$

Let $F = \dfrac{f}{a}$. Then

$$F' - P(z)e^{bz}F = 1. \tag{2.3}$$

If $b \neq 0$, by Lemma 5 we have the order of f is infinite, which is a contradiction. Thus we get $b = 0$ and

$$f' = P(z)f + a, \tag{2.4}$$

it follows from (2.4) that

$$f^{(k)}(z) = P_1(z)f + P_2(z), \tag{2.5}$$

where $P_1(z)$ and $P_2(z)$ are polynomials, $\deg(P_1) = k\deg(P)$, $\deg(P_2) = (k-1)\deg(P)$.

Case 1: If f has finite zeros, we can get $f' - a$ also has finite zeros, therefore f is a polynomial, which is a contradiction.

Case 2: If f has infinite zeros $z_1, z_2, \cdots z_n, \cdots$, and

$$|z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq \cdots, |z_n| \to \infty (n \to \infty).$$

From (2.5), we have $f^{(k)}(z_n) = P_2(z_n)$. By $|f^{(k)}(z_n)| \leq h$, we see that $P_2(z)$ is a constant, thus $P(z)$ is a constant. Let $P(z) = c$, c is a nonzero constant. From (2.4), we obtain

$$\frac{f' - a}{f} = c.$$

This completes the proof of Lemma 6. □

Lemma 7([1]). *Let g be a nonconstant entire function with $\rho(g) \leq 1$; let $k \geq 2$ be an integer, and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

Lemma 8([1]). *There does not exist entire function f satisfying that*

$$f(z) = \sum_{j=0}^{s} C_j \exp(w^j z),$$

where $w = \exp(2\pi i/k)$ and C_j are constants, and

$$f(z) = 0 \Leftrightarrow f'(z) = a.$$

Proof. From the proof of Lemma 7 in [1], we can get the conclusion. □

3. Proof of theorem 1

From the assumption, we see that f is a transcendental entire function. Let us now show that f is of exponential type. Let $F = f - a$, then

$$F = 0 \Leftrightarrow F' = a \Rightarrow F^{(k)} = a.$$

Set $\zeta = \{F(z+w) : w \in C\}$, then ζ is a family of holomorphic functions on the unit disc $\triangle$. By the assumption, for any function $g(z) = F(z+w)$, we have

$$g(z) = 0 \Leftrightarrow g'(z) = a \Rightarrow |g^{(k)}(z)| = |a|,$$

hence by Lemma 1, ζ is normal in $\triangle$. Thus by Lemma 3, there exist $M > 0$ satisfying $f^{\sharp}(z) \leq M$ for all $z \in C$. By Lemma 2, f is of exponential type. Then $\rho(f) = \rho(F) \leq 1$,

$$f(z) = a \Leftrightarrow f'(z) = a \Rightarrow f^{(k)}(z) = a. \tag{3.1}$$

We distinguish the following two cases.

Case 1. If $f' - a$ has finite multiple zeros. We know that f and f' share a IM, so $\frac{f'-a}{f-a}$ have finite zeros, and f is a transcendental entire function, we derive that

$$N(r, \frac{F}{F'-a}) = N(r, \frac{f-a}{f'-a}) = S(r,f) = S(r,F).$$

Therefore by lemma 6, we get

$$\frac{f'-a}{f-a} = \frac{F'-a}{F} = c,$$

where c is a nonzero constant. Consequently, f and f' share a CM, we can get the conclusion by Theorem A.

Case 2. If $f' - a$ has infinite multiple zeros. Then there exists

$$|a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots, |a_n| \to \infty \quad (n \to \infty), \tag{3.2}$$

where a_n is the multiple a-point of f'. We claim:

$$|f^{(k+1)}(a_n)| \leq M_1 \quad (n = 1, 2, 3, \cdots). \tag{3.3}$$

If the inequality (3.3) is not right, we suppose

$$|f^{(k+1)}(a_n)| = b_n \to \infty \quad (n \to \infty). \tag{3.4}$$

Let $g_n(z) = f(z + a_n)$, we know that ζ is normal in $\triangle$, we have $\{f(z+w) : w \in C\}$ is normal in $\triangle$. We see that

$$\{g_n\} \subset \{f(z+w) : w \in C\},$$

thus we get $\{g_n\}$ is normal in $\triangle$, $\forall\ g_n \in \{g_n\}$ we have

$$g_n(0) = f(a_n) = a,$$

hence $\{g_n\}$ is uniformly bounded on compact subsets of $\triangle$. We can get $\{g_n^{(k)}\}$ is uniformly bounded in $|z| \le \frac{1}{2}$. From this we get $\{g_n^{(k)}\}$ is normal in $|z| \le \frac{1}{2}$, but by (3.3) and (3.4), we have

$$|g_n^{(k)\sharp}(0)| = \frac{|g_n^{(k+1)}(0)|}{1+|g_n^{(k)}(0)|^2} = \frac{b_n}{1+|a|^2} \to \infty,$$

which is a contradiction. Thus we prove the claim.

Let

$$f(z) = a + a(z-a_n) + A_3(z-a_n)^3 + \cdots \qquad (n = 1,2,3\cdots). \tag{3.5}$$

Then

$$f'(z) = a + 3A_3(z-a_n)^2 + \cdots \qquad (n = 1,2,3\cdots), \tag{3.6}$$

$$f^{(k)}(z) = a + f^{(k+1)}(a_n)(z-a_n) + \cdots \qquad (n = 1,2,3\cdots). \tag{3.7}$$

Let

$$\varphi = \frac{f^{(k)} - f'}{f-a}. \tag{3.8}$$

We also distinguish the following two cases.

Subcase 2.1. $\varphi \not\equiv 0$. From the assumption and (3.8), we get φ is a entire function and

$$T(r,\varphi) = m(r,\varphi) = S(r,f) = O(\log r) \quad (r \notin E).$$

Hence we can get φ is a polynomial.

From (3.5),(3.6),(3.7) and (3.8), we have

$$\varphi(a_n) = \frac{f^{(k)} - f'}{f-a}\bigg|_{z=a_n} = \frac{1}{a} f^{(k+1)}(a_n),$$

hence

$$|\varphi(a_n)| = |\frac{1}{a} f^{(k+1)}(a_n)| \le M_1. \tag{3.9}$$

We know $\varphi(z)$ is a polynomial and $|a_n| \to \infty$ $(n \to \infty)$, from (3.9) we get φ is a nonzero constant. Let $\varphi = c$, thus we obtain

$$f^{(k)} = f' + c(f-a) \quad (c \ne 0). \tag{3.10}$$

By the assumption, we substitute z_0 into (3.10) and get a contradiction.

Subcase 2.2. $\varphi \equiv 0$, then we get

$$f^{(k)} = f'. \tag{3.11}$$

In the following we deal with the equation (3.11) in the similar way of Lemma 7.

By (3.11), we have

$$f(z) = \sum_{j=0}^{k-2} C_j \exp(w^j z) + D, \tag{3.12}$$

where $w = \exp(2\pi i/k-1)$ and C_j and D are constants.

Since f is transcendental, there exists C_j such that $C_j \neq 0$. We denote the nonzero constants in C_j by $C_{j_m}(0 \leq j_m \leq k-2, m = 0, 1, \cdots s, s \leq k-2)$. Thus we have

$$f(z) = \sum_{m=0}^{s} C_{j_m} \exp(w^{j_m} z) + D, \tag{3.13}$$

Let $z_n = r_n e^{i\theta_n}$, where $0 \leq \theta_n < 2\pi$. Without loss of generality, we may assume that $\theta_n \to \theta_0$ as $n \to \infty$. Let

$$L = \max_{0 \leq m \leq s} \cos(\theta_0 + \frac{2j_m\pi}{k-1}). \tag{3.14}$$

Then, either there exists an index m_0 such that $\cos(\theta_0 + \frac{2j_{m_0}\pi}{k-1}) = L$ or there exist two indices $m_1, m_2 (m_1 \neq m_2)$ such that $\cos(\theta_0 + \frac{2j_{m_1}\pi}{k-1}) = \cos(\theta_0 + \frac{2j_{m_2}\pi}{k-1}) = L$.

We consider these cases separately.

Case 2.2.1. There exists an index m_0 such that

$$\cos(\theta_0 + \frac{2j_{m_0}\pi}{k-1}) = L > \cos(\theta_0 + \frac{2j_m\pi}{k-1}),$$

for $m \neq m_0$. Then there exists $\delta > 0$ such that for n sufficiently large,

$$\cos(\theta_0 + \frac{2j_{m_0}\pi}{k-1}) - \cos(\theta_0 + \frac{2j_m\pi}{k-1}) \geq \delta, \quad \text{for} \quad m \neq m_0. \tag{3.15}$$

We differentiate (3.13) twice and get

$$f'' = \sum_{m=0}^{s} C_{j_m} (w^{j_m})^2 \exp(w^{j_m} z). \tag{3.16}$$

Since $f''(z_n) = 0$, we have

$$(w^{j_{m_0}})^2 C_{j_{m_0}} + \sum_{m \neq m_0} C_{jm} (w^{j_m})^2 \exp(w^{j_m} z_n - w^{j_{m_0}} z_n) = 0. \tag{3.17}$$

By (3.15), we have

$$\begin{aligned} |\exp(w^{j_m}z_n - w^{j_{m_0}}z_n)| &= \exp\left\{r_n\left(\cos(\theta_0+\frac{2j_m\pi}{k-1}) - \cos(\theta_0+\frac{2j_{m_0}\pi}{k-1})\right)\right\} \\ &\le e^{-\delta r_n}\to 0 \quad (n\to\infty). \end{aligned}$$

Thus we obtain $C_{j_{m_0}}=0$, which contradicts our assumption.

Case 2.2.2. There exist two indices $m_1, m_2 (m_1\neq m_2)$ such that

$$\cos(\theta_0+\frac{2j_{m_1}\pi}{k-1}) = \cos(\theta_0+\frac{2j_{m_2}\pi}{k-1}) = L > \cos(\theta_0+\frac{2j_m\pi}{k-1}), \tag{3.18}$$

for $m\neq m_1, m_2$. Then there exists a $\delta>0$ such that for n sufficiently large,

$$\cos(\theta_0+\frac{2j_{m_i}\pi}{k-1}) - \cos(\theta_0+\frac{2j_m\pi}{k-1}) \ge \delta, \qquad \text{for} \quad (m\neq m_1,m_2) \quad (i=1,2). \tag{3.19}$$

Since $f(z_n)=a$, $f'(z_n)=a$ and $f''(z_n)=0$, we have

$$C_{j_{m_1}}\exp(w^{j_{m_1}}z_n) + C_{j_{m_2}}\exp(w^{j_{m_2}}z_n) + \sum_{m\neq m_1,m_2} C_{j_m}\exp(w^{j_m}z_n) + D = a, \tag{3.20}$$

and

$$\begin{aligned} &C_{j_{m_1}}w^{j_{m_1}}\exp(w^{j_{m_1}}z_n) + C_{j_{m_2}}w^{j_{m_2}}\exp(w^{j_{m_2}}z_n) \\ &\quad + \sum_{m\neq m_1,m_2} C_{j_m}w^{j_m}\exp(w^{j_m}z_n) = a, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &C_{j_{m_1}}(w^{j_{m_1}})^2\exp(w^{j_{m_1}}z_n) + C_{j_{m_2}}(w^{j_{m_2}})^2\exp(w^{j_{m_2}}z_n) \\ &\quad + \sum_{m\neq m_1,m_2} C_{j_m}(w^{j_m})^2\exp(w^{j_m}z_n) = 0. \end{aligned} \tag{3.22}$$

Thus we get

$$\begin{aligned} &C_{j_{m_1}}w^{j_{m_1}}(w^{j_{m_1}} - w^{j_{m_2}})\exp(w^{j_{m_1}}z_n) \\ &\quad + \sum_{m\neq m_1,m_2} C_{j_m}w^{j_m}(w^{j_m} - w^{j_{m2}})\exp(w^{j_m}z_n) = aw^{j_{m2}}. \end{aligned} \tag{3.23}$$

Using the same argument as that used in proving $C_{j_{m_0}}=0$ above and the fact that $w^j\neq w^l (j\neq l, 0\le j,l\le k-2)$, we obtain

$$\exp(w^{j_{m_1}}z_n)\to c_0, \qquad (n\to\infty), \tag{3.24}$$

where $c_0\neq 0$ is a constant.

It follows that

$$\cos(\theta_0 + \frac{2j_{m_1}\pi}{k-1}) = \lim_{n\to\infty} \cos(\theta_n + \frac{2j_{m_1}\pi}{k-1}) = 0. \tag{3.25}$$

Similarly, we get

$$\cos(\theta_0 + \frac{2j_{m_2}\pi}{k-1}) = 0.$$

Thus we have

$$\left|\frac{2j_{m_1}\pi}{k-1} - \frac{2j_{m_2}\pi}{k-1}\right| = \pi \quad \text{and} \quad w^{j_{m_2}} = -w^{j_{m_1}}. \tag{3.26}$$

From (3.20), (3.22), (3.25) and (3.26), we can get $D = a$.
Let

$$g = \sum_{j=0}^{s} C_j exp(w^j z), \tag{3.27}$$

then

$$g = 0 \Leftrightarrow g' = a. \tag{3.28}$$

From Lemma 8 and (3.27)-(3.28), we can get a contradiction.

Thus we complete the proof of Theorem 1.

In the similar way, we can prove the Corollary 1 and Corollary 2.

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