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Oscillation of Linear Second Order Delay Dynamic Equations on Time Scales

Hassan Ahmed Agwo
Department of Mathematics, Ain Shams University, Faculty of Education, Roxy, Cairo, Egypt
e-mail : hassanagwa@yahoo.com

Abstract. In this paper, we establish some new oscillation criteria for a second-order delay dynamic equation

$$u^{\triangle\triangle}(t) + p(t)u(\tau(t)) = 0$$

on a time scale $\mathbb{T}$. The results can be applied on differential equations when $\mathbb{T} = \mathbb{R}$, delay difference equations when $\mathbb{T} = \mathbb{N}$ and for delay $q-$difference equations when $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$.

1. Introduction

In the recent years, the theory of time scales has received a lot of attention which was introduced by Stefan Hilger in his Ph.D. thesis in1988 in order to unify continuous and discrete analysis (see [12]). In fact there has been much activities concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales (or measure chains). We refer the reader to a book on the subject of time scales, by Bohner and Peterson [8] which summarizes and organizes much of time scales calculus, see also the book by Bohner and Peterson [9] for advances in dynamic equations on time scales. The problem of establishing sufficient conditions to ensure that all solutions of certain classes of second order dynamic equations are oscillatory has been studied by a number of authors [1], [2], [5]-[7], [14], [17]-[20]. Erbe and Peterson [10] consider the second order dynamic equation without delay

$$(r(t)y^{\triangle}(t))^{\triangle} + p(t)y^{\sigma} = 0 \qquad \text{for} \quad t \in \mathbb{T} \tag{1.1}$$

on a time scale $\mathbb{T}$, where $r(t)$ is bounded above and $\inf \mu(t) > 0$. By using Riccati technique they proved that if

$$\int_{t_0}^{\infty} p(t)\triangle t = \infty,$$

then every solution of (1.1) is oscillatory.

In this paper, we are concerned with the oscillation of the second-order linear delay dynamic equation

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$$u^{\triangle\triangle}(t) + p(t)u(\tau(t)) = 0 \qquad \text{for} \quad t \in \mathbb{T}, \tag{1.2}$$

where (H_1) $p(t)$ is a positive real valued $rd - continuous$ function defined on $\mathbb{T}$, (H_2) $\tau(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$ such that $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Recently, Agarwal, Bohner and Saker in [1] considered equation (1.2) and prove that if

$$\int_{t_0}^{\infty} \sigma(s)p(s)\triangle s = \infty, \tag{1.3}$$

then every bounded solution of (1.2) oscillates. Also they proved that if (1.3) holds and

$$\int_{t_0}^{\infty} \frac{\tau(s)}{\sigma(s)} p(s)\triangle s = \infty \tag{1.4}$$

or

$$\limsup_{t\to\infty} \left\{ t \int_t^{\infty} \frac{\tau(s)}{s} p(s)\triangle s \right\} = \infty, \tag{1.5}$$

then every solution of (1.2) is oscillatory. Also they established some sufficient conditions for oscillation through reducing it to equation of first order and make used the result of Zhang and X. Deng [20], (see also [3]-[6],[18]).

By a solution of equation (1.2), we mean a nontrivial real value function $u(t)$,which has the properties $u(t) \in C^2_{rd}[t_x, \infty), t_x > t_0$ and satisfying equation (1.2) for all $t > t_x$. Our attention is restricted to those solutions of equation (2) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|u(t)| : t > t_1\} > 0$ for any $t_1 > t_x$.

A solution $u(t)$ of (1.2) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Note that if $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t$, $f^{\Delta}(t) = f^{'}(t)$, and (1.2) becomes the second-order delay differential equation

$$u^{''}(t) + p(t)u(\tau(t)) = 0 \quad \text{for } t \in \mathbb{R}. \tag{1.6}$$

If $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta} = \Delta f$, and then (1.2) becomes the second-order delay difference equation

$$\Delta^2 u(t) + p(t)u(\tau(t)) = 0. \tag{1.7}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $f^{\Delta} = \Delta_h f = \frac{f(t+h)-f(t)}{h}$ and (1.2) becomes the second-order delay difference equation

$$\Delta_h^2 u(t) + p(t)u(\tau(t)) = 0. \tag{1.8}$$

If $\mathbb{T}=q^{\mathbb{N}} = \{t : t = q^n,\ n \in \mathbb{N},\ q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x_q^{\Delta}(t) = \frac{x(qt)-x(t)}{(q-1)t}$, and (1.2) becomes the second order $q-$delay difference equation

$$\Delta_q^2 u(t) + p(t)u(\tau(t)) = 0. \tag{1.9}$$

In the present paper the problem of oscillation of all solutions of equation (1.2) is investigated. For this equation a general oscillation criterion is obtained showing the joint contribution of the following two factors: the presence of the delay and the second order nature of the equation. In fact we improving and extending the technique used in [15] to be suitable for second order dynamic equations to obtain new set of sufficient conditions for oscillation of (1.2).

The paper is organized as follows: In section 2, we present some preliminaries on time scales. In section 3, we establish some new useful lemmas. In section 4 oscillation results are presented where the oscillation is due solely to the delay. In section 5 general oscillations criteria are obtained. In section 6 we study oscillation criteria for equation (1.2) due to the second order nature of the equation.

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.On any time scale $\mathbb{T}$, we defined the *forward* and *backward jump operators* by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \text{ and } \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$ is said to be *left-dense* if $\rho(t) = t$, *right-dense* if $t > \sup \mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) > t$. The *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. For the function $f : \mathbb{T} \to \mathbb{R}$ the (*delta*) derivative is defined by

$$f^{\triangle}(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{2.1}$$

f is said to be differentiable if its derivative exists. A useful formula is

$$f^{\sigma} := f(\sigma(t)) = f(t) + \mu(t)f^{\triangle}(t). \tag{2.2}$$

If f, g are differentiable, then fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma} \neq 0$) are differentiable with

$$(fg)^{\triangle} = f^{\triangle}g + f^{\sigma}g^{\triangle} = fg^{\triangle} + f^{\triangle}g^{\sigma}, \tag{2.3}$$

and

$$\left(\frac{f}{g}\right)^{\triangle} := \frac{f^{\triangle}g - fg^{\triangle}}{gg^{\sigma}}. \tag{2.4}$$

If $f^{\triangle}(t) \geq 0$, then f is nondecreasing.

A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *right-dense continuous* if it right continuous at each right-dense point and there exists a finite left limit at all left-dense points. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regressive*, if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all functions $f : \mathbb{T} \rightarrow \mathbb{R}$ which are regressive and $rd-$continuous will be denoted by C_r. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathbb{R}$ by $\mathcal{R}^+ = \{f \in C_r : 1 + \mu(t)f(t) \neq 0,\ t \in \mathbb{T}\}$. A function F with $F^{\triangle} = f$ is called an antiderivative of f and then we define

$$\int_a^b f(t)\triangle t = F(b) - F(a), \tag{2.5}$$

where $a, b \in \mathbb{T}$. It is well known that rd-continuous functions possess antiderivatives. A simple consequence of formula (2.5) is

$$\int_t^{\sigma(t)} f(s)\triangle s = \mu(t)f(t), \tag{2.6}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\triangle t = \lim_{b\rightarrow\infty} \int_a^b f(t)\triangle t. \tag{2.7}$$

If $p \in \mathbb{R}$, then the exponential function is defined as

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\nu)}(p(\nu)\triangle\nu\right), \tag{2.8}$$

for all s, $t \in \mathbb{T}$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) := \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0 \\ z, & h = 0 \end{cases}. \tag{2.9}$$

For the properties of the exponential function we refer to [8].

3. Some useful lemmas

In this section, we establish some useful lemmas which are needed to get our results

Lemma 3.1. *Assume that*

$$\int_{t_0}^{\infty} \tau(s)p(s)\Delta s = \infty \tag{3.1}$$

and $u(t)$ be a positive solution of (1.2) *on* $[T,\infty)$, *then*
(i) $u^{\triangle}(t) > 0$ and $u(t) \geq tu^{\triangle}(t)$ for $t \geq T$,
(ii) u *is nondecreasing while the function* $t \to \frac{u(t)}{t}$ *is nonincreasing on* $[T,\infty)$,
(iii) $u(\tau(t)) \geq (\frac{\tau}{\upsilon})(t)u(\nu(t))$ for $t \geq T_1 > T$, *where*

$$(\frac{\tau}{\nu})(t) = \begin{cases} 1 & \text{if} \quad \tau(t) \geq \nu(t) \\ \frac{\tau(t)}{\nu(t)} & \text{if} \quad \tau(t) \leq \nu(t) \end{cases} \tag{3.2}$$

for any function $\nu(t) : \mathbb{T} \to \mathbb{T}$ *satisfying* $\nu(t) \leq t$ *and* $\lim_{t\to\infty} \nu(t) = \infty$.

Proof. Since $u(t)$ is a positive solution of (1.2) and $p(t) > 0$, then in view of $u^{\triangle\triangle}(t) < 0$ it is obvious that $u^{\triangle}(t) > 0$ for otherwise, we have, $u(t) - u(T_1) = \int_{T_1}^{t} u^{\triangle}(s)\triangle s \leq u^{\triangle}(T_1)(t-T_1)$ and then $u(t) \to -\infty$ as $t \to \infty$, which is a contradiction. Let $\rho(t) := u(t) - tu^{\triangle}(t)$. Since $\rho^{\triangle}(t) = (u(t) - tu^{\triangle}(t))^{\triangle} = -\sigma(t)u^{\triangle\triangle}(t) \geq 0$ for $t \geq T_1$ then , we have either $u(t) - tu^{\triangle}(t) \geq 0$ for $t \geq T_1$ or $u(t) - tu^{\triangle}(t) \geq 0$ for $t \geq t_1$with some for $t_1 \geq T_1$.To prove (i), it suffices to show that the later is impossible. Assume that the second one holds, then

$$\left(\frac{u(t)}{t}\right)^{\triangle} = \frac{tu^{\triangle}(t) - u(t)}{t\sigma(t)} > 0 \qquad \text{for } t \geq t_1 \tag{3.3}$$

and consequently, $u(\tau(t)) \geq c\tau(t)$, $t \geq t_1$ for some $c > 0$. Integrating equation (1.2) from t_2 to t we have

$$u^{\triangle}(t) - u^{\triangle}(t_2) + \int_{t_2}^{t} p(s)u(\tau(s))\triangle s = 0 \tag{3.4}$$

and then

$$u^{\triangle}(t_2) \geq \int_{t_2}^{\infty} p(s)u(\tau(s))\triangle s \geq c\int_{t_2}^{\infty} p(s)\tau(s)\triangle s, \tag{3.5}$$

which contradicts (3.1). Thus (i) is proved. Since $\left(\frac{u(t)}{t}\right)^{\triangle} < 0$, then (ii) follows and (iii) follows from (ii). The proof is complete. ☐

In fact Lemma 3.1 (i) implies

$$u(\tau(t)) \geq \tau(t)u^{\triangle}(\tau(t)) \quad \text{for} \ \ t \geq T. \tag{3.6}$$

This inequality, however, can be improved through the following lemma.

Lemma 3.2. *Assume that* (3.1) *holds and* $u(t)$ *be a positive solution of* (1.2) *on* $[T,\infty)$, *then*

$$u(\tau(t)) \geq \tau_T(t)u^{\triangle}(\tau(t)) \quad \text{for} \;\; t \geq T, \tag{3.7}$$

where

$$\tau_T(t) = \tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)\triangle\xi \quad \text{for } \tau(t) \geq T. \tag{3.8}$$

Proof. Integrate the identity $(u(t) - tu^{\triangle}(t))^{\triangle} = \sigma(t)p(t)u(\tau(t))$ from T to $\tau(t)$, we obtain

$$u(\tau(t)) \geq \tau(t)u^{\triangle}(\tau(t)) + \int_T^{\tau(t)} \sigma(s)p(s)u(\tau(s))\triangle s. \tag{3.9}$$

But,

$$\begin{aligned} \int_T^{\tau(-t)} \sigma(s)p(s)u(\tau(s))\triangle s &\geq \int_T^{\tau(t)} s\; p(s)\tau(s)u^{\triangle}(\tau(s))\triangle s \\ &\geq u^{\triangle}(\tau(t)) \int_T^{\tau(t)} s\; \tau(s)p(s)\triangle s. \end{aligned} \tag{3.10}$$

Then, from (3.9) and (3.10) we obtain (3.7). the proof is complete. □

Lemma 3.2 immediately implies

Lemma 3.3. *Assume that* (3.1) *holds and* $u(t)$ *be a positive solution of* (1.2) *on* $[T,\infty)$, *then the function* $x(t) = u^{\triangle}(t)$ *is a positive solution of the inequality*

$$x^{\triangle}(t) + \tau_T(t)p(t)x(\tau(t)) \leq 0, \tag{3.11}$$

where τ_T *is defined by* (3.8).

Lemma 3.4. *Assume that* (3.1) *holds and* $u(t)$ *be a positive solution of* (1.2) *on* $[T,\infty)$, *then the function* $\vartheta(t) = \frac{u^{\triangle}(\tau(t))}{u^{\triangle}(t)}$ *is a positive solution of the inequality*

$$\vartheta(t) \geq \exp\left(-\int_{\tau(t)}^{t} \xi_{\mu(s)}(-\tau_T(s)p(s)\vartheta(s)\triangle s\right). \tag{3.12}$$

Proof. According to Lemma 3.3, $x(t) = u^{\triangle}(t)$ is a positive solution of (3.11) and then $\vartheta(t) = \frac{x(\tau(t))}{x(t)}$ for $\tau(t) \geq T$. If we rewrite (3.11) as

$$\frac{x^{\triangle}(t)}{x(t)} \leq -\tau_T(t)p(t)\vartheta(t) \qquad \text{for } \tau(t) \geq T \tag{3.13}$$

and integrate from t to $\tau(t)$ then we get (3.12). This complete the proof. □

Lemma 3.5. *Assume that* (3.1) *holds and* $u(t)$ *be a positive solution of* (1.2) *on* $[T,\infty)$, *then there exists a solution* $\vartheta(t)$ *of* (3.12) *such that*

$$u^{\triangle}(s) \geq u^{\triangle}(t)\exp - \int_{\tau(t)}^{t} \xi_{\mu(\eta)}(-\tau_T(\eta)p(\eta)\vartheta(\eta)\triangle\eta, \quad \text{for } t \geq s > T \tag{3.14}$$

Proof. From (3.13) we get

$$\frac{u^{\triangle\triangle}(t)}{u^{\triangle}(t)} \leq - - \tau_T(t)p(t)\vartheta(t) \quad for \tau(t) \geq T, \tag{3.15}$$

where $\vartheta(t)$ is a solution of (3.12). Integrating the inequality (3.15) from s to t we get (3.14). This complete the proof. □

Lemma 3.6. *Assume that the function* $\nu(t)$ *satisfying* $\nu(t) \leq t$ *and* $\lim_{t\to\infty}\nu(t) = \infty$ *and* $u(t)$ *be a positive solution of* (1.2) *on* $[T,\infty)$. *Then the function* $w(t) = \frac{u(\nu(t))}{u^{\triangle}(t)}$ *is a solution of the integral inequality*

$$w(t) \geq \int_{T}^{\nu(t)} \{\exp\{-\int_{s}^{t} \xi_{\mu(\eta)}(-\frac{\tau}{\nu}(\eta)_T(\eta)p(\eta)w(\eta))\triangle\eta\}\}\triangle s \text{ for } t \geq s > T, \tag{3.16}$$

where $\frac{\tau}{\nu}(t)$ *is defined by* (3.2).
Proof. If we rewrite (1.2) as

$$u^{\triangle\triangle}(t) = -p(t)\frac{u(\tau(t))}{u^{\triangle}(t)}u^{\triangle}(t) \qquad \text{for } \tau(t) \geq T, \tag{3.17}$$

then we have

$$u^{\triangle}(t) = u^{\triangle}(T)\exp\left(\int_{T}^{t} \xi_{\mu(\eta)}(-p(\eta)\frac{u(\tau(\eta))}{u^{\triangle}(\eta)})\triangle\eta\right) \tag{3.18}$$

and consequently

$$u(\nu(t)) \geq u^{\triangle}(T)\int_{T}^{\nu(t)} \left\{\exp\left(\int_{T}^{s} \xi_{\mu(\eta)}(-p(\eta)\frac{u(\tau(\eta))}{u^{\triangle}(\eta)})\triangle\eta\right)\right\}\triangle s. \tag{3.19}$$

Dividing (3.19) by (3.18) and using (3.2), we get (3.16). The proof is complete. □

Lemma 3.7. *Assume that the function* $\nu(t)$ *satisfying* $\nu(t) \leq t$ *and* $\lim_{t\to\infty}\nu(t) = \infty$ *and* $u(t)$ *be a positive solution of* (1.2) *on* $[T,\infty)$.*then there exists a solution* $w(t)$ *of* (3.16) *such that*

$$u(t) \geq \left(t + \int_{T}^{s} \xi_{\mu(\eta)}(-p(\eta)\frac{u(\tau(\eta))}{u^{\triangle}(\eta)})\triangle\eta\right)u^{\triangle}(t) \text{ for } \tau(t) \geq T. \tag{3.20}$$

Proof. Integrate the identity $(u(t) - tu^{\triangle}(t))^{\triangle} = \sigma(t)p(t)u(\tau(t))$ from T to t and use (3.2) we obtain

$$\begin{aligned} u(t) &\geq tu^{\triangle}(t) + \int_T^t \sigma(s)p(s)\frac{u(\tau(s))}{u^{\triangle}(s)}u^{\triangle}(s)\triangle s \\ &\geq tu^{\triangle}(t) + \int_T^t sp(s)\frac{u(\tau(s))}{u^{\triangle}(s)}u^{\triangle}(s)\triangle s \\ &\geq u^{\triangle}(t)\left(t + \int_T^t s\ (\frac{\tau}{\nu}(s))p^{\sigma}(s)w(s)\triangle s\right) \quad \text{for } \tau(t) \geq T, \end{aligned} \tag{3.21}$$

where according to Lemma 3.6 w is a solution of (3.16) . Thus (3.20) holds and the proof is complete. □

The following two lemmas give more exact result which will permit us to do without condition (3.12) in section 6.

Lemma 3.8. *Assume that condition* (3.1) *is violated. Then the integral equation corresponding to* (3.12)

$$\vartheta(t) = \exp\left(-\int_{\tau(t)}^t \xi_{\mu(s)}(-\tau_T(s)p(s)\vartheta(s)\triangle s.\right) \tag{3.22}$$

has a bounded solution.

Proof. Let $\delta > 0$ such that $\exp(\delta) \leq 1$. Since condition (3.1) is violated, then there exists $T_0 \geq 0$ such that

$$-\int_{T_0}^{\infty} \xi_{\mu(s)}(-(L+1)\tau(s)p(s))\triangle s \leq \delta,$$

where

$$L = \int_{T_0}^{\infty} \tau(s)p(s)\triangle s.$$

We claim that for any $T \geq T_0$ equation (3.12) has a solution ϑ satisfying $0 \leq \vartheta \leq 1$ for $t \geq T$. To show this consider $\Omega = \{\vartheta \in C([T,\infty) \cap \mathbb{T}, \mathbb{R}^+) : 0 \leq \vartheta \leq 1\}$ which endowed with usual pointwise ordering $\leq$: $\vartheta_1 \leq \vartheta_2 \Leftrightarrow \vartheta_1(t) \leq \vartheta_2(t)$ for all $t \geq T$. Define a mapping S on Ω by

$$(S\vartheta)(t) = \begin{cases} \exp\left(-\int_{\tau(t)}^t \xi_{\mu(s)}(-\tau_T(s)p(s)\vartheta(s)\triangle s.\right) & t \geq T_1 \\ M & T \leq t \leq T_1. \end{cases}$$

Since

$$\int_T^{\tau(t)} \xi\tau(\xi)p(\xi)\triangle\xi \quad \leq \tau(t)\int_{T_0}^{\tau(t)} \tau(\xi)p(\xi)\triangle\xi \leq L\tau(t),$$

it can be easily checked that S maps Ω into itself and S is nondecreasing. Therefore by Knaster's fixed point Theorem there exists $\vartheta \in \Omega$ such that $S\vartheta = \vartheta$ and then (3.22) follows. The proof is complete. □

Lemma 3.9. *Assume that condition* (3.1) *is violated. Then the integral equation corresponding to* (3.16)

$$w(t) = \int_T^{\nu(t)} \left\{ \exp\left\{ -\int_s^t \xi_{\mu(\eta)}(-\frac{\tau}{\nu}(\eta))_T(\eta)p(\eta)w(\eta)\triangle\eta \right\} \right\} \triangle s, \tag{3.23}$$

has a solution w such that $\frac{w}{\nu}$ is *bounded* .

Proof. The proof is similar to the proof of Lemma 3.8, so we omitted it. □

4. Oscillations caused by the delay

In this section oscillation results are obtained for equation (1.2) by reducing it to a first order. Since for the latter the oscillation is due solely to the delay, the criteria hold for delay dynamic equations and do not work in dynamic equation without delay. Lemma 3.3 immediately implies

Theorem 4.1. *Let* (3.1) *be fulfilled and the differential inequality* (3.11) *has no eventually positive solution. Then the equation* (1.2) *is oscillatory.*

Theorem 4.1 reduces the question of oscillation of (1.2) to that of that of the absence of eventually positive solutions of the differential inequality

$$x^{\triangle}(t) + \left(\tau(t) + \int_T^{\tau(t)} s\tau(s)p(s)\triangle s \right) p(t)x(\tau(t)) \le 0. \tag{4.1}$$

So, oscillation results for first order delay dynamic equations can be applied since the oscillation of the equation

$$x^{\triangle}(t) + g(t)x(\delta(t)) = 0 \tag{4.2}$$

is equivalent to the absence of eventually positive solutions of the inequality

$$x^{\triangle}(t) + g(t)x(\delta(t)) \le 0. \tag{4.3}$$

In fact it was proved that (4.2) oscillatory if and only if the inequality (4.3) has no eventually positive solution (see Agwo[6]). Using Corollary 2 in [20] we have the following theorem.

Theorem 4.2. *Let* (3.1) *be fulfilled and*

$$\limsup_{T\to\infty} \sup_{t>T} \sup_{\lambda\in E} \left\{ \lambda \exp\left\{ \int_{\tau(t)}^t \xi_{\mu(s)}(-\lambda\tau_T(s)p(s))\triangle s \right\} \right\} < 1, \tag{4.4}$$

where

$$E = \{\lambda > 0 : 1 - \lambda \tau_T(s) p(s) > 0\}. \tag{4.5}$$

Then the equation (1.2) *is oscillatory.*

5. General oscillations criteria

In this section, we prove a general oscillation theorem for equation (1.2). We first mention two criteria which are immediate consequence of Lemma3.4 and 3.6 respectively.

Theorem 5.1. *Let* (3.1) *be fulfilled and the differential inequality* (3.12) *has no eventually positive solution. Then the equation* (2) *is oscillatory.*

Theorem 5.2. *Let* (3.1) *be fulfilled and the differential inequality* (3.16) *has no eventually positive solution. Then the equation* (1.2) *is oscillatory.*

Theorem 5.3. *Let there exist rd-continuous nondecreasing functions* ν, γ *and* $\delta : \mathbb{T} \to \mathbb{T}$ *such that*

$$\begin{aligned} \nu(t) &\leq t, \tau(t) \leq \delta(t) \leq t, 0 < \gamma(t) \leq \delta(t) \quad \text{for } t \geq T, \\ \nu(t),\ \gamma(t) &\to \infty \text{ as t} \to \infty \end{aligned} \tag{5.1}$$

and for any positive ν *of* (3.22) *and any positive solution* w *of* (3.22) *the inequality*

$$\begin{aligned} \limsup_{t\to\infty}\{ &\int_{\delta(t)}^{t} p(s) \left(\tau(s) + \int_{T}^{\tau(s)} \xi_{\mu(\eta)}(-\eta(\frac{\tau}{\nu}(\eta))p(\eta)w(\eta))\triangle\eta \right) \\ &\times \exp\left\{ - \int_{\delta(s)}^{\delta(t)} \xi_{\mu(\eta)}(-\tau_T(\eta)p(\eta)\vartheta(\eta))\triangle\eta \right\} \triangle s \\ &+ \left(\gamma(t) + \int_{T}^{\gamma(t)} \xi_{\mu(\eta)}(-\eta(\frac{\tau}{\nu}(\eta))p(\eta)w(\eta))\triangle\eta \right) \int_{t}^{\infty} \frac{\tau}{\gamma}(s)p(s)\triangle s\} \end{aligned} \tag{5.2}$$

holds, where τ_T *is defined by* (3.8) *and* $\frac{\tau}{\nu}, \frac{\tau}{\gamma}$ *by* (3.2). *Then the equation* (1.2) *is oscillatory.*

Proof. Suppose that (3.1) is violated. Then by Lemma 3.8 and Lemma 3.9 the integral equations (3.22) and (3.23) have solutions ϑ_0 and w_0, respectively such that $\vartheta_0(t) \leq 1$ and $w_0(t) \leq \tau(t)$. Suppose that contrary to assertion of the theorem, the equation (1.2) has a non oscillatory solution u which we may and will assume to be positive. By Lemma 3.2, we have

$$u(\tau(t)) \geq (\frac{\tau}{\gamma}(t))u(\gamma(t)) \quad \text{for } t \geq t_1. \tag{5.3}$$

On the other hand, according to Lemma 3.5 and 3.6 and because $u^{\triangle}$ is nonincreasing, there exist positive solutions ϑ and w of the integral inequalities (3.12), (3.14) and (3.20) respectively such that

$$u^{\triangle}(\tau(s)) \geq u^{\triangle}(\delta(s)) \geq E(\vartheta)(s,t)u^{\triangle}(\delta(t)) \quad \text{for} \quad t \geq s \geq t_2, \tag{5.4}$$

$$u(\tau(s)) \geq F_T(w)(s)u^{\triangle}(\tau(s)) \quad \text{for} \quad s \geq t_2, \tag{5.5}$$

$$u(\gamma(s)) \geq F_T(w)(s)u^{\triangle}(\gamma(s)) \;\geq F_T(w)(s)u^{\triangle}(\delta(s)) \;\text{ for }\; s \geq t_2, \tag{5.6}$$

where for any $\alpha : \mathbb{T} \to \mathbb{T}$, we get

$$E(\vartheta)(s,t) = \exp\left\{-\int_{\delta(s)}^{\delta(t)} \xi_{\mu(\eta)}(-\tau_T(\eta)p(\eta)\vartheta(\eta)\triangle\eta\right\}, \tag{5.7}$$

$$F_\alpha(w)(t) = \left\{\alpha(t) + \int_T^{\alpha(t)} \xi_{\mu(\eta)}(-\eta(\frac{\tau}{\nu}(\eta))p(\eta)w(\eta)\triangle\eta\right\}. \tag{5.8}$$

Integrating (1.2) from $\delta(t)$ to ∞ and taking into account (4.1) and (4.3)-(4.6) along with the nondecreasing character of $u,\ \gamma$ and δ we get

$$\begin{aligned} u^{\triangle}(\delta(t)) \;&\geq\; \int_{\delta(t)}^{t} p(s)u(u(s))\triangle s + \int_t^\infty p(s)u(\tau(s))\triangle s \\ &\geq\; \int_{\delta(t)}^{t} p(s)F_T(w)(s)u^{\triangle}(\tau(s))\triangle s + u(\gamma(t))\int_t^\infty \frac{\tau}{\gamma}(s)p(s)\triangle s \\ &\geq\; u^{\triangle}(\delta(t))\int_{\delta(t)}^{t} p(s)F_T(w)(s)E(\vartheta)(s,t)\triangle s \\ &\quad + F_\gamma(w)(t))\int_t^\infty \frac{\tau}{\gamma}(s)p(s)\triangle s \end{aligned} \tag{5.9}$$

for large t. But this contradicts (4.2). The proof is complete.

Now we formulate some corollaries of the Theorem 4.1. We begin with one which shows the joint effect of the delay and the second order nature of (1.2) in its simplest form.

Corollary 5.1. *Let τ be nondecreasing and*

$$\limsup_{t\to\infty}\left\{\int_{\tau(t)}^{t} p(s)\tau(s)\triangle s + \tau(t)\int_t^\infty p(s)\triangle s\right\} > 1. \tag{5.10}$$

Then the equation (1.2) *is oscillatory.*

In fact the above result not only new but also improves Theorem 3.6 in [1]. Taking the first in (5.2) with $\nu(t) \equiv t$ and using the obvious estimate $w(t) \geq t-T$, we obtain

Corollary 5.2. *Let there exists a nondecreasing function* $\delta : \mathbb{T} \to \mathbb{T}$ satisfying $\tau(t) \le \delta(t) \le t$ for $t \ge t_0$ and such that for any solution ϑ of (3.12) the inequality

$$\limsup_{t\to\infty}\{\int_{\delta(t)}^{t} p(s)\tau_0(s)\exp\{-\int_{\delta(s)}^{\delta(t)} \xi_{\mu(\eta)}(-\tau_T(\eta)p(\eta)\vartheta(\eta)\triangle\eta\}\triangle s\} > 1 \tag{5.11}$$

holds, where τ_0 *is defined by* (3.8). *Then the equation* (1.2) *is oscillatory.*

Analogously, taking the second term in (5.2) with $\nu(t) \equiv t$ and using the estimate $w(t) \ge t - T$, we obtain

Corollary 5.3. *Let there exists a nondecreasing function* $\gamma : \mathbb{T} \to \mathbb{T}$ satisfying $t_0 < \gamma(t) \le \tau(t) \le t$ for $t \ge t_0$ and such that for any solution ϑ of (3.12) the inequality

$$\limsup_{t\to\infty}\left\{\left(\gamma(t) + \int_{t_0}^{\gamma(t)} \xi_{\mu(\eta)}(-\tau_T(\eta)p(\eta))\triangle\eta\right)\int_t^\infty p(s)\triangle s\right\} > 1 \tag{5.12}$$

holds. Then the equation (1.2) *is oscillatory.*

The following corollary follows by taking $\delta(t) \equiv t$ and $\gamma(t) = \nu(t)$ in (5.2).

Corollary 5.4. *Let there exists a nondecreasing function* $\nu : \mathbb{T} \to \mathbb{T}$ satisfying $t_0 < \nu(t) \le \tau(t) \le t$ for $t \ge t_0$ *and such that for any solution* w *of* (3.16) *the inequality*

$$\limsup_{t\to\infty}\left\{(\nu(t) + \int_{t_0}^{\nu(t)} \xi_{\mu(\eta)}(-\eta(\frac{\tau}{\nu}(\eta))p(\eta)w(\eta))\triangle\eta)\int_t^\infty \frac{\tau}{\nu}(s)p(s)\triangle s\right\} > 1 \tag{5.13}$$

holds. Then the equation (1.2) *is oscillatory.*

Remark 5.5. In the case of ordinary second order dynamic equations (i.e., $\tau(t) = t$ in (1.2)), Corollary 5.1 yields to if

$$\limsup_{t\to\infty} t\int_t^\infty p(s)\triangle s > 1 \tag{5.14}$$

holds. Then every solution of

$$u^{\triangle\triangle}(t) + p(t)u(t) = 0 \quad \text{for } t \in \mathbb{T} \tag{5.15}$$

oscillates.

Remark 5.6. In the case of ordinary second order dynamic equations (i.e., $\tau(t) = t$ in (1.2)), Corollary 5.3 yields to if

$$\limsup_{t\to\infty} \left\{ \left(t + \int_{t_0}^{t} \xi_{\mu(\eta)}(-\eta^2 p(\eta))\triangle\eta \right) \int_{t}^{\infty} p(s)\triangle s \right\} > 1 \tag{5.16}$$

holds. Then every solution of (5.15) oscillates.

Remark 5.7. When $\mathbb{T} = \mathbb{R}$, (5.14) yields to the well-known E. Hille [13] oscillation criteria

$$\limsup_{t\to\infty} t \int_{t}^{\infty} p(s)ds > 1 \tag{5.17}$$

for the equation $u''(t) + p(t)u(t) = 0$.but (5.16) yields to

$$\limsup_{t\to\infty} \left\{ \left(t + \int_{t_0}^{t} \eta^2 p(\eta)d\eta \right) \int_{t}^{\infty} p(s)ds \right\} > 1, \tag{5.18}$$

which improves (5.17).

Also, when $\mathbb{T} = \mathbb{R}$, then according to Corollary 5.1, equation (1.6) is oscillatory if

$$\limsup_{t\to\infty} \left\{ \int_{\tau(t)}^{t} p(s)\tau(s)ds + \tau(t) \int_{t}^{\infty} p(s)ds \right\} > 1. \tag{5.19}$$

This improves the result J. Wei [19], (see also [11],[16]).

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