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On a Reverse Hardy-Hilbert's Inequality

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Abstract. This paper deals with a reverse Hardy-Hilbert's inequality with a best constant factor by introducing two parameters λ and α. We also consider the equivalent form and the analogue integral inequalities. Some particular results are given.

1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0\,(n \in N)$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the well known Hardy-Hilbert's inequality is as follows (see Hardy et al. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{sin(\frac{\pi}{p})}\{\sum_{n=1}^{\infty} a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} b_n^q\}^{\frac{1}{q}}; \tag{1.1}$$

where the constant factor $\pi/sin(\pi/p)$ is the best possible. The analogue integral form of (1.1) is:

If $f(x), g(x) \geq 0$, such that $0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(x)dx < \infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{sin(\frac{\pi}{p})}\{\int_0^{\infty} f^p(x)dx\}^{\frac{1}{p}}\{\int_0^{\infty} g^q(x)\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\pi/sin(\pi/p)$ is still the best possible.

Inequalities (1.1) and (1.2) are important in analysis and its applications (see Mitrinovic et al. [2]). In recent years, (1.1) was strengthened by [3], [4] as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \{\sum_{n=1}^{\infty}[\frac{\pi}{sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{\frac{1}{p}}}]a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}[\frac{\pi}{sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{\frac{1}{q}}}]b_n^q\}^{\frac{1}{q}}, \tag{1.3}$$

where γ is Euler constant and $1-\gamma(= 0.42278433^+)$ is the best value.

By introducing a parameter, Yang [5], [6] gave two extensions of (1.1) as:

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If the series in the right of the following inequalities converge to positive numbers, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < k_{\lambda}(p)\Big\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty} n^{1-\lambda}b_n^q\Big\}^{\frac{1}{q}}, \tag{1.4}$$

where the constant factor $k_{\lambda}(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ $(2 - min\{p, q\} < \lambda \leq 2)$ is the best possible (B(u,v) is the β function); and the other is

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\alpha}+n^{\alpha}} < \frac{\pi}{\alpha sin(\frac{\pi}{p})}\Big\{\sum_{n=1}^{\infty} n^{(p-1)(1-\alpha)}a_n^p\Big\}^{\frac{1}{p}}\Big\{\sum_{n=1}^{\infty} n^{(q-1)(1-\alpha)}b_n^q\Big\}^{\frac{1}{q}}, \tag{1.5}$$

where the constant factor $\pi/[\alpha\sin(\pi/p)]$ $(0 < \alpha \leq \min\{p, q\})$ is the best possible. For $\lambda = \alpha = 1$, both (1.4) and (1.5) reduce to (1.1).

In 2003, Yang et al. [7] summarized the way of weight coefficient on research for Hilbert-type inequalities. More recently, Zhao [8] consider some inverses of Pachpatte's inequalities. But the problem on how to build the reverse of (1.1) is still unsolved.

The main objective of this paper is to deal with a reverse inequality of (1.1) with a best constant factor by introducing two parameters λ and α. The equivalent form, the analogue integral inequalities and some particular results are considered.

2. A reverse Hardy-Hilbert's integral inequality

For $\lambda > 2 - min\{p, q\}$, the analogue integral inequality of (1.4) is (see [9]):

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x+y)^{\lambda}}dxdy \\ < k_{\lambda}(p)\Big\{\int_0^{\infty} x^{1-\lambda}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^{\infty} y^{1-\lambda}g^q(y)dy\Big\}^{\frac{1}{q}},\end{aligned} \tag{2.1}$$

where the constant factor $k_{\lambda}(p) = B\big(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\big)$ is the best possible. For $\lambda = 1$, (2.1) reduces to (1.2). The expression of the β function B(p,q) is as follows (see [10]):

$$B(p, q) = B(q, p) = \int_0^{\infty}\frac{u^{p-1}}{(1+u)^{p+q}}du \ (p, q > 0). \tag{2.2}$$

Setting $x = X^{\alpha}, y = Y^{\alpha}$ $(\alpha > 0)$ in (2.1), and putting $F(X) = X^{\alpha-1}f(X^{\alpha}), G(Y) = Y^{\alpha-1}G(Y^{\alpha})$, by simplification, one has

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty}\frac{F(X)G(Y)}{(X^{\alpha}+Y^{\alpha})^{\lambda}}dXdY < \frac{k_{\lambda}(p)}{\alpha} \\ \times\Big\{\int_0^{\infty}\frac{(X^{1-\alpha}F(X))^p}{X^{1+\alpha(\lambda-2)}}dX\Big\}^{\frac{1}{p}}\Big\{\int_0^{\infty}\frac{(Y^{1-\alpha}G(Y))^q}{Y^{1+\alpha(\lambda-2)}}dY\Big\}^{\frac{1}{q}}.\end{aligned} \tag{2.3}$$

For $\alpha = 1$, (2.3) reduces to the form of (2.1). We can conclude that inequalities (2.3) and (2.1) are equivalent, and so the constant factor $\frac{1}{\alpha}k_\lambda(p)$ in (2.3) is still the best possible. For $\lambda = 1\,(\alpha > 0)$ in (2.3), one has (see [6])

$$\int_0^\infty\int_0^\infty \frac{F(X)G(Y)}{X^\alpha + Y^\alpha}dXdY < \frac{\pi}{\alpha\sin(\frac{\pi}{p})} \times\Big\{\int_0^\infty X^{(p-1)(1-\alpha)}F^p(X)dX\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty Y^{(q-1)(1-\alpha)}G^q(Y)dY\Big\}^{\frac{1}{q}}. \tag{2.4}$$

Inequality (2.4) is an analogue integral form of (1.5), which is an extension of (1.2) with a parameter $\alpha > 0$.

We discover that for $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $2 - p < \lambda < 2 - q$, the constant factor $k_\lambda(p)$ in (2.1) is defined. In the following, a reverse of (2.3) is considered.

Lemma 2.1. *If $0 < r < 1$ (or $r < 0$), $\frac{1}{r} + \frac{1}{s} = 1$ and $2 - max\{r, s\} < \lambda < 2 - min\{r, s\}$, define the weight function $\omega_{\lambda,\alpha}(r, x)\,(x \in (0,\infty))$ as*

$$\omega_{\lambda,\alpha}(r,x) := x^{\alpha(\frac{s+\lambda-2}{s})}\int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^\lambda}\Big(\frac{1}{y}\Big)^{1-\alpha\frac{r+\lambda-2}{r}}dy, \tag{2.5}$$

then we have

$$\omega_{\lambda,\alpha}(r,x) = \frac{1}{\alpha}B\Big(\frac{r+\lambda-2}{r}, \frac{s+\lambda-2}{s}\Big)\ (x \in (0,\infty)). \tag{2.6}$$

Proof. Setting $u = (y/x)^\alpha$ in the integral of (2.5), we obtain

$$\omega_{\lambda,\alpha}(r,x) = \frac{1}{\alpha}\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{r+\lambda-2}{r}-1}du.$$

Since $\frac{r+\lambda-2}{r} + \frac{s+\lambda-2}{s} = \lambda$, by (2.2), we have (2.6). The lemma is proved. □

Lemma 2.2. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $2 - p < \lambda < 2 - q$ and $0 < \varepsilon < \alpha(1 - q)(p + \lambda - 2)$, one has*

$$\begin{aligned} I &:= \int_1^\infty\int_1^\infty \frac{x^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}}{(x^\alpha + y^\alpha)^\lambda}y^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}dydx \\ &< \frac{1}{\varepsilon\alpha}B\Big(\frac{q+\lambda-2}{q} - \frac{\varepsilon}{q\alpha}, \frac{p+\lambda-2}{p} + \frac{\varepsilon}{q\alpha}\Big). \end{aligned} \tag{2.7}$$

Proof. For fixed x, setting $u = (y/x)^\alpha$, we find $dy = \frac{x}{\alpha}u^{\frac{1}{\alpha}-1}du$, and

$$\begin{aligned} I &< \int_1^\infty\Big[\int_0^\infty \frac{y^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}}{(x^\alpha + y^\alpha)^\lambda}dy\Big]x^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}dx \\ &= \frac{1}{\alpha}\int_1^\infty\Big[\int_0^\infty \frac{u^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q\alpha}-1}}{(1+u)^\lambda}du\Big]x^{-1-\varepsilon}dx = \frac{1}{\varepsilon\alpha}\int_0^\infty \frac{u^{(\frac{q+\lambda-2}{q}-\frac{\varepsilon}{q\alpha})-1}}{(1+u)^\lambda}du. \end{aligned}$$

By (2.2), we have (2.7). The lemma is proved. □

Lemma 2.3. (*The reverse Hölder′s integral inequality*) *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, $f(t), g(t) \geq 0$ *and* $f \in L^p(E), g \in L^q(E)$, *then*

$$\int_E f(t)g(t)dt \geq (\int_E f^p(t)dt)^{1/p}(\int_E g^q(t)dt)^{1/q}, \tag{2.8}$$

where the equality holds if and only if there exists real numbers a and b, such that they are not all zero and $af^p(t) = bg^q(t)$, *a.e. in* E *(see* [11], *p.29).*

Theorem 2.4. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, 2 - p < \lambda < 2 - q$, $f(x), g(x) \geq 0$, $0 < \int_0^\infty \frac{(x^{1-\alpha}f(x))^p}{x^{1+\alpha(\lambda-2)}}dx < \infty$ *and* $0 < \int_0^\infty \frac{(x^{1-\alpha}g(x))^q}{x^{1+\alpha(\lambda-2)}}dx < \infty$, *then*

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\alpha)^\lambda}dxdy &> \frac{1}{\alpha}k_\lambda(p) \\ &\times \Big\{ \int_0^\infty \frac{(x^{1-\alpha}f(x))^p}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{p}} \Big\{ \int_0^\infty \frac{(x^{1-\alpha}g(x))^q}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{q}}, \end{aligned} \tag{2.9}$$

where the constant factor $\frac{1}{\alpha}k_\lambda(p)$ $(k_\lambda(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}))$ *is the best possible. In particular, for* $\alpha = 1$, *one has*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy \\ &> k_\lambda(p)\Big\{ \int_0^\infty x^{1-\lambda}f^p(x)dx\Big\}^{\frac{1}{p}} \Big\{ \int_0^\infty x^{1-\lambda}g^q(x)dx\Big\}^{\frac{1}{q}}. \end{aligned} \tag{2.10}$$

Proof. By the reverse *Hölder′s* inequality (2.8), we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \Big[\frac{f(x)}{(x^\alpha + y^\alpha)^{\frac{\lambda}{p}}}\Big(\frac{x^{p-\alpha(p+\lambda-2)}}{y^{q-\alpha(q+\lambda-2)}}\Big)^{\frac{1}{pq}}\Big]\Big[\frac{g(y)}{(x^\alpha + y^\alpha)^{\frac{\lambda}{q}}}\Big(\frac{y^{q-\alpha(q+\lambda-2)}}{x^{p-\alpha(p+\lambda-2)}}\Big)^{\frac{1}{pq}}\Big]dxdy \\ \geq\ &\Big\{\int_0^\infty \int_0^\infty \frac{f^p(x)}{(x^\alpha + y^\alpha)^\lambda}\Big(\frac{x^{p-\alpha(p+\lambda-2)}}{y^{q-\alpha(q+\lambda-2)}}\Big)^{\frac{1}{q}}dxdy\Big\}^{\frac{1}{p}} \\ &\times \Big\{\int_0^\infty \int_0^\infty \frac{g^q(y)}{(x^\alpha + y^\alpha)^\lambda}\Big(\frac{y^{q-\alpha(q+\lambda-2)}}{x^{p-\alpha(p+\lambda-2)}}\Big)^{\frac{1}{p}}dxdy\Big\}^{\frac{1}{q}}. \end{aligned} \tag{2.11}$$

We conclude that (2.11) takes the form of strict inequality. Otherwise (see Lemma 2.3), there exist real numbers a and b, such that they are not all zero and

$$a\frac{f^p(x)}{(x^\alpha + y^\alpha)^\lambda}\Big(\frac{x^{p-\alpha(p+\lambda-2)}}{y^{q-\alpha(q+\lambda-2)}}\Big)^{\frac{1}{q}} = b\frac{g^q(y)}{(x^\alpha + y^\alpha)^\lambda}\Big(\frac{y^{q-\alpha(q+\lambda-2)}}{x^{p-\alpha(p+\lambda-2)}}\Big)^{\frac{1}{p}},$$

$$a.e.\ \text{in}\,(0,\infty) \times (0,\infty).$$

It follows that $ax^{p-\alpha(p+\lambda-2)}f^p(x) = by^{q-\alpha(q+\lambda-2)}g^q(y)$, $a.e.in\,(0,\infty)\times(0,\infty)$. Hence there exists a real number c, such that

$$ax^{p-\alpha(p+\lambda-2)}f^p(x) = c,\ by^{q-\alpha(q+\lambda-2)}g^q(y) = c,\ a.e.in\,(0,\infty).$$

Suppose $a \neq 0$. One has $x^{p-1-\alpha(p+\lambda-2)}f^p(x) = \frac{c}{a}x^{-1}$, $a.e.in\,(0,\infty)$, which contradicts the face that $0 < \int_0^\infty x^{p-1-\alpha(p+\lambda-2)}f^p(x)dx < \infty$. Hence by (2.5), we can reduce (2.11) as

$$\begin{aligned}(2.12)\qquad &\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^\alpha+y^\alpha)^\lambda}dxdy \\ &> \Big\{\int_0^\infty \omega_{\lambda,\alpha}(q,x)\frac{(x^{1-\alpha}f(x))^p}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{p}}\times\Big\{\int_0^\infty \omega_{\lambda,\alpha}(p,y)\frac{(y^{1-\alpha}g(y))^q}{y^{1+\alpha(\lambda-2)}}dy\Big\}^{\frac{1}{q}}.\end{aligned}$$

By (2.6), we have (2.9).

For $0 < \varepsilon < \alpha(1-q)(p+2-\lambda)$, setting $\tilde{f}(x), \tilde{g}(x)$ as: $f(x) = g(x) = 0, x \in (0,1)$;

$$f(x) = x^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1},\ g(x) = x^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1},\ x \in [1,\infty),$$

then we obtain

$$(2.13)\qquad \Big\{\int_0^\infty \frac{(x^{1-\alpha}\tilde{f}(x))^p}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty \frac{(x^{1-\alpha}\tilde{g}(x))^q}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{q}} = \frac{1}{\varepsilon}.$$

If there exist parameters α and λ, such that the constant factor $\frac{1}{\alpha}k_\lambda(p)$ in (2.9) is not the best possible, then there exists a positive number k, with $k > \frac{1}{\alpha}k_\lambda(p)$, such that (2.9) is still valid if one replaces $\frac{1}{\alpha}k_\lambda(p)$ by k. In particular, one has

$$\begin{aligned}&\varepsilon\int_0^\infty\int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{(x^\alpha+y^\alpha)^\lambda}dxdy > \varepsilon k \\ &\times\Big\{\int_0^\infty \frac{(x^{1-\alpha}\tilde{f}(x))^p}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty \frac{(x^{1-\alpha}\tilde{g}(x))^q}{x^{1+\alpha(\lambda-2)}}dx\Big\}^{\frac{1}{q}}.\end{aligned}$$

By (2.7) and (2.13), it follows that

$$\frac{1}{\alpha}B(\frac{q+\lambda-2}{q} - \frac{\varepsilon}{q\alpha}, \frac{p+\lambda-2}{p} + \frac{\varepsilon}{q\alpha}) > k,$$

and then $\frac{1}{\alpha}k_\lambda(p) \geq k\,(\varepsilon \to 0^+)$. This contradicts the face that $k > \frac{1}{\alpha}k_\lambda(p)$. Hence the constant factor $\frac{1}{\alpha}k_\lambda(p)$ in (2.9) is the best possible. The theorem is proved.

Remark 2.5. Following (2.9) and (2.3) for $\lambda = 2$, one can get a two-sides inequality as

$$\begin{aligned}(2.14)\qquad &\Big\{\int_0^\infty \frac{(x^{1-\alpha}f(x))^p}{x}dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty \frac{(x^{1-\alpha}g(x))^q}{x}dx\Big\}^{\frac{1}{q}} \\ &< \alpha\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^\alpha+y^\alpha)^2}dxdy \\ &< \Big\{\int_0^\infty \frac{(x^{1-\alpha}f(x))^r}{x}dx\Big\}^{\frac{1}{r}}\Big\{\int_0^\infty \frac{(x^{1-\alpha}g(x))^s}{x}dx\Big\}^{\frac{1}{s}},\end{aligned}$$

where $\alpha > 0, 0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, and $r > 1, \frac{1}{r} + \frac{1}{s} = 1$.

Theorem 2.6. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, 2 - p < \lambda < 2 - q$, $f(x) \geq 0$, *such that* $0 < \int_0^\infty \frac{(x^{1-\alpha} f(x))^p}{x^{1+\alpha(\lambda-2)}} dx < \infty$, *then*

$$\int_0^\infty y^{\alpha[(p-1)(\lambda-2)+1]-1} \Big[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \Big]^p dy > \Big[\frac{k_\lambda(p)}{\alpha}\Big]^p \int_0^\infty \frac{(x^{1-\alpha} f(x))^p}{x^{1+\alpha(\lambda-2)}} dx, \tag{2.15}$$

where the constant factor $[\frac{1}{\alpha} k_\lambda(p)]^p$ $(k_\lambda(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}))$ *is the best possible. Inequality* (2.15) *is equivalent to* (2.9). *In particular, for* $\alpha = 1$, *one has*

$$\int_0^\infty y^{(p-1)(\lambda-2)} \Big[\int_0^\infty \frac{f(x)}{(x+y)^\lambda} dx \Big]^p dy > [k_\lambda(p)]^p \int_0^\infty x^{1-\lambda} f^p(x) dx. \tag{2.16}$$

If $p < 0$ *and* $2 - q < \lambda < 2 - p$, *one has the reversions of* (2.15) *and* (2.16), *and the constant factors in the two reversions are all the best possible.*

Proof. Setting g(y) as

$$g(y) := y^{\alpha[(p-1)(\lambda-2)+1]-1} \Big[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \Big]^{p-1},$$

then by (2.9), one has

$$\begin{aligned} 0 &< \int_0^\infty \frac{(y^{1-\alpha} g(y))^q}{y^{1+\alpha(\lambda-2)}} dy \\ &= \int_0^\infty y^{\alpha[(p-1)(\lambda-2)+1]-1} \Big[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \Big]^p dy \\ &= \int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x^\alpha + y^\alpha)^\lambda} dx dy \geq \frac{1}{\alpha} k_\lambda(p) \\ &\quad \times \Big\{ \int_0^\infty \frac{(x^{1-\alpha} f(x))^p}{x^{1+\alpha(\lambda-2)}} dx \Big\}^{\frac{1}{p}} \Big\{ \int_0^\infty \frac{(x^{1-\alpha} g(x))^q}{x^{1+\alpha(\lambda-2)}} dx \Big\}^{\frac{1}{q}}, \end{aligned} \tag{2.17}$$

and then

$$\begin{aligned} &\Big\{ \int_0^\infty \frac{(y^{1-\alpha} g(y))^q}{y^{1+\alpha(\lambda-2)}} dy \Big\}^{1-\frac{1}{q}} \\ &= \Big\{ \int_0^\infty y^{\alpha[(p-1)(\lambda-2)+1]-1} \Big[\int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^\lambda} dx \Big]^p dy \Big\}^{\frac{1}{p}} \\ &\geq \frac{1}{\alpha} k_\lambda(p) \Big\{ \int_0^\infty \frac{(x^{1-\alpha} f(x))^p}{x^{1+\alpha(\lambda-2)}} dx \Big\}^{\frac{1}{p}}. \end{aligned} \tag{2.18}$$

If $\int_0^\infty \frac{(y^{1-\alpha}g(y))^q}{y^{1+\alpha(\lambda-2)}}dy < \infty$, then in view of (2.9), (2.17) takes strict inequality; so does (2.18). If $\int_0^\infty \frac{(y^{1-\alpha}g(y))^q}{y^{1+\alpha(\lambda-2)}}dy = \infty$, then (2.18) takes strict inequality. Hence we have (2.15).

On the other hand, if (2.15) is valid, by the reverse $H\ddot{o}lder's$ inequality (2.8), one has

$$\begin{aligned}(2.19)\qquad & \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^\alpha+y^\alpha)^\lambda}dxdy \\ = & \int_0^\infty [y^{\frac{1+\alpha(q+\lambda-2)-q}{q}}\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^\lambda}dx][y^{\frac{q-1-\alpha(q+\lambda-2)}{q}}g(y)]dy \\ \geq & \{\int_0^\infty y^{\alpha[(p-1)(\lambda-2)+1]-1}\big[\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^\lambda}dx\big]^p dy\}^{\frac{1}{p}} \\ & \times\{\int_0^\infty \frac{(y^{1-\alpha}g(y))^q}{y^{1+\alpha(\lambda-2)}}dy\}^{1-\frac{1}{q}}.\end{aligned}$$

By (2.15), one has (2.9). Hence (2.15) and (2.9) are equivalent. If the constant factor in (2.15) is not the best possible, then by using (2.19), we can get a contradiction that the constant factor in (2.9) is not the best possible.

For $p < 0$ and $2-q < \lambda < 2-p$, we still have (2.17), (2.18) and (2.19). By using (2.18), since $p < 0$, we can get a reversion of (2.15). By using (2.19), we can conclude that the constant factor in the reversion of (2.15) is the best possible. The theorem is proved.

3. A reverse Hardy-Hilbert's inequality

Lemma 3.1. *If* $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, 2-p < \lambda < 2-q, 0 < \alpha \leq \min\{\frac{p}{p+\lambda-2}, \frac{q}{q+\lambda-2}\}$ *and* $0 < \varepsilon < \alpha(p+\lambda-2)$, *then*

$$\begin{aligned}(3.1)\qquad J \ := & \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{m^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}n^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}}{(m^\alpha+n^\alpha)^\lambda} \\ < & \frac{1}{\alpha}B\Big(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p\alpha}, \frac{q+\lambda-2}{q}+\frac{\varepsilon}{p\alpha}\Big)\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}.\end{aligned}$$

Proof. Since $0 < p < 1, \lambda > 2-p > 0$ and $0 < \alpha \leq \frac{p}{p+\lambda-2} < \frac{p}{p+\lambda-2-(\varepsilon/\alpha)}$, we find $\frac{\alpha(p+\lambda-2)-\varepsilon}{p} - 1 < 0$ and

$$\sum_{m=1}^\infty \frac{m^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}n^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}}{(m^\alpha+n^\alpha)^\lambda} < \int_0^\infty \frac{x^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}n^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}}{(x^\alpha+n^\alpha)^\lambda}dx.$$

Setting $y=(x/n)^\alpha$ in the above integral, we obtain $dx=\frac{n}{\alpha}y^{\frac{1}{\alpha}-1}dy$ and

$$\begin{aligned} J &< \sum_{n=1}^{\infty}\int_0^\infty \frac{x^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}n^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}}{(x^\alpha+n^\alpha)^\lambda}dx \\ &= \frac{1}{\alpha}\int_0^\infty \frac{y^{\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p\alpha}-1}}{(1+y)^\lambda}dy\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}. \end{aligned}$$

Hence by (2.2), we have (3.1). The lemma is proved. □

Lemma 3.2. *If* $0<p<1, \frac{1}{p}+\frac{1}{q}=1, 2-p<\lambda<2-q, 0<\alpha\le\min\{\frac{p}{p+\lambda-2},\frac{q}{q+\lambda-2}\}$ *and* $\theta_{\lambda,\alpha}(q,n)$ *is defined by*

$$\theta_{\lambda,\alpha}(q,n):=\frac{1}{k_\lambda(p)}\int_{\frac{1}{n^\alpha}}^\infty \frac{y^{\frac{q+\lambda-2}{q}-1}}{(1+y)^\lambda}dy\,(n\in N), \tag{3.2}$$

then $0<\theta_{\lambda,\alpha}(q,n)<1$ *and*

$$\theta_{\lambda,\alpha}(q,n)=1-O(\frac{1}{n^{\alpha(q+\lambda-2)/q}}). \tag{3.3}$$

In particular, for $\lambda=2$, *one has*

$$\theta_{2,\alpha}(q,n)=\frac{n^\alpha}{n^\alpha+1}\,(n\in N). \tag{3.4}$$

Proof. It is obvious that $0<\theta_{\lambda,\alpha}(q,n)<1$, since

$$k_\lambda(p)=B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})=\int_0^\infty\frac{y^{\frac{q+\lambda-2}{q}-1}}{(1+y)^\lambda}dy.$$

Setting

$$\theta_{\lambda,\alpha}(q,n)=1-\frac{1}{k_\lambda(p)}\int_0^{\frac{1}{n^\alpha}}\frac{y^{\frac{q+\lambda-2}{q}-1}}{(1+y)^\lambda}dy,$$

we obtain

$$\begin{aligned} \frac{1}{2^\lambda}(\frac{q}{q+\lambda-2})\frac{1}{n^{\alpha(q+\lambda-2)/q}} &= \frac{1}{2^\lambda}\int_0^{\frac{1}{m^\alpha}} y^{\frac{q+\lambda-2}{q}-1}dy \\ < \int_0^{\frac{1}{m^\alpha}}\frac{y^{\frac{q+\lambda-2}{q}-1}}{(1+y)^\lambda}dy &< \int_0^{\frac{1}{n^\alpha}} y^{\frac{q+\lambda-2}{q}-1}dy=(\frac{q}{q+\lambda-2})\frac{1}{n^{\alpha(q+\lambda-2)/q}}. \end{aligned}$$

Then (3.3) is valid. Since for $\lambda=2$, $k_2(p)=1$, by (3.2), one has

$$\theta_{2,\alpha}(q,n)=1-\int_0^{\frac{1}{n^\alpha}}\frac{1}{(1+y)^2}dy=\frac{n^\alpha}{n^\alpha+1}.$$

The lemma is proved. □

Note. By (3.3), since $\alpha(q+\lambda-2)/q>0$, it is obvious that

$$0<\sum_{n=1}^{\infty}O(\frac{1}{n^{\alpha(q+\lambda-2)/q}})\frac{1}{n}<\infty. \tag{3.5}$$

Theorem 3.3. *If* $0<p<1, \frac{1}{p}+\frac{1}{q}=1, 2-p<\lambda<2-q, 0<\alpha\leq\min\{\frac{p}{p+\lambda-2},\frac{q}{q+\lambda-2}\}$ $a_n, b_n\geq 0$, *such that*

$$0<\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}}<\infty \text{ and } 0<\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}}<\infty,$$

then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} > \frac{k_\lambda(p)}{\alpha}\{\sum_{n=1}^{\infty}\theta_{\lambda,\alpha}(q,n)\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{q}}, \tag{3.6}$$

where $0<\theta_{\lambda,\alpha}(q,n)<1$, *and the constant factor* $\frac{1}{\alpha}k_\lambda(p)(k_\lambda(p)=B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}))$ *is the best possible. In particular, for* $\lambda=2$, *one has* $0<\alpha\leq 1$ *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^2}>\frac{1}{\alpha}\{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}a_n)^p}{n+n^{1-\alpha}}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n}\}^{\frac{1}{q}}. \tag{3.7}$$

Proof. By the reverse $H\ddot{o}lder's$ inequality, one has

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} \\ =\ &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\frac{a_m}{(m^\alpha+n^\alpha)^{\frac{\lambda}{p}}}(\frac{m^{p-\alpha(p+\lambda-2)}}{n^{q-\alpha(q+\lambda-2)}})^{\frac{1}{pq}}][\frac{b_n}{(m^\alpha+n^\alpha)^{\frac{\lambda}{q}}}(\frac{n^{q-\alpha(q+\lambda-2)}}{m^{p-\alpha(p+\lambda-2)}})^{\frac{1}{pq}}] \\ \geq\ &\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{(m^\alpha+n^\alpha)^\lambda}(\frac{m^{p-\alpha(p+\lambda-2)}}{n^{q-\alpha(q+\lambda-2)}})^{\frac{1}{q}}\}^{\frac{1}{p}} \\ &\times\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n^q}{(m^\alpha+n^\alpha)^\lambda}(\frac{n^{q-\alpha(q+\lambda-2)}}{m^{p-\alpha(p+\lambda-2)}})^{\frac{1}{p}}\}^{\frac{1}{q}} \\ =\ &\{\sum_{m=1}^{\infty}[\sum_{n=1}^{\infty}\frac{m^{\alpha\frac{p+\lambda-2}{p}}n^{\alpha\frac{q+\lambda-2}{q}-1}}{(m^\alpha+n^\alpha)^\lambda}]\frac{(m^{1-\alpha}a_m)^p}{m^{1+\alpha(\lambda-2)}}\}^{\frac{1}{p}} \end{aligned} \tag{3.8}$$

$$\times \{ \sum_{n=1}^{\infty} [\sum_{m=1}^{\infty} \frac{n^{\alpha \frac{q+\lambda-2}{q}} m^{\alpha \frac{q+\lambda-2}{q}-1}}{(m^{\alpha}+n^{\alpha})^{\lambda}}] \frac{(n^{1-\alpha} b_n)^q}{n^{1+\alpha(\lambda-2)}} \}^{\frac{1}{q}}$$
$$= \{ \sum_{m=1}^{\infty} \varpi_{\lambda,\alpha}(q,m) \frac{(m^{1-\alpha} a_m)^p}{m^{1+\alpha(\lambda-2)}} \}^{\frac{1}{p}} \{ \sum_{n=1}^{\infty} \varpi_{\lambda,\alpha}(p,n) \frac{(n^{1-\alpha} b_n)^q}{n^{1+\alpha(\lambda-2)}} \}^{\frac{1}{q}},$$

where, the weight coefficient $\varpi_{\lambda,\alpha}(r,j)\,(r=p,q)$ is defined by

$$\varpi_{\lambda,\alpha}(r,j) := \sum_{k=1}^{\infty} \frac{j^{\alpha \frac{s+\lambda-2}{s}} k^{\alpha \frac{r+\lambda-2}{r}-1}}{(j^{\alpha}+k^{\alpha})^{\lambda}} \;\; (r>1, \frac{1}{r}+\frac{1}{s}=1, j=m,n). \tag{3.9}$$

Since $0<\alpha\le \min\{\frac{p}{p+\lambda-2}, \frac{q}{q+\lambda-2}\}, \alpha\frac{p+\lambda-2}{p}-1\le 0$, and $\lambda>0$, setting $y=(x/n)^{\alpha}$, we find

$$\begin{aligned} \varpi_{\lambda,\alpha}(p,n) &< \int_0^{\infty} \frac{n^{\alpha \frac{q+\lambda-2}{q}} x^{\alpha\frac{p+\lambda-2}{p}-1}}{(n^{\alpha}+x^{\alpha})^{\lambda}} dx \\ &= \frac{1}{\alpha}\int_0^{\infty} \frac{y^{\frac{p+\lambda-2}{p}-1}}{(1+y)^{\lambda}} dy = \frac{1}{\alpha} k_{\lambda}(p); \end{aligned} \tag{3.10}$$

$$\begin{aligned} \varpi_{\lambda,\alpha}(q,m) &> \int_1^{\infty} \frac{m^{\alpha \frac{p+\lambda-2}{p}} x^{\alpha\frac{q+\lambda-2}{q}-1}}{(m^{\alpha}+x^{\alpha})^{\lambda}} dx \\ &= \frac{1}{\alpha}\int_{\frac{1}{m^{\alpha}}}^{\infty} \frac{y^{\frac{q+\lambda-2}{q}-1}}{(1+y)^{\lambda}} dy = \frac{1}{\alpha} k_{\lambda}(p)\theta_{\lambda,\alpha}(q,m), \end{aligned} \tag{3.11}$$

where, $\theta_{\lambda,\alpha}(q,m)$ is defined by (3.2)

Then, in view of $0<p<1, q<0$, (3.8), (3.10) and (3.11), we have (3.6). For $\lambda=2$, by (3.4) and (3.6), we have (3.7).

For $0<\varepsilon<\alpha(p+\lambda-2)$, setting $\tilde{a}_n, \tilde{b}_n$ as: $\tilde{a}_n = n^{\frac{\alpha(p+\lambda-2)-\varepsilon}{p}-1}; \tilde{b}_n = n^{\frac{\alpha(q+\lambda-2)-\varepsilon}{q}-1}$, then by (3.5), we find

$$\begin{aligned} &\{ \sum_{n=1}^{\infty} \theta_{\lambda,\alpha}(q,n) \frac{(n^{1-\alpha}\tilde{a}_n)^p}{n^{1+\alpha(\lambda-2)}} \}^{\frac{1}{p}} \{ \sum_{n=1}^{\infty} \frac{(n^{1-\alpha}\tilde{b}_n)^q}{n^{1+\alpha(\lambda-2)}} \}^{\frac{1}{q}} \\ &= \{ \sum_{n=1}^{\infty} [1-O(\frac{1}{n^{\alpha(q+\lambda-2)/q}})] \frac{1}{n^{1+\varepsilon}} \}^{\frac{1}{p}} \{ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \}^{\frac{1}{q}} \\ &> \{ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} - \sum_{n=1}^{\infty} O(\frac{1}{n^{\alpha(q+\lambda-2)/q}}) \frac{1}{n} \}^{\frac{1}{p}} \{ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \}^{\frac{1}{q}} \\ &= (1-o(1))^{\frac{1}{p}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \;\; (\varepsilon\to 0^+). \end{aligned} \tag{3.12}$$

If the constant factor $\frac{1}{\alpha}k_\lambda(p)$ in (3.6) is not the best possible, then there exists a positive number $k > \frac{1}{\alpha}k_\lambda(p)$, such that (3.6) is still valid if one replaces $\frac{1}{\alpha}k_\lambda(p)$ by k. In particular, one has

$$(3.13)\quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(m^\alpha+n^\alpha)^\lambda} > k\{\sum_{n=1}^{\infty}\theta_{\lambda,\alpha}(q,n)\frac{(n^{1-\alpha}\tilde{a}_n)^p}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}\tilde{b}_n)^q}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{q}}.$$

By (3.1), (3.12) and (3.13), it follows that

$$\frac{1}{\alpha}B\Big(\frac{p+\lambda-2}{p}-\frac{\varepsilon}{p\alpha},\frac{q+\lambda-2}{q}+\frac{\varepsilon}{p\alpha}\Big) > k(1-o(1))^{\frac{1}{p}},$$

and then $\frac{1}{\alpha}k_\lambda(p) \geq k\,(\varepsilon \to 0^+)$. This contradicts the fact that $k > \frac{1}{\alpha}k_\lambda(p)$. Hence the constant factor $\frac{1}{\alpha}k_\lambda(p)$ in (3.6) is the best possible. The theorem is proved. □

Theorem 3.4. *If* $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1, 2-p < \lambda < 2-q, 0 < \alpha \leq \min\{\frac{p}{p+\lambda-2},\frac{q}{q+\lambda-2}\}$ $a_n, b_n \geq 0$, *such that* $0 < \sum_{n=1}^{\infty}\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}} < \infty$, *then we have*

$$(3.14)\quad \sum_{n=1}^{\infty}n^{\alpha[p(\lambda-1)-\lambda+2]-1}\Big[\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}\Big]^p > \Big[\frac{1}{\alpha}k_\lambda(p)\Big]^p\sum_{n=1}^{\infty}\theta_{\lambda,\alpha}(q,n)\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}},$$

where $0 < \theta_{\lambda,\alpha}(q,n) < 1$, *and the constant factor* $[\frac{1}{\alpha}k_\lambda(p)]^p$ $(k_\lambda(p) = B(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}))$ *is the best possible; inequality* (3.14) *is equivalent to* (3.6). *In particular, for* $\lambda = 2$, *one has* $0 < \alpha \leq 1$ *and*

$$(3.15)\quad \sum_{n=1}^{\infty}n^{p\alpha-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^2}\Big]^p > \Big(\frac{1}{\alpha}\Big)^p\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}a_n)^p}{n+n^{1-\alpha}};$$

$$(3.16)\quad \sum_{n=1}^{\infty}n^{p-1}\Big[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^2}\Big]^p > \sum_{n=1}^{\infty}\frac{a_n^p}{n+1}\ (\alpha=1).$$

Proof. Setting b_n as

$$b_n := n^{\alpha[p(\lambda-1)-\lambda+2]-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}\right]^{p-1}\quad (n \in N),$$

then by (3.6), we find

$$(3.17)\quad 0 < \sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}} = \sum_{n=1}^{\infty} n^{\alpha[p(\lambda-1)-\lambda+2]-1}[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}]^p$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda} \geq \frac{k_\lambda(p)}{\alpha}\{\sum_{n=1}^{\infty}\theta_{\lambda,\alpha}(q,n)\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{p}}$$
$$\times\{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{q}},$$

and then

$$(3.18)\quad \{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}}\}^{1-\frac{1}{q}}$$
$$= \{\sum_{n=1}^{\infty} n^{\alpha[p(\lambda-1)-\lambda+2]-1}[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}]^p\}^{\frac{1}{p}}$$
$$\geq \frac{k_\lambda(p)}{\alpha}\{\sum_{n=1}^{\infty}\theta_{\lambda,\alpha}(q,n)\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{p}}.$$

If $\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}} < \infty$, then by (3.6), (3.17) keeps strict inequality; so does (3.18). If $\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}} = \infty$, then (3.18) keeps naturally strict inequality, since

$$0 < \sum_{n=1}^{\infty}\theta_{\lambda,\alpha}(q,n)\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}} < \sum_{n=1}^{\infty}\frac{(n^{1-\alpha}a_n)^p}{n^{1+\alpha(\lambda-2)}} < \infty.$$

Hence for $0 < p < 1$, inequality (3.14) is valid.

On the other hand, if (3.14) is valid, by the reverse *Hölder's* inequality, one has

$$(3.19)\quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\alpha)^\lambda}$$
$$= \sum_{n=1}^{\infty}[n^{-\frac{q-1+\alpha(2-\lambda-q)}{q}}\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}][n^{\frac{q-1+\alpha(2-\lambda-q)}{q}}b_n]$$
$$\geq \{\sum_{n=1}^{\infty} n^{\alpha[p(\lambda-1)-\lambda+2]-1}[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\alpha)^\lambda}]^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{(n^{1-\alpha}b_n)^q}{n^{1+\alpha(\lambda-2)}}\}^{\frac{1}{q}}.$$

Then by (3.14), we have (3.6). It follows that inequality (3.14) is equivalent to (3.6).

If the constant factor in (3.14) is not the best possible, we can conclude a contradiction that the constant factor in (3.6) is not the best possible by using (3.19). Thus we complete the proof of the theorem. □

Remark 3.5. By (3.7) for $\alpha = 1$, and (1.4) for $\lambda = 2$, one can get a two-sides inequality as follows:

$$\{\sum_{n=1}^{\infty}\frac{a_n^p}{n+1}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{b_n^q}{n}\}^{\frac{1}{q}} < \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^2} < \{\sum_{n=1}^{\infty}\frac{a_n^r}{n}\}^{\frac{1}{r}}\{\sum_{n=1}^{\infty}\frac{b_n^s}{n}\}^{\frac{1}{s}}, \tag{3.20}$$

where $r > 1, \frac{1}{r} + \frac{1}{s} = 1$ and $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$.

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