KYUNGPOOK Math. J. 47(2007), 403-410

Fuzzy Subalgebras of Type (α, β) in BCK/BCI-Algebras

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Abstract. Using the *belongs to* relation $(\in)$ and *quasi-coincidence with* relation (q) between fuzzy points and fuzzy sets, the concept of (α, β)-fuzzy subalgebras where α and β areany two of $\{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$ with $\alpha \neq \in\wedge\mathrm{q}$ was already introduced, and related properties were investigated (see [3]). In this paper, we give a condition for an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra to be an $(\in, \in)$-fuzzy subalgebra. We provide characterizations of an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra. We show that a proper $(\in, \in)$-fuzzy subalgebra $\mathscr{A}$ of X with additional conditions can be expressed as the union of two proper non-equivalent $(\in, \in)$-fuzzy subalgebras of X. We also prove that if $\mathscr{A}$ is a proper $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of a CK/BCI-algebra X such that $\#\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} \geq 2$, then there exist two proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy subalgebras of$X$ such that $\mathscr{A}$ can be expressed as the union of them.

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [5], has played a vital role in generating some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [2]. In particular, an $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, the author [3] introduced the concept of (α, β)-fuzzy subalgebra of a BCK/BCI-algebra and investigated related results. This paper is a continuation of the paper [3]. We give a condition for an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra to be an $(\in, \in)$-fuzzy subalgebra. We consider the homomorphic (pre) image of $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra. We provide characterizations of an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra. We show that a proper $(\in, \in)$-fuzzy subalgebra $\mathscr{A}$ of X with additional conditions can be expressed as the union of two proper non-equivalent $(\in, \in)$-fuzzy subalgebras of X. We also prove that if $\mathscr{A}$ is a proper $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of a BCK/BCI-algebra X such that $\#\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} \geq 2$, then there exist

Received May 23, 2006.
2000 Mathematics Subject Classification: 03G10, 03B05, 03B52.
Key words and phrases: belong to, quasi-coincident with, $(\in, \in)$-fuzzy subalgebra, $(\in, \mathrm{q})$-fuzzy subalgebra, $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra.

two proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy subalgebras of X such that $\mathscr{A}$ can be expressed as the union of them.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(i) $(\forall x, y, z \in X)\,(((x * y) * (x * z)) * (z * y) = 0)$,

(ii) $(\forall x, y \in X)\,((x * (x * y)) * y = 0)$,

(iii) $(\forall x \in X)\,(x * x = 0)$,

(iv) $(\forall x, y \in X)\,(x * y = y * x = 0 \Rightarrow x = y)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra.* In what follows let X denote a BCK/BCI-algebra unless otherwise specified. A nonempty subset S of X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the book [4] for further information regarding BCK/BCI-algebras.

A fuzzy set $\mathscr{A}$ in a set X of the form

$$\mathscr{A}(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set $\mathscr{A}$ in a set X, Pu and Liu [5] gave meaning to the symbol $x_t \alpha \mathscr{A}$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$.

To say that $x_t \in \mathscr{A}$ (resp. $x_t \mathrm{q} \mathscr{A}$) means that $\mathscr{A}(x) \geq t$ (resp. $\mathscr{A}(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\mathscr{A}$.

To say that $x_t \in\vee\mathrm{q}\, \mathscr{A}$ (resp. $x_t \in\wedge\mathrm{q}\, \mathscr{A}$) means that $x_t \in \mathscr{A}$ or $x_t \mathrm{q} \mathscr{A}$ (resp. $x_t \in \mathscr{A}$ and $x_t \mathrm{q} \mathscr{A}$).

A fuzzy set $\mathscr{A}$ in X is called a *fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)\,(\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y)\}). \tag{1}$$

Proposition 2.1. *Let $\mathscr{A}$ be a fuzzy set in X. Then $\mathscr{A}$ is a fuzzy subalgebra of X if and only if $U(\mathscr{A}; t) := \{x \in X \mid \mathscr{A}(x) \geq t\}$ is a subalgebra of X for all $t \in (0, 1]$, for our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X.*

3. (α, β)-fuzzy subalgebras

In what follows let α and β denote any one of $\in$, q, $\in\vee\mathrm{q}$, or $\in\wedge\mathrm{q}$ unless otherwise specified. To say that $x_t \overline{\alpha} \mathscr{A}$ means that $x_t \alpha \mathscr{A}$ does not hold.

Proposition 3.1 ([3]). *For any fuzzy set $\mathscr{A}$ in X, the condition* (1) *is equivalent to the following condition*

$$(2) \qquad (\forall x, y \in X)\,(\forall t_1, t_2 \in (0,1])\,(x_{t_1}, y_{t_2} \in \mathscr{A} \;\Rightarrow\; (x * y)_{\min\{t_1,t_2\}} \in \mathscr{A}).$$

Definition 3.2 ([3]). A fuzzy set $\mathscr{A}$ in X is said to be an (α, β)-*fuzzy subalgebra* of X, where $\alpha \neq\, \in\!\wedge\, \mathrm{q}$, if it satisfies the following conditions:

$$(3) \qquad (\forall x, y \in X)\,(\forall t_1, t_2 \in (0,1])\,(x_{t_1} \alpha \mathscr{A},\, y_{t_2} \alpha \mathscr{A} \;\Rightarrow\; (x * y)_{\min\{t_1,t_2\}} \beta \mathscr{A}.)$$

Let $\mathscr{A}$ be a fuzzy set in X such that $\mathscr{A}(x) \leq 0.5$ for all $x \in X$. Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in\!\wedge\, \mathrm{q}\mathscr{A}$. Then $\mathscr{A}(x) \geq t$ and $\mathscr{A}(x) + t > 1$. It follows that

$$1 < \mathscr{A}(x) + t \leq \mathscr{A}(x) + \mathscr{A}(x) = 2\mathscr{A}(x)$$

so that $\mathscr{A}(x) > 0.5$. This means that $\{x_t \mid x_t \in\!\wedge\, \mathrm{q}\mathscr{A}\} = \emptyset$. Therefore the case $\alpha = \in\!\wedge\, \mathrm{q}$ in Definition 3.2 will be omitted.

Example 3.3 ([3]). Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table :

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathscr{A}$ be a fuzzy set in X defined by $\mathscr{A}(0) = 0.6$, $\mathscr{A}(a) = 0.7$, and $\mathscr{A}(b) = \mathscr{A}(c) = 0.3$. Then $\mathscr{A}$ is an $(\in, \in\!\vee\, \mathrm{q})$-fuzzy subalgebra of X. But

(1) $\mathscr{A}$ is not an $(\in, \in)$-fuzzy subalgebra of X since $a_{0.62} \in \mathscr{A}$ and $a_{0.66} \in \mathscr{A}$, but $(a * a)_{\min\{0.62,\, 0.66\}} = 0_{0.62}\, \overline{\in}\, \mathscr{A}$.

(2) $\mathscr{A}$ is not a $(\mathrm{q}, \in\!\vee\, \mathrm{q})$-fuzzy subalgebra of X since $a_{0.41}\, \mathrm{q}\, \mathscr{A}$ and $b_{0.77}\, \mathrm{q}\, \mathscr{A}$, but $(a * b)_{\min\{0.41,\, 0.77\}} = c_{0.41}\, \overline{\in\!\vee\, \mathrm{q}}\, \mathscr{A}$.

(3) $\mathscr{A}$ is not an $(\in\!\vee\, \mathrm{q}, \in\!\vee\, \mathrm{q})$-fuzzy subalgebra of X since $a_{0.5} \in\!\vee\, \mathrm{q}\, \mathscr{A}$ and $c_{0.8} \in\!\vee\, \mathrm{q}\, \mathscr{A}$, but $(a * c)_{\min\{0.5,\, 0.8\}} = b_{0.5}\, \overline{\in\!\vee\, \mathrm{q}}\, \mathscr{A}$.

The following lemma is useful in the sequel.

Lemma 3.4 ([3]). *A fuzzy set $\mathscr{A}$ in X is an $(\in, \in\!\vee\, \mathrm{q})$-fuzzy subalgebra of X if and only if it satisfies:*

$$(4) \qquad (\forall x, y \in X)\,(\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}).$$

Theorem 3.5. *Let S be a subalgebra of X. For any $t \in (0, 0.5]$, there exists an $(\in, \in\!\vee\, \mathrm{q})$-fuzzy subalgebra $\mathscr{A}$ of X such that $U(\mathscr{A}; t) = S$.*

Proof. Let $\mathscr{A}$ be a fuzzy set in X defined by

$$\mathscr{A}(x) = \begin{cases} t & \text{if } \ x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$ where $t \in (0, 0.5]$. Obviously, $U(\mathscr{A}; t) = S$. Assume that $\mathscr{A}(x * y) < \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$ for some $x, y \in X$. Since $\#\mathrm{Im}(\mathscr{A}) = 2$, it follows that $\mathscr{A}(x * y) = 0$ and $\min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\} = t$, and so $\mathscr{A}(x) = t = \mathscr{A}(y)$, so that $x, y \in S$ but $x * y \notin S$. This is a contradiction, and so $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$. Using Lemma 3.4, we know that $\mathscr{A}$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X. □

Theorem 3.6. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras and let $\mathscr{A}$ and $\mathscr{B}$ be $(\in, \in\vee\mathrm{q})$-fuzzy subalgebras of X and Y, respectively. Then*

(i) *$f^{-1}(\mathscr{B})$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X.*

(ii) *If $\mathscr{A}$ satisfies the sup property, i.e., for any subset T of X there exists $x_0 \in T$ such that*

$$\mathscr{A}(x_0) = \bigvee\{\mathscr{A}(x) \mid x \in T\},$$

then $f(\mathscr{A})$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of Y when f is onto.

Proof. (i) Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in f^{-1}(\mathscr{B})$ and $y_{t_2} \in f^{-1}(\mathscr{B})$. Then $(f(x))_{t_1} \in \mathscr{B}$ and $(f(y))_{t_2} \in \mathscr{B}$. Since $\mathscr{B}$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of Y, it follows that

$$(f(x * y))_{\min\{t_1,\, t_2\}} = (f(x) * f(y))_{\min\{t_1,\, t_2\}} \in\vee\,\mathrm{q}\,\mathscr{B}$$

so that $(x * y)_{\min\{t_1,\, t_2\}} \in\vee\,\mathrm{q}\, f^{-1}(\mathscr{B})$. Therefore $f^{-1}(\mathscr{B})$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X.

(ii) Let $a, b \in Y$ and $t_1, t_2 \in (0, 1]$ be such that $a_{t_1} \in f(\mathscr{A})$ and $b_{t_2} \in f(\mathscr{A})$. Then $(f(\mathscr{A}))(a) \geq t_1$ and $(f(\mathscr{A}))(b) \geq t_2$. Since $\mathscr{A}$ has the sup property, there exists $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that

$$\mathscr{A}(x) = \bigvee\{\mathscr{A}(z) \mid z \in f^{-1}(a)\}$$

and

$$\mathscr{A}(y) = \bigvee\{\mathscr{A}(w) \mid w \in f^{-1}(b)\}.$$

Then $x_{t_1} \in \mathscr{A}$ and $y_{t_2} \in \mathscr{A}$. Since $\mathscr{A}$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X, we have $(x * y)_{\min\{t_1,\, t_2\}} \in\vee\,\mathrm{q}\,\mathscr{A}$. Now $x * y \in f^{-1}(a * b)$ and so $(f(\mathscr{A}))(a * b) \geq \mathscr{A}(x * y)$. Thus

$$(f(\mathscr{A}))(a * b) \geq \min\{t_1,\, t_2\} \ \text{ or } \ (f(\mathscr{A}))(a * b) + \min\{t_1,\, t_2\} > 1$$

which means that $(a * b)_{\min\{t_1,\, t_2\}} \in\vee\,\mathrm{q}\, f(\mathscr{A})$. Consequently, $f(\mathscr{A})$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of Y. □

Theorem 3.7. *Let $\mathscr{A}$ be a* $(\mathrm{q}, \in\vee\mathrm{q})$*-fuzzy subalgebra of a BCK-algebra X such that that $\mathscr{A}$ is not constant on X_0. If $\mathscr{A}(0) = \bigvee\{\mathscr{A}(x) \mid x \in X\}$, then $\mathscr{A}(x) \geq 0.5$ for all $x \in X_0$.*

Proof. Assume that $\mathscr{A}(x) < 0.5$ for all $x \in X$. Since $\mathscr{A}$ is not constant on X_0, there exists $y \in X_0$ such that $t_y = \mathscr{A}(y) \neq \mathscr{A}(0) = t_0$. Then $t_y < t_0$. Choose $t_1 > 0.5$ such that $t_y + t_1 < 1 < t_0 + t_1$. Then $0_{t_1}\mathrm{q}\mathscr{A}$ and $y_1\mathrm{q}\mathscr{A}$. Since $\mathscr{A}(x) + t_1 = t_y + t_1 < 1$, we get $y_{t_1}\overline{\mathrm{q}}\mathscr{A}$ and so $(y*0)_{\min\{1,\, t_1\}} = y_{t_1}\overline{\in\vee\mathrm{q}}\mathscr{A}$. This contradicts the fact that $\mathscr{A}$ is a $(\mathrm{q}, \in\vee\mathrm{q})$-fuzzy subalgebra of X. Therefore $\mathscr{A}(x) \geq 0.5$ for some $x \in X$. Now if possible, let $t_0 = \mathscr{A}(0) < 0.5$. Then there exists $x \in X$ such that $t_x = \mathscr{A}(x) \geq 0.5$. Thus $t_0 < t_x$. Take $t_1 > t_0$ such that $t_0 + t_1 < 1 < t_x + t_1$. Then $x_{t_1}\mathrm{q}\mathscr{A}$ and $0_1\mathrm{q}\mathscr{A}$, but $(0*x)_{\min\{1,\, t_1\}} = 0_{t_1}\overline{\in\vee\mathrm{q}}\mathscr{A}$, a contradiction. Hence $\mathscr{A}(0) \geq 0.5$. Finally let $t_x = \mathscr{A}(x) < 0.5$ for some $x \in X_0$. Taking $t_1 > 0$ such that $t_x + t_1 < 0.5$, then $x_1\mathrm{q}\mathscr{A}$ and $0_{0.5+t_1}\mathrm{q}\mathscr{A}$ since $\mathscr{A}(0) \geq 0.5$. But

$$\mathscr{A}(x) + 0.5 + t_1 = t_x + 0.5 + t_1 < 0.5 + 0.5 = 1,$$

which implies that $x_{0.5+t_1}\overline{\mathrm{q}}\mathscr{A}$. Thus $(x*0)_{\min\{1,\, 0.5+t_1\}} = x_{0.5+t_1}\overline{\in\vee\mathrm{q}}\mathscr{A}$, a contradiction. Therefore $\mathscr{A}(x) \geq 0.5$ for all $x \in X_0$. □

The following is our question: Does Theorem 3.7 hold in a BCI-algebra?

A fuzzy set $\mathscr{A}$ in X is said to be *proper* if $\mathrm{Im}(\mathscr{A})$ has at least two elements. Two fuzzy sets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.8. *Let X be a BCK/BCI-algebra. Then a proper* $(\in, \in)$*-fuzzy subalgebra $\mathscr{A}$ of X such that $\#\mathrm{Im}(\mathscr{A}) \geq 3$ can be expressed as the union of two proper non-equivalent* $(\in, \in)$*-fuzzy subalgebras of X.*

Proof. Let $\mathscr{A}$ be a proper $(\in, \in)$-fuzzy subalgebra of X with $\mathrm{Im}(\mathscr{A}) = \{t_0, t_1, \cdots, t_n\}$, where $t_0 > t_1 > \cdots > t_n$ and $n \geq 2$. Then

$$U(\mathscr{A}; t_0) \subseteq U(\mathscr{A}; t_1) \subseteq \cdots \subseteq U(\mathscr{A}; t_n) = X$$

is the chain of $\in$-level subalgebras of $\mathscr{A}$. Define fuzzy sets $\mathscr{B}$ and $\mathscr{C}$ in X by

$$\mathscr{B}(x) = \begin{cases} r_1 & \text{if } x \in U(\mathscr{A}; t_1), \\ t_2 & \text{if } x \in U(\mathscr{A}; t_2) \setminus U(\mathscr{A}; t_1), \\ \cdots & \\ t_n & \text{if } x \in U(\mathscr{A}; t_n) \setminus U(\mathscr{A}; t_{n-1}), \end{cases}$$

and

$$\mathscr{C}(x) = \begin{cases} t_0 & \text{if } x \in U(\mathscr{A}; t_0), \\ t_1 & \text{if } x \in U(\mathscr{A}; t_1) \setminus U(\mathscr{A}; t_0), \\ r_2 & \text{if } x \in U(\mathscr{A}; t_3) \setminus U(\mathscr{A}; t_1), \\ t_4 & \text{if } x \in U(\mathscr{A}; t_4) \setminus U(\mathscr{A}; t_3), \\ \cdots & \\ t_n & \text{if } x \in U(\mathscr{A}; t_n) \setminus U(\mathscr{A}; t_{n-1}), \end{cases}$$

respectively, where $t_2 < r_1 < t_1$ and $t_4 < r_2 < t_2$. Then $\mathscr{B}$ and $\mathscr{C}$ are $(\in, \in)$-fuzzy subalgebras of X with

$$U(\mathscr{A}; t_1) \subseteq U(\mathscr{A}; t_2) \subseteq \cdots \subseteq U(\mathscr{A}; t_n) = X$$

and

$$U(\mathscr{A}; t_0) \subseteq U(\mathscr{A}; t_1) \subseteq U(\mathscr{A}; t_3) \subseteq \cdots \subseteq U(\mathscr{A}; t_n) = X$$

as respective chains of $\in$-level subalgebras, and $\mathscr{B}, \mathscr{C} \leq \mathscr{A}$. Thus $\mathscr{B}$ and $\mathscr{C}$ are non-equivalent, and obviously $\mathscr{B} \cup \mathscr{C} = \mathscr{A}$. This completes the proof. □

Note that every $(\in, \in)$-fuzzy subalgebra is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra, but the converse is not true in general (see [3]). Now we give a condition for an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra to be an $(\in, \in)$-fuzzy subalgebra.

Theorem 3.9. *Let $\mathscr{A}$ be an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X such that $\mathscr{A}(x) < 0.5$ for all $x \in X$. Then $\mathscr{A}$ is an $(\in, \in)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mathscr{A}$ and $y_{t_2} \in \mathscr{A}$. Then $\mathscr{A}(x) \geq t_1$ and $\mathscr{A}(y) \geq t_2$. It follows from Lemma 3.4 that

$$\mathscr{A}(xy) \geq \min\{\mathscr{A}(x),\, \mathscr{A}(y),\, 0.5\} = \min\{\mathscr{A}(x),\, \mathscr{A}(y)\} \geq \min\{t_1,\, t_2\}$$

so that $(x * y)_{\min\{t_1,\, t_2\}} \in \mathscr{A}$. Hence $\mathscr{A}$ is an $(\in, \in)$-fuzzy subalgebra of X. □

Theorem 3.10. *A fuzzy set $\mathscr{A}$ in X is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X if and only if the set*

$$U(\mathscr{A}; t) := \{x \in X \mid \mathscr{A}(x) \geq t\}$$

is a subalgebra of X for all $t \in (0,\, 0.5]$.

Proof. Assume that $\mathscr{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X. Let $x, y \in U(\mathscr{A}; t)$ for $t \in (0, 0.5]$. Then $\mathscr{A}(x) \geq t$ and $\mathscr{A}(y) \geq t$. It follows from Lemma 3.4 that

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x),\, \mathscr{A}(y),\, 0.5\} \geq \min\{t,\, 0.5\} = t$$

so that $x * y \in U(\mathscr{A}; t)$. Therefore $U(\mathscr{A}; t)$ is a subalgerba of X. Conversely, let $\mathscr{A}$ be a fuzzy set in X such that the set

$$U(\mathscr{A}; t) := \{x \in X \mid \mathscr{A}(x) \geq t\}$$

is a subalgebra of X for all $t \in (0,\, 0.5]$. If there exist $x, y \in X$ such that $\mathscr{A}(x * y) < \min\{\mathscr{A}(x),\, \mathscr{A}(y),\, 0.5\}$, then we can take $t \in (0, 1)$ such that $\mathscr{A}(x * y) < t < \min\{\mathscr{A}(x),\, \mathscr{A}(y),\, 0.5\}$. Thus $x, y \in U(\mathscr{A}; t)$ and $t < 0.5$, and so $x * y \in U(\mathscr{A}; t)$, i.e., $\mathscr{A}(x * y) \geq t$. This is a contradiction. Therefore

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x),\, \mathscr{A}(y),\, 0.5\}$$

for all $x, y \in X$. Using Lemma 3.4, we conclude that $\mathscr{A}$ is an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of X. □

For any fuzzy set $\mathscr{A}$ in X and $t \in (0,1]$, we denote

$$\mathscr{A}_t := \{x \in X \mid x_t \mathrm{q} \mathscr{A}\} \text{ and } [\mathscr{A}]_t := \{x \in X \mid x_t \in\vee\mathrm{q}\,\mathscr{A}\}.$$

Obviously, $[\mathscr{A}]_t = U(\mathscr{A}; t) \cup \mathscr{A}_t$.

Theorem 3.11. *A fuzzy set $\mathscr{A}$ in X is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X if and only if $[\mathscr{A}]_t$ is a subalgebra of X for all $t \in (0,1]$.*

We call $[\mathscr{A}]_t$ an $(\in\vee\mathrm{q})$-*level subalgebra* of $\mathscr{A}$.

Proof. Let $\mathscr{A}$ be an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X and let $x, y \in [\mathscr{A}]_t$ for $t \in (0,1]$. Then $x_t \in\vee\mathrm{q}\,\mathscr{A}$ and $y_t \in\vee\mathrm{q}\,\mathscr{A}$, that is, $\mathscr{A}(x) \geq t$ or $\mathscr{A}(x) + t > 1$, and $\mathscr{A}(y) \geq t$ or $\mathscr{A}(y) + t > 1$. Since $\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$ by Lemma 3.4, we have $\mathscr{A}(x * y) \geq \min\{t, 0.5\}$. Otherwise, $x_t \overline{\in\vee\mathrm{q}}\,\mathscr{A}$ or $y_t \overline{\in\vee\mathrm{q}}\,\mathscr{A}$, a contradiction. If $t \leq 0.5$, then $\mathscr{A}(x * y) \geq \min\{t, 0.5\} = t$ and so $x * y \in U(\mathscr{A}; t) \subseteq [\mathscr{A}]_t$. If $t > 0.5$, then $\mathscr{A}(x * y) \geq \min\{t, 0.5\} = 0.5$ and thus $\mathscr{A}(x * y) + t > 0.5 + 0.5 = 1$. Hence $(x * y)_t \mathrm{q} \mathscr{A}$, and so $x * y \in \mathscr{A}_t \subseteq [\mathscr{A}]_t$. Therefore $[\mathscr{A}]_t$ is a subalgebra of . Conversely, let $\mathscr{A}$ be a fuzzy set in X and $t \in (0,1]$ be such that $[\mathscr{A}]_t$ is a subalgebra of X. If possible, let

$$\mathscr{A}(x * y) < t < \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$$

for some $t \in (0, 0.5)$ and $x, y \in X$. Then $x, y \in U(\mathscr{A}; t) \subseteq [\mathscr{A}]_t$, which implies that $x * y \in [\mathscr{A}]_t$. Hence $\mathscr{A}(x * y) \geq t$ or $\mathscr{A}(x * y) + t > 1$, a contradiction. Therefore

$$\mathscr{A}(x * y) \geq \min\{\mathscr{A}(x), \mathscr{A}(y), 0.5\}$$

for all $x, y \in X$. Using Lemma 3.4, we conclude that $\mathscr{A}$ is an $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X. □

Theorem 3.12. *Let $\mathscr{A}$ be a proper $(\in, \in\vee\mathrm{q})$-fuzzy subalgebra of X such that $\#\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} \geq 2$. Then there exist two proper non-equivalent $(\in, \in\vee\mathrm{q})$-fuzzy subalgebras of X such that $\mathscr{A}$ can be expressed as the union of them.*

Proof. Let $\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} = \{t_1, t_2, \cdots, t_r\}$, where $t_1 > t_2 > \cdots > t_r$ and $r \geq 2$. Then the chain of $(\in\vee\mathrm{q})$-level subalgebras of $\mathscr{A}$ is

$$[\mathscr{A}]_{0.5} \subseteq [\mathscr{A}]_{t_1} \subseteq [\mathscr{A}]_{t_2} \subseteq \cdots \subseteq [\mathscr{A}]_{t_r} = X.$$

Let $\mathscr{B}$ and $\mathscr{C}$ be fuzzy sets in X defined by

$$\mathscr{B}(x) = \begin{cases} t_1 & \text{if } x \in [\mathscr{A}]_{t_1}, \\ t_2 & \text{if } x \in [\mathscr{A}]_{t_2} \setminus [\mathscr{A}]_{t_1}, \\ \cdots & \\ t_r & \text{if } x \in [\mathscr{A}]_{t_r} \setminus [\mathscr{A}]_{t_{r-1}}, \end{cases}$$

and

$$\mathscr{C}(x) = \begin{cases} A(x) & \text{if } x \in [\mathscr{A}]_{0.5}, \\ k & \text{if } x \in [\mathscr{A}]_{t_2} \setminus [\mathscr{A}]_{0.5}, \\ t_3 & \text{if } x \in [\mathscr{A}]_{t_3} \setminus [\mathscr{A}]_{t_2}, \\ \cdots & \\ t_r & \text{if } x \in [\mathscr{A}]_{t_r} \setminus [\mathscr{A}]_{t_{r-1}}, \end{cases}$$

respectively, where $t_3 < k < t_2$. Then $\mathscr{B}$ and $\mathscr{C}$ are $(\in, \in \vee \mathrm{q})$-fuzzy subalgebras of X, and $\mathscr{B}, \mathscr{C} \leq \mathscr{A}$. The chains of $(\in \vee \mathrm{q})$-level subalgebras of $\mathscr{B}$ and $\mathscr{C}$ are, respectively, given by

$$[\mathscr{A}]_{t_1} \subseteq [\mathscr{A}]_{t_2} \subseteq \cdots \subseteq [\mathscr{A}]_{t_r}$$

and

$$[\mathscr{A}]_{0.5} \subseteq [\mathscr{A}]_{t_2} \subseteq \cdots \subseteq [\mathscr{A}]_{t_r}.$$

Therefore $\mathscr{B}$ and $\mathscr{C}$ are non-equivalent and clearly $\mathscr{A} = \mathscr{B} \cup \mathscr{C}$. This completes the proof. □

4. Conclusions

In the notion of an (α, β)-fuzzy subalgebra, we can consider twelve different types of such structures resulting from three choices of α and four choices of β. In [3], the author discussed mainly $(\in, \in)$-type, (q, q)-type, $(\mathrm{q}, \in \vee \mathrm{q})$-type, $(\in \vee \mathrm{q}, \in \vee \mathrm{q})$-type and $(\in, \in \vee \mathrm{q})$-type. In this paper, we considered more properties of an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra in a BCK/BCI-algebra. We considered the homomorphic (pre)image of $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra, and we gave a condition for an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra to be an $(\in, \in)$-fuzzy subalgebra. We provided characterizations of an $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra. We showed that a proper $(\in, \in)$-fuzzy subalgebra $\mathscr{A}$ of X with additional conditions can be expressed as the union of two proper non-equivalent $(\in, \in)$-fuzzy subalgebras of X. We also proved that if $\mathscr{A}$ is a proper $(\in, \in \vee \mathrm{q})$-fuzzy subalgebra of a BCK/BCI-algebra X such that $\#\{\mathscr{A}(x) \mid \mathscr{A}(x) < 0.5\} \geq 2$, then there exist two proper non-equivalent $(\in, \in \vee \mathrm{q})$-fuzzy subalgebras of X such that $\mathscr{A}$ can be expressed as the union of them.

Future research will focus on considering other types together with relations among them.

References

[1] S. A. Bhatti, M. A. Chaudhry and B. Ahmad, *On classification of BCI-algebras,* Math. Jpn., **34(6)**(1989), 865-876.

[2] S. K. Bhakat and P. Das, $(\in, \in \vee \mathrm{q})$-*fuzzy subgroup,* Fuzzy Sets and Systems, **80**(1996), 359-368.

[3] Y. B. Jun, *On* (α, β)*-fuzzy subalgerbas of BCK/BCI-algebras,* Bull. Korean Math. Soc., **42(4)**(2005), 703-711.

[4] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co., Korea (1994).

[5] P. M. Pu and Y. M. Liu, *Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., **76**(1980), 571-599.