

## On Subclasses of P-Valent Analytic Functions Defined by Integral Operators

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ABSTRACT. In the present paper we introduce the subclass  $S_{a,c}^\lambda(p, A, B)$  of analytic functions and then we investigate some interesting properties of functions belonging to this subclass. Our results generalize many results known in the literature and especially generalize some of the results obtained by Ling and Liu [5].

### 1. Introduction

Let  $\mathcal{A}_p$  denote the class of functions

$$(1.1) \quad f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic and p-valent in the open unit disc  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ . Set  $\mathcal{A}_1 = \mathcal{A}$ . For  $-1 \leq B < A \leq 1$  the function  $f \in \mathcal{A}_p$  is said to be in the class  $S_p^*(A, B)$ , if

$$(1.2) \quad \frac{zf'(z)}{f(z)} \prec p \frac{1 + Az}{1 + Bz},$$

where the symbol “ $\prec$ ” stands for subordination. In particular, we note that  $S_p^*(1 - \frac{2\alpha}{p}, -1) := S_p^*(\alpha)$ ,  $(0 \leq \alpha < p)$  is the class of p-valently starlike functions of order  $\alpha$ . Also let  $C_p(A, B)$  denote the subclass of  $\mathcal{A}_p$  consisting of all functions  $f$  such that  $zf'(z) \in S_p^*(A, B)$ . In particular, a function in the class  $C_p(1 - \frac{2\alpha}{p}, -1) := C_p(\alpha)$ ,  $(0 \leq \alpha < p)$  is said to be p-valently convex function of order  $\alpha$ . Furthermore a function  $f \in \mathcal{A}_p$  is said to be in the class  $\rho_p(A, B)$ ,  $(-1 \leq B < A \leq 1)$ , if

$$\frac{z^p}{(1-z)^{p(A-B)}} * f(z) \in S_p^*(A, B), \quad z \in \mathbb{U},$$

where “ $*$ ” denote the Hadamard product (or convolution).

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Set  $\rho_1(1 - \frac{2\alpha}{p}, -1) = \rho(\alpha)$ , ( $0 \leq \alpha < 1$ ). Class  $\rho(\alpha)$  is called the class of prestarlike functions of order  $\alpha$ . In [8], it is shown  $\rho(\alpha) \subset \rho(\beta)$  for  $\alpha \leq \beta \leq 1$ . For real or complex numbers  $a, b, c$  ( $c \neq 0, -1, -2, \dots$ ), the hypergeometric function is defined by

$$(1.3) \quad {}_2F_1(a, b; c; z) = 1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots = 1 + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where  $(a)_n$  is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\dots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3, \dots\}). \end{cases}$$

We note that the series in (1.3) converges absolutely for  $z \in \mathbb{U}$  and hence represents an analytic function in  $\mathbb{U}$ .

Making use of the Hadamard product (or convolution), we define the linear operator  $L_p(a, c)f(z) : \mathcal{A}_p \mapsto \mathcal{A}_p$  by

$$(1.4) \quad L_p(a, c)f(z) := \phi_p(a, c; z) * f(z) \quad (f \in \mathcal{A}_p),$$

where

$$(1.5) \quad \phi_p(a, c; z) = z^p {}_2F_1(a, 1; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p},$$

$$(z \in \mathbb{U}; a \in \mathbb{R}; c \in \mathbb{R} - \mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0, -1, -2, -3, \dots\})$$

The operator  $L_p(a, c)$  was introduced and studied by Saitoh [10]. This operator is an extension of the Carlson-Shaffer operator  $L_1(a, c)$  and familiar fractional derivative operator  $D_z^\lambda$ .

Corresponding to the function  $\phi_p(a, c; z)$  defined by (1.5), we introduce a function  $\phi_p^\lambda(a, c; z)$  given by

$$\phi_p(a, c; z) * \phi_p^\lambda(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p),$$

which leads us to the following family of linear operators  $\tau_p^\lambda(a, c)$ :

$$(1.6) \quad \tau_p^\lambda(a, c)f(z) = \phi_p^\lambda(a, c; z) * f(z) \\ (a, c \in \mathbb{R} - \mathbb{Z}_0^-; \lambda > -p; z \in \mathbb{U}; f \in \mathcal{A}_p).$$

It is readily verified from the definition (1.6) that

$$(1.7) \quad \tau_p^1(p+1, 1)f(z) = f(z) \quad \tau_p^1(p, 1)f(z) = \frac{zf'(z)}{p}$$

$$(1.8) \quad z(\tau_p^\lambda(a+1, c)f(z))' = a\tau_p^\lambda(a, c)f(z) - (a-p)\tau_p^\lambda(a+1, c)f(z),$$

$$(1.9) \quad z(\tau_p^\lambda(a, c)f(z))' = (\lambda+p)\tau_p^{\lambda+1}(a, c)f(z) - \lambda\tau_p^\lambda(a, c)f(z).$$

The operator  $\tau_p^\lambda(a, c)$  were recently investigated by cho et al. [2], Ling et al. [5]. Also the special case of  $\tau_p^\lambda(a, c)$  were earlier considered by Choi et al. [3], Liu [4] and Noor et al. [7].

By using the general linear operator  $\tau_p^\lambda(a, c)$ , we define a new subclass of  $\mathcal{A}_p$  by

$$(1.10) \quad S_{a,c}^\lambda(p, A, B) = \left\{ f : f \in \mathcal{A}_p \text{ and } \frac{z(\tau_p^\lambda(a, c)f(z))'}{p \tau_p^\lambda(a, c)f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

Thus, for some suitably chosen parameters  $a, c, \lambda, p$  and  $A, B$  the class  $S_{a,c}^\lambda(p, A, B)$  reduce to several known subclasses of univalent and multivalent analytic functions. For example, we have  $S_{p+1,1}^1(p, A, B) = S_p^*(A, B), S_{p,1}^1(p, A, B) = C_p(A, B)$  and  $S_{1,1}^{A-B-1}(p, A, B) = \rho_p(A, B)$ .

In the present paper, we investigate some interesting properties of functions in the class  $\mathcal{A}_p$  as those belonging to the subclass  $S_{a,c}^\lambda(p, A, B)$ . Our results generalize many results known in the literature, especially the recent work of Ling and Liu [5].

To prove our main results, we need the following lemmas.

**Lemma 1.1 ([9]).** *If  $f \in C_1(0)$  and  $g \in S_1^*(0)$ , then for each function  $F$ , analytic in  $\mathbb{U}$ , the image of  $\mathbb{U}$  under  $(f * Fg)/(f * g)$  is a subset of the convex hull of  $F(\mathbb{U})$ .*

**Lemma 1.2.** *If  $-1 \leq B < A \leq 1, \beta > 0$  and the complex number  $\gamma$  satisfies  $\Re(\gamma) \geq -\beta(1-A)/(1-B)$ , then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{A})$$

has a univalent solution in  $\mathcal{A}$  given by

$$(1.11) \quad q(z) := \begin{cases} \frac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta \int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B} dt} - \frac{\gamma}{\beta}, & (B \neq 0); \\ \frac{z^{\beta+\gamma} \exp(\beta Az)}{\beta \int_0^z t^{\beta+\gamma-1} \exp(\beta At) dt} - \frac{\gamma}{\beta}, & (B = 0). \end{cases}$$

If  $\phi(z) = 1 + c_1z + c_2z^2 + \dots$  is analytic in  $\mathbb{U}$  and satisfies

$$(1.12) \quad \phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

then

$$\phi(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

and  $q(z)$  is the best dominant of (1.12).

The above lemma is due to Miller and Mocanu [6].

**Lemma 1.3 ([11]).** Let  $\nu$  be a positive measure on  $[0, 1]$ . Let  $h$  be a complex-valued function defined on  $\mathbb{U} \times [0, 1]$  such that  $h(\cdot, t)$  is analytic in  $\mathbb{U}$  for each  $t \in [0, 1]$ , and  $h(z, \cdot)$  is  $\nu$ -integrable on  $[0, 1]$  for all  $z \in \mathbb{U}$ . In addition, suppose that  $\Re\{h(z, t)\} > 0$ ,  $h(-r, t)$  is real and  $\Re\{1/h(z, t)\} \geq 1/h(-r, t)$  for  $|z| \leq r < 1$  and  $t \in [0, 1]$ . If  $h(z) = \int_0^1 h(z, t) d\nu(t)$ , then  $\Re\{1/h(z)\} \geq 1/h(-r)$ .

**Lemma 1.3 ([12]).** For real numbers  $a, b, c$  ( $c \neq 0, -1, -2, \dots$ ), we have

$$(1.13) \quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z) \quad (c > b > 0)$$

$$(1.14) \quad {}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$$

$$(1.15) \quad {}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1(a, c-b; c; z/1-z).$$

## 2. Main Results

We begin with the following:

**Theorem 2.1.** Let  $\lambda > -1$  and  $\phi(z) \in S_{1,1}^1(1, 1, -1)$ . If  $f(z) \in S_{a,c}^\lambda(1, A, B)$ , then  $(\phi * f)(z) \in S_{a,c}^\lambda(1, A, B)$ .

*Proof.* we have  $f(z) \in S_{a,c}^\lambda(1, A, B)$  if and only if

$$F(z) := \frac{z(\tau_1^\lambda(a, c)f)'(z)}{\tau_1^\lambda(a, c)f(z)} \prec \frac{1 + Az}{1 + Bz} := G(z).$$

Since  $G$  is convex, an application of Lemma (1.1) yields

$$\begin{aligned} & \frac{\phi(z) * F(z)(\tau_1^\lambda(a, c)f(z))}{\phi(z) * \tau_1^\lambda(a, c)f(z)} = \frac{\phi(z) * z(\tau_1^\lambda(a, c)f)'(z)}{\phi(z) * \tau_1^\lambda(a, c)f(z)} \\ & = \frac{z(\tau_1^\lambda(a, c)f * \phi)'(z)}{\tau_1^\lambda(a, c)(f * \phi)(z)} \prec p \frac{1 + Az}{1 + Bz}, \end{aligned}$$

and so  $f * \phi \in S_{a,c}^\lambda(1, A, B)$ .  $\square$

By taking special values for  $a, c, \lambda$ , we have the following well-known results as corollaries of Theorem 2.1.

**Corollary 2.1.** For  $-1 \leq B < A \leq 1$ , if  $\phi \in C_1(0)$ ,  $f \in S_1^*(A, B)$ , then  $\phi * f \in S_1^*(A, B)$ .

**Corollary 2.2.** For  $-1 \leq B < A \leq 1$ , if  $\phi \in C_1(0), f \in C_1(A, B)$ , then  $\phi * f \in C_1(A, B)$ .

**Corollary 2.3.** For  $-1 \leq B < A \leq 1$ , if  $\phi \in C_1(0), f \in \rho_1(A, B)$ , then  $\phi * f \in \rho_1(A, B)$ .

**Theorem 2.2.** Let  $a \geq p, \lambda \geq 0, -1 \leq B < 0$  and  $0 \leq A < \frac{-B}{p}$ . Then

$$S_{a,c}^\lambda(p, A, B) \subset S_{a+1,c}^\lambda(p, 1 - \frac{2\alpha}{p}, -1),$$

where

$$\alpha = a \left\{ {}_2F_1\left(1, \frac{p(B-A)}{B}; a+1; \frac{B}{B-1}\right) \right\}^{-1} + p - a.$$

The result is best possible.

*Proof.* Let  $f \in S_{a,c}^\lambda(p, A, B)$  and set

$$(2.1) \quad \psi(z) = \frac{z(\tau_p^\lambda(a+1, c)f)'(z)}{p\tau_p^\lambda(a+1, c)f(z)}, \quad (z \in \mathbb{U}).$$

We note that  $\psi(z) = 1 + c_1z + \dots$  is analytic in  $\mathbb{U}$ . By logarithmically differentiating both sides of (2.1) and multiplying the resulting equation by  $z$ , we have

$$\frac{z(\tau_p^\lambda(a, c)f)'(z)}{p\tau_p^\lambda(a, c)f(z)} = \psi(z) + \frac{z\psi'(z)}{p\psi(z) + (a-p)} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathbb{U}).$$

Thus,  $\psi(z)$  satisfies the differential subordination (1.12) and hence by Lemma 1.2, we get

$$\psi(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

where  $q(z)$  is given by

$$(2.2) \quad q(z) = \frac{z^a(1 + Bz) \frac{p(A-B)}{B}}{p \int_0^z t^{a-1}(1 + Bt) \frac{p(A-B)}{B} dt} - \frac{a-p}{p}.$$

Let us define  $Q(z)$  by  $\frac{1}{Q(z)} := p q(z) + a - p$ . For completing our proof it is sufficient to show that

$$\inf_{|z| < 1} \Re \left\{ \frac{1}{Q(z)} \right\} = \frac{1}{Q(-1)}.$$

If we set  $d = \frac{p(A-B)}{B}$ ,  $e = a + 1$ ,  $b = a$ , then  $e > b > 0$  and our hypotheses  $a \geq p$ ,  $0 \leq A \leq \frac{-B}{p}$  implies  $e > d > 0$ . From (2.2), by using (1.13), (1.14), (1.15) and Lemma 1.4 we find that

$$\begin{aligned} Q(z) &= (1+Bz)^d \int_0^1 s^{b-1}(1+Bs z)^{-d} ds = \frac{\Gamma(b)}{\Gamma(e)} {}_2F_1(1, d; e; \frac{Bz}{Bz+1}) \\ &= \frac{\Gamma(b)}{\Gamma(e)} (1+Bz) {}_2F_1(1, e-d; e; -Bz) \\ &= \frac{\Gamma(b)}{\Gamma(e-d)\Gamma(d)} (1+Bz) \int_0^1 t^{e-d-1}(1-t)^{d-1}(1+Btz)^{-1} dt \\ &= \frac{\Gamma(b)}{\Gamma(e-d)\Gamma(d)} (1+Bz) \int_0^1 (1-t)^{e-d-1} t^{d-1} (1+B(1-t)z)^{-1} dt \\ &= \int_0^1 h(z, t) d\nu(t), \end{aligned}$$

where

$$h(z, t) := \frac{1+Bz}{1+(1-t)Bz} \quad (0 \leq t \leq 1) \quad \text{and} \quad d\nu(t) := \frac{\Gamma(b)}{\Gamma(d)\Gamma(e-d)} t^{d-1} (1-t)^{e-d-1} dt.$$

It is easily verified that  $\nu(t)$  is a positive measure on  $[0, 1]$  and  $\Re\{h(z, t)\} > 0$ , for  $-1 \leq B < 0$ , and  $h(-r, t)$  is real for  $0 \leq r < 1$ ,  $t \in [0, 1]$ . Now

$$\Re \left\{ \frac{1}{h(z, t)} \right\} = \Re \left\{ \frac{1+(1-t)Bz}{1+Bz} \right\} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{h(-r, t)},$$

for  $|z| \leq r < 1$  and  $t \in [0, 1]$ . Therefore, by using Lemma 1.3, we have  $\Re \left\{ \frac{1}{Q(z)} \right\} \geq \frac{1}{Q(-r)}$ ,  $|z| \leq r < 1$  and by letting  $r \mapsto 1^-$ , we obtain  $\Re \left\{ \frac{1}{Q(z)} \right\} \geq \frac{1}{Q(-1)}$ . Hence the proof is complete.  $\square$

The proof of Theorem 2.3 below is much akin to that of Theorem 2.2 and so the details involved may be omitted.

**Theorem 2.3.** *Let  $\lambda \geq 0$ ,  $-1 \leq B < 0$  and  $0 \leq A < \frac{-B(\lambda+1)}{p}$ . Then*

$$S_{a,c}^{\lambda+1}(p, A, B) \subset S_{a,c}^{\lambda}(p, 1 - \frac{2\gamma}{p}, -1),$$

where

$$\gamma = (\lambda + p) \left\{ {}_2F_1\left(1, \frac{p(B-A)}{B}; \lambda + p + 1; \frac{B}{B-1}\right) \right\}^{-1} - \lambda.$$

The result is best possible.

**Theorem 2.4.** Let  $\lambda > -p, -1 \leq B < A \leq 1, a > 0, c > 0$  and  $p \in \mathbb{N} := \{1, 2, 3, \dots\}$ . If  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in S_{a,c}^{\lambda}(p, A, B)$ , then

$$(2.3) \quad |a_n| \leq \frac{(pA - pB)_n (a)_n}{(c)_n (\lambda + p)_n} \quad (n = 1, 2, 3, \dots).$$

When  $B = -1$ , the result is sharp for the function given by  $\tau_p^{\lambda}(a, c)f(z) = \frac{z^p}{(1 - z)^{p(1+A)}}$ .

*Proof.* Since  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in S_{a,c}^{\lambda}(p, A, B)$ , it follows that

$$(2.4) \quad \frac{z(\tau_p^{\lambda}(a, c)f)'(z)}{p\tau_p^{\lambda}(a, c)f(z)} := h(z),$$

where  $h(z) = 1 + p_1z + p_2z^2 + \dots$  is analytic in  $\mathbb{U}$  and  $h(z) \prec \frac{1 + Az}{1 + Bz}$ . Substituting the series expansion of  $f(z)$  and  $h(z)$  in (2.4) and equating the coefficients of  $z^n$  on both sides of the resulting equation, we obtain

$$(2.5) \quad nk_n = p_n p k_0 + p_{n-1} p k_1 + \dots + p_1 p k_{n-1} \quad (n = 1, 2, 3, \dots),$$

where  $p_0 := k_0 := 1$  and  $k_n := \frac{(c)_n (\lambda + p)_n}{(a)_n (1)_n} a_n$ . Using the well known coefficient estimates (see for details [1])

$$|p_n| \leq A - B, \quad (n \geq 1)$$

in (2.5), we get the required result (2.3). It is easily verified the result is sharp for the function  $f$  defined by  $\tau_p^{\lambda}(a, c)f(z) = \frac{z^p}{(1 - z)^{p(1+A)}}$ , when  $B = -1$ .  $\square$

Letting  $\lambda = 1, a = p + 1$  and  $c = 1$  in the above theorem, yields the following corollary.

**Corollary 2.4.** Let  $-1 \leq B < A \leq 1$ . If  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in S_p^*(A, B)$ , then

$$|a_n| \leq \frac{(pA - pB)_n}{(1)_n} \quad n = 1, 2, 3, \dots,$$

when  $B = -1$ , the result is sharp for the function defined by  $f(z) = \frac{z^p}{(1 - z)^{p(1+A)}}$ .

Also by taking  $\lambda = 1, a = p$  and  $c = 1$  in the Theorem 2.4 we get

**Corollary 2.5.** Let  $-1 \leq B < A \leq 1$ . If  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in C_p(A, B)$ , then

$$|a_n| \leq \frac{(pA - pB)_n}{(1)_n (p + n)} \quad n = 1, 2, 3, \dots,$$

and the result is sharp for the function defined by  $f(z) = \frac{z^p}{(1-z)^{p(1+A)}}$  when  $B = -1$ .

**Corollary 2.6.** Let  $-1 \leq B < A \leq 1$ . If  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in \rho_p(A, B)$ , then

$$|a_n| \leq 1 \quad n = 1, 2, 3, \dots,$$

and the result is sharp for the function defined by  $f(z) = \frac{z^p}{(1-z)^{p(1+A)}}$  when  $B = -1$ .

**Theorem 2.5.** Let  $a > 0, c > 0, -1 \leq B < A \leq 1$  and  $\lambda > -p$ . If  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in S_{a,c}^{\lambda}(p, A, B)$ , then

$$(2.6) \quad \left| f^{(k)}(z) \right| \leq \frac{1}{r^k} \{ L_p(a, c) L_p(1, \lambda + p) r^k (z {}_2F_1(pA - pB, 1; 1; z))_{z=r}^{(k)} \},$$

$$(|z| = r < 1, k \leq p, p \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

When  $B = -1$ , the result is sharp for the function  $f(z)$  given by  $f(z) = L_p(a, c) L_p(1, \lambda + p) \frac{z^p}{(1-z)^{p(1+A)}}$ .

*Proof.* Let  $f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in S_{a,c}^{\lambda}(p, A, B)$ . Then, using Theorem 2.4, we obtain that

$$\begin{aligned} \left| z^k f^{(k)}(z) \right| &\leq p(p-1) \cdots (p-k+1) |z^p| + \sum_{n=1}^{\infty} (n+p) \cdots (n+p-k+1) |a_n| |z|^{n+p} \\ &\leq p(p-1) \cdots (p-k+1) r^p \\ &\quad + \sum_{n=1}^{\infty} (n+p) \cdots (n+p-k+1) \frac{(pA - pB)_n (a)_n}{(c)_n (\lambda + p)_n} r^{n+p} \\ &= \sum_{n=0}^{\infty} (n+p) \cdots (n+p-k+1) \frac{(pA - pB)_n (a)_n}{(c)_n (\lambda + p)_n} r^{n+p} \\ &= L_p(a, c) L_p(1, \lambda + p) r^k (z {}_2F_1(pA - pB, 1; 1; z))_{z=r}^{(k)}. \end{aligned}$$

Thus

$$\left| f^{(k)}(z) \right| \leq \frac{1}{r^k} L_p(a, c) L_p(1, \lambda + p) r^k (z {}_2F_1(pA - pB, 1; 1; z))_{z=r}^{(k)}.$$

By considering the function  $f(z) = L_p(a, c) L_p(1, \lambda + p) \frac{z^p}{(1-z)^{p(1+A)}}$ , one can show that this result is sharp when  $B = -1$ . □

**Corollary 2.7.** Let  $-1 \leq B < A \leq 1$ . If  $f(z) \in \rho_p(A, B)$ , then

$$\left| f^{(k)}(z) \right| \leq \left( \frac{z^p}{1-z} \right)_{z=r}^{(k)}, \quad k = 0, 1, 2, \dots, p,$$



where  $r = |z|$ . This result is sharp and the extreme function is  $f(z) = \frac{z^p}{1-z}$  when  $B = -1$ .

**Corollary 2.6.** Let  $-1 \leq B < A \leq 1$ . If  $f(z) \in S_p^*(A, B)$ , then

$$\left| f^{(k)}(z) \right| \leq \left( \frac{z^p}{(1-z)^{p(A-B)}} \right)_{z=r}^{(k)}, \quad k = 0, 1, 2, \dots, p,$$

where  $r = |z|$ . This result is sharp for the function given by  $f(z) = \frac{z^p}{(1-z)^{p(1+A)}}$  when  $B = -1$ .

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