KYUNGPOOK Math. J. 47(2007), 381-392

Vortex Filament Equation and Non-linear Schrӧdinger Equation in S^3

Hongning Zhang
Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, P.R. China
e-mail : hongning-zhang@163.com

Faen Wu
Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, P.R. China
e-mail : wufaen@yahoo.com

Abstract. In 1906, da Rios, a student of Leivi-Civita, wrote a master's thesis modeling the motion of a vortex in a viscous fluid by the motion of a curve propagating in R^3, in the direction of its binormal with a speed equal to its curvature. Much later, in 1971 Hasimoto showed the equivalence of this system with the non-linear Schrӧdinger equation (NLS)

$$q_t = i(q_{ss} + \frac{1}{2}|q|^2 q).$$

In this paper, we use the same idea as Terng used in her lecture notes but different technique to extend the above relation to the case of S^3, and obtained an analogous equation that

$$q_t = i[q_{ss} + (\frac{1}{2}|q|^2 + 1)q].$$

1. Introduction

The material of this section was taken from [2] with a minor modification.

1.1. A special orthogonal frame field on S^3

S^3 is the unit sphere in R^4 i.e.,

$$S^3 = \{x \in R^4 | |x| = 1\}. \tag{1.1}$$

For any $x, y \in S^3$, the distance $d(x, y)$ between x and y is defined by

$$\cos d(x, y) = x \cdot y, \tag{1.2}$$

Received May 22, 2006, and, in revised form, December 5, 2006.
2000 Mathematics Subject Classification: 53C45, 53C40.
Key words and phrases: parallel frame Vortex filament equation non-linear schrӧdinger equation Frenet frame.
This paper was supported by project No.10571088 of NSFC.

where $x \cdot y$ is the inner product of x and y. For any constant a,$a \in (0,1)$,there exists $A \in O(4)$ such that

$$d(x, Ax) = a \ \forall x \in S^3. \tag{1.3}$$

For example we may take

$$A = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} or A = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}, \tag{1.4}$$

and $a^2 + b^2 + c^2 + d^2 = 1$.

We can also regard S^3 as a set of all the unit quaternions and regard R^4 as a non-commutative division algebra. Its unit element is 1=(1,0,0,0), and its generators are $i = (0,1,0,0)$, $j = (0,0,1,0,)$, $k = (0,0,0,1)$, where i, j, k satisfy

$$\begin{cases} i \cdot j = k = -j \cdot i \\ j \cdot k = i = -k \cdot j \\ k \cdot i = j = -i \cdot k \\ i^2 = j^2 = k^2 = -1. \end{cases} \tag{1.5}$$

Define the module of a quaternion $x = x_1 1 + x_2 i + x_3 j + x_4 k \in R^4$ by

$$|x|^2 = \sum_{i=1}^{4} x_i^2, \tag{1.6}$$

and the product of two quaternions has the property:

$$|x \cdot y| = |x| \cdot |y|, \tag{1.7}$$

for any $x, y \in R^4$. So the set of all the unit quaternions i.e S^3 is a non-commutative Lie group. The two matrices in (1.4) just correspond to the left and right translation of $a1 + bi + cj + dk \in S^3$. That is to say, for $g = a1 + bi + cj + dk \in S^3$ we have

$$Lg, Rg : S^3 \to S^3$$

$$Lg(x) = g \cdot x; \quad Rg(x) = x \cdot g; \quad for \ x \in S^3. \tag{1.8}$$

The mapping

$$f : S^3 \to o(4)$$

$$a1 + bi + cj + dk \mapsto \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}, \tag{1.9}$$

gives an isomorphism from S^3 to a subgroup of O(4) corresponding to the left translation which is determined by the element in S^3. It will be convenient to regard S^3 as this subgroup for computation.

In the following we'll find the tangent space of S^3 at the unit element. It is spaned by $x_1 = (0,1,0,0), x_2 = (0,0,1,0), x_3 = (0,0,0,1)$. Notice that x_1 is the tangent vector of the curve $c(t) = (\cos t, \sin t, 0, 0) \in S^3$ at 1=(1,0,0,0). Since

$$c(t) = \cos t \cdot 1 + \sin t \cdot i = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}, \tag{1.10}$$

so we can regard x_1 as

$$\frac{d}{dt}c(t)|_{t=0} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in O(4), \tag{1.11}$$

similarly regard x_2 , x_3 as

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.12}$$

respectively. It's easy to verify that

$$[x_1, x_2] = 2x_3, [x_2, x_3] = 2x_1, [x_3, x_1] = 2x_2. \tag{1.13}$$

If we ignore the first commponent of $x_i, i = 1, 2, 3$, and regard it as the vector of R^3 then

$$\begin{cases} x_1 \times x_2 = x_3 \\ x_2 \times x_3 = x_1 \\ x_3 \times x_1 = x_2 \end{cases}, \tag{1.14}$$

or use the usual inner product and orientation in S^3, we can also define the above relation.

Let $\tilde{x}_i$ be the vector field which is obtained by the left translation of x_i similarly we have

$$\begin{cases} \tilde{x}_1 \times \tilde{x}_2 = \tilde{x}_3 \\ \tilde{x}_2 \times \tilde{x}_3 = \tilde{x}_1 \\ \tilde{x}_3 \times \tilde{x}_1 = \tilde{x}_2 \end{cases}. \tag{1.15}$$

The cross product "$\times$" in the tangent space at each piont in S^3 is defined by ordinary inner product and orientation.

1.2. The Frenet frame of curves on S^3

In this section we want to build the Frenet frame of curves in S^3. The theory of curves in S^3 has a special treatment. In other words we can use left invariant vector field $\tilde{x}_i$ to express all the tangent vector fields on S^3.

Let $c : [0, l] \to S^3$ be a curve and parametrized by its arc length. Its tangent vector is

$$\frac{d}{ds}c(s) = t(s), \tag{1.16}$$

as $c(s) \in S^3$, then

$$c(s) \cdot c(s) = 1. \tag{1.17}$$

Differentiating both sides of (1.17), we get

$$\frac{d}{ds}c(s) \cdot c(s) = 0. \tag{1.18}$$

So $t(s) = \frac{d}{ds}c(s)$ is the tangent vector field on S^3 along $c(s)$, it can be expressed as

$$t(s) = \sum_{i=1}^{3} f_i(s)\tilde{x}_i(c(s)), \tag{1.19}$$

where $f_i(s)$ are some smooth functions on $c(s)$. As $c(s)$ is parametrized by its arc length, so

$$\sum_{i=1}^{3} f_i^2(s) = 1. \tag{1.20}$$

Differentiating both sides of (1.20), we get

$$\sum_{i=1}^{3} f_i(s)f_i'(s) = 0. \tag{1.21}$$

Let ∇' denotes covariant differentiation on S^3. Any vector fields along $c(s)$ can be expressed as

$$\sum_{i=1}^{3} h_i(s)\tilde{x}_i(c(s)). \tag{1.22}$$

Then

$$\begin{aligned}(1.23)\qquad & \frac{\nabla'}{ds}\{\sum_{i=1}^{3} h_i(s)\tilde{x}_i(c(s))\} \\ = & \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \sum_{i=1}^{3} h_i(s)\frac{\nabla'}{ds}\tilde{x}_i(c(s)) \\ = & \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \sum_{i=1}^{3} h_i(s)\sum_{j=1}^{3} f_j(s)\nabla'^{\tilde{x}_i(c(s))}_{\tilde{x}_j} \\ = & \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} h_i(s)f_j(s)[\tilde{x}_j, \tilde{x}_i](c(s)) \\ = & \sum_{i=1}^{3} h_i'(s)\tilde{x}_i(c(s)) + \det\begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ f_1 & f_2 & f_3 \\ h_1 & h_2 & h_3 \end{pmatrix}.\end{aligned}$$

In particular

$$(1.24)\qquad \frac{\nabla'}{ds}t(s) = \sum_{i=1}^{3} f_i'(s)\tilde{x}_i(c(s)).$$

Define curvature function of curve $c(s)$by

$$(1.25)\qquad k = |\frac{\nabla'}{ds}t(s)| = (\sum_{i=1}^{3} f_i'^2(s))^{\frac{1}{2}}.$$

Assume that $k \neq 0$ then the normal vector field along $c(s)$ is difined by

$$(1.26)\qquad n = \frac{1}{k}\frac{\nabla'}{ds}t(s) = \frac{1}{k}\sum_{i=1}^{3} f_i'(s)\tilde{x}_i(c(s)).$$

Then n is a unit vector of the tangent space of S^3 at $c(s)$, and n is perpendicular to t.

Binormal vector field along $c(s)$ is given by:

$$(1.27)\qquad b = t \times n = \frac{1}{k}\det\begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \end{pmatrix} \equiv \frac{1}{k}\sum_{i=1}^{3} g_i(s)\tilde{x}_i(c(s)).$$

So b is still a unit vector of the tangent space of S^3 at $c(s)$, and b is perpendicular to both t and n. By (1.26), (1.24) can be written as

$$(1.28)\qquad \frac{\nabla'}{ds}t(s) = kn.$$

Let

$$\tau = \frac{\nabla'}{ds} n(s) \cdot b(s), \tag{1.29}$$

the function τ is called the torsion of the curve $c(s)$. By direct computation we get

$$\frac{\nabla'}{ds} n(s) = -kt(s) + \tau b(s), \tag{1.30}$$

$$\frac{\nabla'}{ds} b(s) = -\tau n(s). \tag{1.31}$$

So along the curve $c(s)$ there is an orthognal frame field $\{c(s); t(s), n(s), b(s)\}$ which is called Frenet frame of curves on S^3. (1.28), (1.30), (1.31) are called Frenet formula. We rewrite it in the matrix form

$$\frac{\nabla'}{ds} \begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \tag{1.32}$$

1.3. The parallel frame of curves on S^3

We want to change the Frenet frame $(t, n, b)^T$ to $(e_1, e_2, e_3)^T$ so that the 2,3-th entry of the coefficient matrix of $\frac{\nabla'}{ds}(e_1, e_2, e_3)^T$ is zero. To do this, we follow the method as described in [1]. Rotate the Frenet frame (n, b) by an angle $\beta(s)$ satisfy that

$$\beta'(s) = -\tau(s), \tag{1.33}$$

then

$$\frac{\nabla'}{ds} e_2 \cdot e_3 = 0. \tag{1.34}$$

So that we get the new o.n frame $(e_1, e_2, e_3)^T$, and it satisfies

$$\frac{\nabla'}{ds} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{1.35}$$

where

$$\begin{cases} k_1 = k \cos\beta(s) \\ k_2 = -k \sin\beta(s) \end{cases}. \tag{1.36}$$

The o.n frame $(e_1, e_2, e_3)^T$ is called parallel frame along $c(s)$. The function $k_1(s), k_2(s)$ are called the principal curvatures along e_2, e_3 respectively. However, the choice of parallel frame is not unique, because we can replace $\beta(s)$ by $\beta(s)$ plus a constant β_0, get another o.n frame $(e_1, v_2, v_3)^T$. It is again parallel, but the principal curvature $\tilde{k}_1, \tilde{k}_2$ along v_2, v_3 are satisfies

$$\frac{\nabla'}{ds}e_1 = k_1e_2 + k_2e_3 = \tilde{k}_1v_2 + \tilde{k}_2v_3. \tag{1.37}$$

2. Vortex filament equation and the NLS in S^3

In 1906, da.Rios, a graduate student of Levi-Civita, wrote a master degree thesis, in which he modeled the movement of a thin vortex in a vicious fluid by the motion of a curve propagating in R^3 in the direction of its binormal with a speed equal to its curvature according to

$$\gamma_t = \gamma_s \times \gamma_{ss}. \tag{2.1}$$

This is called the vortex filament equation or smoke ring equation, and it can be regarded as a dynamical system on the space of curves in R^3. Much later, in 1971, Hasimoto showed the equivalence of this system with the NLS

$$q_t = i(q_{ss} + \frac{1}{2}|q|^2q). \tag{2.2}$$

In this section we'll build the vortex filament equation in S^3 similar to that as in R^3, and study the relationship between vortex filament equation and the NLS in S^3. For any $\gamma(s,t)$ belongs to S^3, $\gamma(s,t)$ is a surface in S^3 so

$$\gamma(s,t) \cdot \gamma(s,t) = 1. \tag{2.3}$$

Differentiating both sides of (2.3) with respect to s and t respectively

$$\begin{cases} \gamma_s(s,t) \cdot \gamma(s,t) = 0 \\ \gamma_t(s,t) \cdot \gamma(s,t) = 0. \end{cases} \tag{2.4}$$

So both $\gamma_s(s,t)$ and $\gamma_t(s,t)$ are the vectors of the tangent space of S^3 at $\gamma(s,t)$.

We can use left invariant vector fields $\tilde{x}_i$ to express all the tangent vector fields on S^3, let

$$\gamma_s(s,t) = \sum_{i=1}^{3} f_i(s,t)\tilde{x}_i(s,t), \tag{2.5}$$

then similar to (1.24),

$$\frac{\nabla'}{ds}\gamma_s(s,t) = \sum_{i=1}^{3} \frac{d}{ds}f_i(s,t)\tilde{x}_i(\gamma(s,t)). \tag{2.6}$$

So $\gamma_s(s,t)\times\frac{\nabla'}{ds}\gamma_s(s,t)$ is still a vector of tangent space of S^3 at $\gamma(s,t)$. If $\gamma(s,t)$satisfy

$$\gamma_t(s,t)=\gamma_s(s,t)\times\frac{\nabla'}{ds}\gamma_s(s,t), \tag{2.7}$$

we call $\gamma(s,t)$ is the vortex filament surface in S^3 . (2.7) is called the vortex filament equation on S^3.

Equation (2.7) has the property:

Proposition. *If $\gamma_s(s,t)$ is a solution of* (2.7) *and $|\gamma_s(s,0)|=1$ for all s, then $|\gamma_s(s,t)|=1$ for all (s,t). In other words if $\gamma(\cdot,0)$ is parametrized by arc length then so is $\gamma(\cdot,t)$ for all t.*

Proof. It suffices to prove that

$$\frac{d}{dt}\langle\gamma_s(s,t),\gamma_s(s,t)\rangle=0 \tag{2.8}$$

Remark: $\langle,\rangle$ is the inner product in S^3 . To see (2.8) , we compute directly to get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle\gamma_s(s,t),\gamma_s(s,t)\rangle=\langle\frac{\nabla'}{dt}\gamma_s,\gamma_s\rangle \qquad (2.9)\\
=\;&\langle\frac{d}{dt}\gamma_s-a\gamma,\gamma_s\rangle\;=\langle\frac{d}{dt}\gamma_s,\gamma_s\rangle\\
=\;&\langle\frac{d}{ds}\gamma_t,\gamma_s\rangle=\langle\frac{d}{ds}(\gamma_s\times\frac{\nabla'}{ds}\gamma_s),\gamma_s\rangle\\
=\;&\langle\frac{d}{ds}\gamma_s\times\frac{\nabla'}{ds}\gamma_s+\gamma_s\times\frac{d}{ds}(\frac{\nabla'}{ds}\gamma_s),\gamma_s\rangle=\langle\frac{d}{ds}\gamma_s\times\frac{\nabla'}{ds}\gamma_s,\gamma_s\rangle\\
=\;&(\gamma_s,\frac{d}{ds}\gamma_s,\frac{\nabla'}{ds}\gamma_s)=(\frac{d}{ds}\gamma_s,\frac{\nabla'}{ds}\gamma_s,\gamma_s)\\
=\;&(\frac{\nabla'}{ds}\gamma_s+b\gamma,\frac{\nabla'}{ds}\gamma_s,\gamma_s)=(\frac{\nabla'}{ds}\gamma_s,\frac{\nabla'}{ds}\gamma_s,\gamma_s)\\
=\;&0.
\end{aligned}$$

Both a and b are some functions on $\gamma(s,t)$. So for a solution $\gamma(s,t)$ of (2.7), we may assume that $\gamma(\cdot,t)$ is parametrized by arc length for all t.

Next we will explain the geometric meaning of the evolution equation on the space of curves in S^3. Let $(t,n,b)(\cdot,t)$ denote the Frenet frame of the curve $\gamma(\cdot,t)$. Since $\gamma_s=t$, and $\frac{\nabla'}{ds}\gamma_s=\frac{\nabla'}{ds}t=kn$ then the curve flow (2.7) becomes:

$$\gamma_t=kt\times n=kb. \tag{2.10}$$

In other words, the curve flow (2.7) moves in the direction of binormal with curvature as its speed in S^3. In the following, we write equation (2.7) in terms of parallel frame $(e_1,e_2,e_3)^T$. Recall that if we rotate the Frenet frame (n,b) by an angle $\beta(s,t)$ satisfy

$$\frac{d}{ds}\beta(s,t)=-\tau(s,t), \tag{2.11}$$

then

$$b = \sin\beta e_2 + \cos\beta e_3. \tag{2.12}$$

Hence

$$\gamma_t = kb = k(\sin\beta e_2 + \cos\beta e_3) = -k_2 e_2 + k_1 e_3. \tag{2.13}$$

In fact vortex filament equation (2.7) and NLS

$$q_t = i[q_{ss} + (\frac{1}{2}|q|^2 + 1)q]$$

shown the same motion equation. We will give the demonstration below.

Suppose $\gamma(s,t)$ is a solution of (2.7), choose a parallel frame $(e_1, e_2, e_3)^T(\cdot, t)$ for each curve $\gamma(\cdot, t)$. Let $k_1(\cdot, t)$ and $k_2(\cdot, t)$ denote the principal curvature of $\gamma(\cdot, t)$ along $e_2(\cdot, t), e_3(\cdot, t)$ respectively. Then we get

$$\frac{\nabla'}{ds}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.14}$$

We want to compute $\frac{\nabla'}{dt}(e_1, e_2, e_3)^T$, so we first compute $\frac{\nabla'}{dt}e_1(\cdot, t)$,

$$\begin{aligned} \frac{\nabla'}{dt}e_1 = \frac{\nabla'}{dt}t &= \frac{\nabla'}{dt}\gamma_s = \frac{\nabla'}{ds}\gamma_t \\ &= \frac{\nabla'}{ds}(-k_2 e_2 + k_1 e_3) \\ &= -(k_2)_s e_2 + (-k_2)\frac{\nabla'}{ds}e_2 + (k_1)_s e_3 + k_1\frac{\nabla'}{ds}e_3 \\ &= -(k_2)_s e_2 + (k_1)_s e_3 + (-k_2)(-k_1 e_1) + k_1(-k_2 e_1) \\ &= -(k_2)_s e_2 + (k_1)_s e_3. \end{aligned} \tag{2.15}$$

Since $e_i(s,t)$ are orthogonal, there exists a function $u(s,t)$ so that

$$\frac{\nabla'}{dt}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & -(k_2)_s & (k_1)_s \\ (k_2)_s & 0 & u \\ -(k_1)_s & -u & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.16}$$

Now, how to compute the function $u(s,t)$ is the key step. If we can find $u(s,t)$, then we can get the coefficient matrix of $\frac{\nabla'}{dt}(e_1, e_2, e_3)^T$. Before computing $u(s,t)$ we give some preparative knowledge.

(1): Since

$$\begin{aligned} \frac{\nabla'}{ds}\frac{\nabla'}{dt} - \frac{\nabla'}{dt}\frac{\nabla'}{ds} &= \nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}} - \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}} - \nabla_{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]} + \nabla_{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]} \\ &= R(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) + \nabla_{[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}]}, \end{aligned} \tag{2.17}$$

where $\nabla_{[\frac{\partial}{\partial s},\frac{\partial}{\partial t}]} = 0$, so

$$\frac{\nabla'}{dt}\frac{\nabla'}{ds} = \frac{\nabla'}{ds}\frac{\nabla'}{dt} - R(\frac{\partial}{\partial s},\frac{\partial}{\partial t}). \tag{2.18}$$

$R(\frac{\partial}{\partial s},\frac{\partial}{\partial t})$ is the curvature operator of S^3.

(2): Let M be a Rieman manifold of constant curvature K, for any X,Y, Z ,W $\in T_pM$ we have

$$R(X,Y,Z,W) = \langle R(X,Y)Z,W\rangle = -K(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle). \tag{2.19}$$

This can refer to [3]. Since S^3 is a space of constant curvature, and the sectional curvature K=1. So

$$\begin{aligned} &\langle R(\frac{\partial}{\partial s},\frac{\partial}{\partial t})e_2,e_3\rangle \\ =\ & \langle R(e_1,-k_2e_2+k_1e_3)e_2,e_3\rangle \\ =\ & (-1)(\langle e_1,e_2\rangle\langle -k_2e_2+k_1e_3,e_3\rangle - \langle e_1,e_3\rangle\langle -k_2e_2+k_1e_3,e_2\rangle) \\ =\ & 0 \end{aligned} \tag{2.20}$$

similarly

$$\langle R(\frac{\partial}{\partial s},\frac{\partial}{\partial t})e_1,e_2\rangle = k_2, \tag{2.21}$$

$$\langle R(\frac{\partial}{\partial s},\frac{\partial}{\partial t})e_1,e_3\rangle = -k_1 \tag{2.22}$$

by (2.18), (2.20), (2.21), (2.22) we get

$$\langle\frac{\nabla'}{dt}\frac{\nabla'}{ds}e_2,e_3\rangle = \langle\frac{\nabla'}{ds}\frac{\nabla'}{dt}e_2,e_3\rangle, \tag{2.23}$$

$$\langle\frac{\nabla'}{dt}\frac{\nabla'}{ds}e_1,e_2\rangle = \langle\frac{\nabla'}{ds}\frac{\nabla'}{dt}e_1,e_2\rangle - k_2, \tag{2.24}$$

$$\langle\frac{\nabla'}{dt}\frac{\nabla'}{ds}e_1,e_3\rangle = \langle\frac{\nabla'}{ds}\frac{\nabla'}{dt}e_1,e_3\rangle + k_1. \tag{2.25}$$

In the following we begin to compute $u(s,t)$,

$$\begin{aligned} \langle\frac{\nabla'}{dt}(\frac{\nabla'}{ds}e_2),e_3\rangle &= \langle\frac{\nabla'}{dt}(-k_1e_1),e_3\rangle \\ &= \langle -(k_1)_te_1 - k_1\frac{\nabla'}{dt}e_1,e_3\rangle \\ &= \langle -k_1(-(k_2)_se_2+(k_1)_se_3),e_3\rangle \\ &= -k_1(k_1)_s \end{aligned} \tag{2.26}$$

similarly

$$\langle \frac{\nabla'}{ds}(\frac{\nabla'}{dt}e_2), e_3\rangle = (k_2)_s k_2 + u_s, \tag{2.27}$$

by (2.23)

$$-k_1(k_1)_s = (k_2)_s k_2 + u_s. \tag{2.28}$$

So

$$u = -\frac{1}{2}(k_1^2 + k_2^2) + c(t), \tag{2.29}$$

for some smooth function $c(t)$. Remember that for each fixed t, we can rotate $(e_2, e_3)(\cdot, t)$by a constant angle $\theta(t)$ to another parallel normal frame $(v_2, v_3)(\cdot, t)$of $\gamma(\cdot, t)$. If we choose $\theta(t)$ so that $\theta'(t) = -c(t)$, then the new parallel frame satisfies

$$\frac{\nabla'}{dt}\begin{pmatrix} e_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & -(\tilde{k}_2)_s & (\tilde{k}_1)_s \\ (\tilde{k}_2)_s & 0 & -\frac{\tilde{k}_1^2+\tilde{k}_2^2}{2} \\ -(\tilde{k}_1)_s & \frac{\tilde{k}_1^2+\tilde{k}_2^2}{2} & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{2.30}$$

So we have proved the first part of the following theorem:

Theorem. *Suppose $\gamma(s,t)$ is a solution of the vortex filament equation (2.7) and $|\gamma(s,0)| = 1$ for all s. Then*

(1) *there exists a parallel normal frame $(e_1, e_2, e_3)^T(\cdot, t)$ for each curve $\gamma(\cdot, t)$ so that*

$$\frac{\nabla'}{ds}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{2.31}$$

$$\frac{\nabla'}{dt}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & -(k_2)_s & (k_1)_s \\ (k_2)_s & 0 & -\frac{k_1^2+k_2^2}{2} \\ -(k_1)_s & \frac{k_1^2+k_2^2}{2} & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{2.32}$$

where $k_1(\cdot, t)$ and $k_2(\cdot, t)$ are the principal curvatures of $\gamma(\cdot, t)$ along $e_2(\cdot, t)$and $e_3(\cdot, t)$ respectively.

(2) *$q = k_1 + ik_2$ is a solution of the NLS*

$$q_t = i[q_{ss} + (\frac{1}{2}|q|^2 + 1)q].$$

Proof. We have proved (1). For (2), we use (2.31),(2.32) to compute the evolution of $(k_1)_t$ and $(k_2)_t$.

$$
\begin{aligned}
(2.33)\quad (k_1)_t &= \frac{\nabla'}{dt}k_1 = \frac{\nabla'}{dt}\langle\frac{\nabla'}{ds}e_1, e_2\rangle \\
&= \langle\frac{\nabla'}{dt}\frac{\nabla'}{ds}e_1, e_2\rangle + \langle\frac{\nabla'}{ds}e_1, \frac{\nabla'}{dt}e_2\rangle \\
&= \langle\frac{\nabla'}{ds}\frac{\nabla'}{dt}e_1, e_2\rangle - k_2 + \langle k_1e_2 + k_2e_3, (k_2)_se_1 - \frac{k_1^2+k_2^2}{2}e_3\rangle \\
&= \langle\frac{\nabla'}{ds}(-(k_2)_se_2 + (k_1)_se_3), e_2\rangle - k_2 - k_2\frac{k_1^2+k_2^2}{2} \\
&= -(k_2)_{ss} - k_2 - k_2\frac{k_1^2+k_2^2}{2}
\end{aligned}
$$

similarly

$$
(2.34)\qquad (k_2)_t = (k_1)_{ss} + k_1 + k_1\frac{k_1^2+k_2^2}{2}.
$$

Therefore

$$
\begin{aligned}
(2.35)\quad q_t &= (k_1)_t + i(k_2)_t \\
&= -(k_2)_{ss} - k_2 - k_2\frac{k_1^2+k_2^2}{2} + i(k_1)_{ss} + ik_1 + ik_1\frac{k_1^2+k_2^2}{2} \\
&= i[q_{ss} + (\frac{1}{2}|q|^2 + 1)q].
\end{aligned}
$$

□

References

[1] Chuu-Lian Terng, Lecture 1 Peking University summer school, 2005.

[2] M. do Carmo, Riemannian Geometry, Birkhauser, Boston, 1992.

[3] M. Spivak, A Comprehensive Introduction to Differential Geometry, Vol. IV, Publish or Perish, 1979.

[4] W. Chen and S. S Chern, Lecture Notes On Differential Geometry Peking University press, 2004.

[5] H. Hasimoto, *A soliton on a vortex filament*, J. Fluid Mech., **51**(1972), 477-485.

[6] Da Rios, L. S., *On the motion of an unbounded fluid flow with an isolated vortex filament*, (in Italian), Rend. Circ. Mat. Palermo, **22**(1906), 117.