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Some Theorems on Generating Functions

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Abstract. In this paper, we derive some generating relations involving Konhauser polynomials, Gauss, Humbert, Appell and Kampé de Fériet hypergeometric functions with the help of four general theorems on generating functions (partly unilateral and partly bilateral) of one and two variables.

1. Introduction

Let $f(z_1, \cdots, z_r)$ be a function of r independent complex variables defined in some domain $\mathbb{C}^r$ as the sum of a confluent multiple series

$$f(z_1, \cdots, z_r) = \sum_{k_1, \cdots, k_r = 0}^{\infty} A(k_1, \cdots, k_r)\, z_1^{k_1} \cdots z_r^{k_r} \tag{1.1}$$

and let

$$\Delta_n(m_1, \cdots, m_r; z_1, \cdots, z_r) = \sum_{k_1, \cdots, k_r = 0}^{M \leq n} (-n)_M\, A(k_1, \cdots, k_r)\, z_1^{k_1} \cdots z_r^{k_r}, \tag{1.2}$$

where $\{A(k_1, \cdots ., k_r) \mid k_j \in \mathbb{N},\ j = 1, \cdots, r\}$ is a bounded multiple complex sequence and M is defined by $M = m_1 k_1 + \cdots + m_r k_r$, $m_1, \cdots, m_r$ representing positive integers.

Konhauser [5] defined the polynomial $Z_n^{\alpha}(x; k)$ by

$$Z_n^{\alpha}(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}, \tag{1.3}$$

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where k is a positive integer.

Agarwal and Manocha [1] have obtained the following theorem for Kounhasuer polynomials.

Let a and b are complex constants, not both zero. Then

$$\sum_{n=0}^{\infty} Z_n^{\alpha}\left(\frac{x}{(a+bn)^{1/k}};k\right)\frac{[(a+bn)t]^n}{(1+\alpha)_{kn}} = \frac{e^{a\nu}}{1-b\nu}\,{}_0F_k\left[\begin{matrix}\text{———} & ; \\ & & -\nu\left(\frac{x}{k}\right)^k \\ \Delta(k;1+\alpha) & ; \end{matrix}\right], \tag{1.4}$$

where $\nu = te^{b\nu}$, $\Delta(k;\alpha)$ denote the sequence of n parameters α/k, $(\alpha+1)/k, \cdots,$ $(\alpha+k-1)/k$, $k \geq 1$ and ${}_0F_k$ is hypergeometric function [11, p.42(1)].

From Lagrange expansion formula [8], we have

$$\frac{e^{a\nu}}{1-b\nu} = \sum_{n=0}^{\infty}\frac{(a+bn)^n}{n!}t^n, \tag{1.5}$$

where ν is a function of t defined implicitly by $\nu = te^{b\nu}$, $\nu(0) = 0$.

An interesting (partly bilateral and partly unilateral) generating function for Laguerre polynomials $L_n^{\alpha}(x)$ [9] due to Exton [3, p.147(3)], is recalled here in the following (modified) form (see [6], [7])

$$\exp(s+t-xt/s) = \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty}\frac{s^m\,t^n}{(m+n)!}L_n^{(m)}(x) \tag{1.6}$$

or equivalently,

$$\exp(s+t-xt/s) = \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty}\frac{s^m\,t^n}{m!\,n!}\,{}_1F_1(-n;\,m+1;\,x), \tag{1.7}$$

where

$$m^\star := \max\{0,-m\} \quad (m\in\mathbb{Z} := \{0,\pm1,\pm2,\cdots\}) \tag{1.8}$$

In subsequent sections of this paper, we shall encounter the following polynomial $\theta_{n,k}^{a,b}(x,y)$ involving Konhauser polynomial and its special cases

$$\theta_{n,k}^{a,b}(x,y) = \sum_{p=0}^{n}\frac{(a+b(n-p))^{n-p}\,(-x)^p}{(p+m)!\;p!\;(1+\alpha)_{k(n-p)}}\,Z_{n-p}^{\alpha}\left(\frac{y}{(a+b(n-p))^{1/k}};\,k\right) \tag{1.9}$$

$$\theta_{n,1}^{a,b}(x,y) = \sum_{p=0}^{n} \frac{(a+b(n-p))^{n-p}\,(-x)^p}{(p+m)!\;p!\;(1+\alpha)_{n-p}}\,L_{n-p}^{\alpha}\left(\frac{y}{a+b(n-p)}\right) \tag{1.10}$$

$$\theta_{n,1}^{a,b}(x,0) = \sum_{p=0}^{n} \frac{(a+b(n-p))^{n-p}\,(-x)^p}{(p+m)!\;p!\;(n-p)!} \tag{1.11}$$

$$\theta_{n,1}^{1,0}(x,0) = \frac{1}{n!\;m!}\;{}_1F_1[-n;\,1+m;\,x] = \frac{1}{\Gamma(1+m+n)}\;L_n^m(x), \tag{1.12}$$

where $L_n^{\alpha}(x)$ is Laguerre polynomial [9] and m and k are positive integers.

The article is organized as follows. In the main Section 2, we derive two theorems on partly bilateral and partly unilateral generating functions of general nature. Two more theorems on multiple generating functions involving Appell and Kampé de Fériet series are proved in Section 3. Later in Section 4, it is shown as to how these theorems lead to a number of generating functions for certain classical polynomials.

2. Generating functions involving bilateral series

Theorem 1. *Let the function* $f(z_1,\cdots,z_r)$ *be defined by* (1.1) *and let* $\Delta_n(m_1,\cdots,m_r;\,z_1,\cdots,z_r)$ *be defined by* (1.2). *Also let* $m^\star$ *be defined by* (1.8). *Then*

$$\begin{aligned} &\frac{e^{s-xt/s+a\nu}}{1-b\nu}\,f[(-\nu)^{m_1}z_1,\cdots,(-\nu)^{m_r}z_r] \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty} s^m\,t^n \sum_{p=0}^{n} \frac{(-x)^p(a+b(n-p))^{n-p}}{(p+m)!\,p!\,(n-p)!} \\ &\quad\times\;\Delta_{n-p}\left[m_1,\cdots,m_r;\,\frac{z_1}{(a+b(n-p))^{m_1}},\cdots,\frac{z_r}{(a+b(n-p))^{m_r}}\right], \end{aligned} \tag{2.1}$$

where $\nu = te^{b\nu}$, *a and b are complex constants, not both zero and provided that each member of* (2.1) *exists.*

Proof. Denote for convenience, the first member of the assertion (2.1) by Ω. Then using the following expansion formula due to Agarwal and Manocha [2, p.276(1.4)]

$$\begin{aligned} &\frac{e^{a\nu}}{1-b\nu}\,f[(-\nu)^{m_1}z_1,\cdots,(-\nu)^{m_r}z_r] \\ &= \sum_{n=0}^{\infty}\frac{(a+bn)^n\;t^n}{n!}\times\;\Delta_n\left[m_1,\cdots,m_r;\,\frac{z_1}{(a+bn)^{m_1}},\cdots,\frac{z_r}{(a+bn)^{m_r}}\right], \end{aligned} \tag{2.2}$$

where a and b are arbitrary complex constants, not simultaneously equal to zero and $\nu = te^{b\nu}$ and expanding the exponential function, we obtain

$$\Omega = \sum_{m=0}^{\infty} \frac{s^m}{m!} \sum_{p=0}^{\infty} \frac{(-x)^p\, t^p}{k!\, s^p} \sum_{n=0}^{\infty} \frac{(a+bn)^n\, t^n}{n!} \times \Delta_n \left[m_1, \cdots, m_r; \frac{z_1}{(a+bn)^{m_1}}, \cdots, \frac{z_r}{(a+bn)^{m_r}} \right] \tag{2.3}$$

□

Upon replacing the summation indices m and n in (2.3) by $(m+p)$ and $(n-p)$ respectively, if we rearrange the resulting triple series (which can be justified by absolute convergence of the series involved), we are led finally to the generating function (2.1).

Remark. The above theorem provides us a class of generating relations for the functions Δ_n. A large variety of special cases including [2, p.276(1.4)] (when $x = 0$) may be deduced from it by assigning particular values to variables and parameters.

Theorem 2. *Let a and b be complex constants, not both zero and let $\nu = te^{b\nu}$. Also let $m^\star$ be defined by* (1.8). *Then*

$$\frac{e^{a\nu+s-xt/s}}{1-b\nu}\, {}_0F_k \left[\_;\, \Delta(k; 1+\alpha);\, -\nu \left(\frac{y}{k}\right)^k \right] = \sum_{m=-\infty}^{\infty} \sum_{n=m^\star}^{\infty} s^m\, t^n\, \theta_{n,k}^{a,b}(x,y), \tag{2.4}$$

where $\theta_{n,k}^{a,b}(x,y)$ is given by (1.9).

The derivation of (2.4) runs parallel to that of (2.1) except that we use (1.4) in place of (2.2) and we skip the details.

3. Multiple generating functions

Theorem 3. *The Appell's hypergeometric polynomials* [11]

$$f_{m,n}(x,y) = F_2 \left[\alpha,\, -n,\, -m;\, \beta, \gamma;\, \frac{x}{a_1+b_1 n}, \frac{y}{a_2+b_2 m} \right] \tag{3.1}$$

are generated by

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a_1+b_1 n)^n\, (a_2+b_2 m)^m\, f_{m,n}(x,y)\, \frac{t^n\, T^m}{n!\; m!} = \frac{e^{a_1\nu+a_2 w}}{(1-b_1\nu)\,(1-b_2 w)}\; \psi_2[\alpha;\, \beta, \gamma;\, -x\nu, -yw], \tag{3.2}$$

where $\nu = te^{b_1\nu}$, $w = Te^{b_2 w}$ and ψ_2 denotes Humbert's confluent hypergeometric function of two variables [11].

Proof. Starting with the left hand side of (3.2), using series expansion of Appell's function F_2 (see [11, p.53(5)]) and

$$(-m)_k = \frac{(-1)^k\ m!}{(m-k)!}\ , \qquad 0 \le k \le m \tag{3.3}$$

we have

$$\sum_{n,m=0}^{\infty} (a_1{+}b_1 n)^n\,(a_2{+}b_2 m)^m\,\frac{t^n\,T^m}{n!\ m!}\,\sum_{k=0}^{n}\frac{(-1)^k\ k!}{(n-k)!}\,\frac{x^k}{(a_1+b_1 n)^k}\,\sum_{r=0}^{m}\frac{(-1)^r\ r!}{(m-r)!}\,\frac{(\alpha)_{k+r}}{(\beta)_k\,(\gamma)_r}\,\frac{y^r}{(a_2+b_2 m)^r}$$

Now replacing n by $n+k$ and m by $m+r$ and using (1.5), we get

$$\begin{aligned}
&\frac{e^{a_1\nu+a_2 w}}{(1-b_1\nu)(1-b_2 w)}\sum_{k,r=0}^{\infty}\frac{(\alpha)_{k+r}}{(\beta)_k\,(\gamma)_r}\,\frac{(-xte^{b_1\nu})^k}{k!}\,\frac{(-yTe^{b_2 w})^r}{r!}\\
=\;&\frac{e^{a_1\nu+a_2 w}}{(1-b_1\nu)\,(1-b_2 w)}\,\psi_2[\alpha;\ \beta,\gamma;\ -x\nu,-yw]
\end{aligned}$$

by [11, p.59(42)]. □

Following the method of proof of the formula (3.2), we can readily obtain the following theorem involving Kampé de Fériet series of two variables $F^{p:l;m}_{q:r;s}$ (see [11, p.63(16)]).

Theorem 4. *The Kampé de Fériet's hypergeometric polynomials*

$$f_{m,n}(x,y) \;=\; \mathrm{F}^{p:1;1}_{q:0;1}\left[\begin{array}{ll}(\alpha_p):-n;-m\ \ ; & \\ & \dfrac{x}{a_1+b_1 n},\ \dfrac{y}{a_2+b_2 m}\\ (\beta_q):\_\_\_;\_\_\_\ \ ; & \end{array}\right] \tag{3.4}$$

are generated by

$$\begin{aligned}
&\sum_{n,m=0}^{\infty}(a_1+b_1 n)^n\,(a_2+b_2 m)^m\,f_{m,n}(x,y)\,\frac{t^n\,T^m}{n!\,m!} \qquad (3.5)\\
=\;&\frac{e^{a_1\nu+a_2 w}}{(1-b_1\nu)\,(1-b_2 w)}\;{}_pF_q\left[\begin{array}{ll}(\alpha_p)\ \ ; & \\ & -(x\nu+yw)\\ (\beta_q)\ \ ; & \end{array}\right],
\end{aligned}$$

where $\nu = te^{b_1\nu}$, $w = Te^{b_2 w}$, (α_p) *denotes* $\alpha_1,\cdots,\alpha_p$ *and* ${}_pF_q$ *denotes generalized hypergeometric function* [11, *p.* 42(1)].

4. Applications

First of all in its special cases when $z_1,\cdots,z_r = 0$, (2.1) reduces to

$$\frac{e^{s-xt/s+a\nu}}{1-b\nu} \;=\; \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty} s^m\,t^n\,\theta^{a,b}_{n,1}(x,0), \tag{4.1}$$

where $\theta_{n,1}^{a,b}(x,0)$ is given by (1.11). Secondly upon setting $a = 1$ and $b = 0$ and using (1.12), (4.1) would obviously correspond to the generating function of Exton given by (1.6) (or (1.7)).

For $r = 1$ and $f(-\nu z_1) = {}_0F_1[-;\ \alpha+1;\ -\nu y]$, (2.1) reduces to

$$(4.2)\qquad \frac{e^{s-xt/s+a\nu}}{1-b\nu}\,{}_0F_1[\_;\ \alpha+1;\ -\nu y] \;=\; \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty}\frac{s^m\,t^n}{m!\,n!}\,\theta_{n,1}^{a,b}(x,y),$$

where $\theta_{n,1}^{a,b}(x,y)$ is given by (1.10). Further for $y = 0$, (4.2) reduces to (4.1) and for $x = 0$, we have

$$(4.3)\qquad \frac{e^{a\nu}}{1-b\nu}\,{}_0F_1[\_;\ \alpha+1;\ -y\nu] = \sum_{n=0}^{\infty}\frac{t^n\,(a+bn)^n}{(1+\alpha)_n}\,L_n^{\alpha}\left(\frac{y}{a+bn}\right)$$

Another special case of (2.1) would occur when we set $r = 1$ and $f(-\nu z_1) = {}_1F_1[1+\beta;\ 1+\alpha;\ \left(\frac{z-1}{2}\right)\nu]$. Thus we have

$$(4.4)\qquad \frac{e^{s-xt/s+a\nu}}{1-b\nu}\,{}_1F_1\left[1+\beta;\ 1+\alpha;\ \left(\frac{z-1}{2}\right)\nu\right]$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty} s^m\,t^n \times \sum_{p=0}^{n}\frac{(-x)^p(a+b(n-p))^{n-p}}{(p+m)!\,p!\,(n-p)!\,(1+\alpha)_{n-p}}$$
$$\cdot\, P_{n-p}^{\alpha,\beta-\alpha-n+p}\left(\frac{z}{a+b(n-p)}\right),$$

where $P_n^{(\alpha,\beta)}(x)$ is Jacobi polynomial (11, p.91).

The generating function (2.1) for $x = 0$ corresponds to the main result of Agarwal and Manocha [2, p.276].

Next, we consider some applications of assertion (2.4). First of all by setting $k = 1$, (2.4) immediately yields (4.2).

If in the assertion (2.4), we set $x = 0$, then it reduces to (1.4).

For $a = 1$ and $b = 0$, (2.4) yields the generating function

$$(4.5)\qquad e^{t+s-xt/s}\,{}_0F_k\left[\_;\ \Delta(k;1+\alpha);\ -t\left(\frac{y}{k}\right)^k\right]$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=m^\star}^{\infty} s^m\ t^n\ \sum_{p=0}^{n}\frac{(-x)^p}{(p+m)!\ p!\ (1+\alpha)_{k(n-p)}}\,Z_{n-p}^{\alpha}(y;k),$$

which further for $x = 0$ and $t = s$ reduces to the following known generating function [1, p.116(18), see also 10, pp. 243-244]

$$(4.6)\qquad \sum_{n=0}^{\infty} Z_n^{\alpha}(y;k)\,\frac{t^n}{(1+\alpha)_{kn}} \;=\; e^t\,{}_0F_k\left[\_;\ \Delta(k;1+\alpha);\ -t\left(\frac{y}{k}\right)^k\right]$$

Setting $\alpha = \beta = \gamma$ in (3.1) and using [11, p.53(5)], we get

$$\sum_{n,m=0}^{\infty} (a_1 + b_1 n - x)^n \, (a_2 + b_2 m - y)^m \times 2F_1 \left[-n, -m;\, \alpha;\, \frac{xy}{(a_1 + b_1 n - x)\,(a_2 + b_2 m - y)} \right] t^n \, T^m = \frac{e^{a_1\nu + (a_2 - y)w}}{(1 - b_1\nu)\,(1 - b_2 w)} \, {}_1F_1[\alpha;\, 2\alpha;\, yw - x\nu], \tag{4.7}$$

where $\nu = te^{b_1\nu}$ and $w = Te^{b_2 w}$.

For $p = q = 1$, (3.5) yields

$$\sum_{n,m=0}^{\infty} (a_1 + b_1 n)^n \, (a_2 + b_2 m)^m \times F_1 \left[\alpha, -n, -m;\, \beta;\, \frac{x}{(a_1 + b_1 n)} \, \frac{y}{(a_2 + b_2 m)} \right] \frac{t^n \, T^m}{n! \; m!} = \frac{e^{a_1\nu + a_2 w}}{(1 - b_1\nu)\,(1 - b_2 w)} \, {}_1F_1[\alpha;\, \beta;\, -(x\nu + yw)], \tag{4.8}$$

where $\nu = te^{b_1\nu}$, $w = Te^{b_2 w}$ and F_1 is Appell's function [11, p.53(4)].

For $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$, (3.2) yields

$$\sum_{n,m=0}^{\infty} F_2 \, [\alpha, -n, -m;\, \beta, \gamma;\, x\, y] \, \frac{t^n \, T^m}{n! \; m!} = e^{t+T} \, \psi_2[\alpha;\, \beta, \gamma;\, -xt, -yt] \tag{4.9}$$

which is a generalization of the well known generating function due to Rainville [9]

$$\sum_{n=0}^{\infty} {}_2F_1 \, [-n, \alpha;\, \beta;\, x] \, \frac{t^n}{n!} = e^t \, {}_1F_1[\alpha;\, \beta;\, -xt] \tag{4.10}$$

For $y = 0$, (3.2) reduces to a result [3, p.159(3.4)]

$$\sum_{n=0}^{\infty} \frac{(a + bn)^n}{(1 + \alpha)_n} \, P_n^{\alpha,\, \beta - n} \left(\frac{x}{a + bn} \right) t^n = \frac{e^{a\nu}}{1 - b\nu} \, {}_1F_1 \left[1 + \alpha + \beta;\, 1 + \alpha;\, -\left(\frac{1 - x}{2} \right) \nu \right], \qquad \nu = te^{b\nu}, \tag{4.11}$$

which further for $\alpha = 0$ yields a known result [3, p.160(3.18)]

$$\sum_{n=0}^{\infty} \frac{(a + bn)^n}{n!} \, Y_n^{(\beta - n)} \left(\frac{x}{a + bn} \right) t^n = \frac{e^{a\nu}}{1 - b\nu} \left(1 - \frac{x\nu}{2} \right)^{-\alpha - 1}, \qquad \nu = te^{b\nu}, \tag{4.12}$$

where $Y_n^{(\alpha)}(x)$ are generalized Bessel polynomials of Krall and Frink (see [11], p.75(1)).

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