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$\pi-$Morphic Rings

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Abstract. An element a in a ring R is called left morphic if $R/Ra \cong \mathbf{l}(a)$. A ring R is called left morphic if every element is left morphic. In this paper, an element a in a ring R is called left π-morphic (resp. left G-morphic) if there exists a positive number n such that a^n (resp. $a^n \neq 0$) is left morphic. A ring R is called left π-morphic (resp. left G-morphic) if every element is left π-morphic (resp. left G-morphic). The Morita invariance of left π-morphic (resp. left G-morphic) rings is discussed. Several relevant properties are proved. In particular, it is shown that a left Noetherian ring R with $M_4(R)$ left G-morphic or $M_2(R)$ left morphic is QF. Some known results of left morphic rings are extended to left G-morphic rings and left π-morphic rings.

1. Introduction

Throughout the paper, all rings are associative with identity and all modules are unitary. Let R be a ring. The right (resp. left) annihilator of a subset X of R is denoted by $\mathbf{r}(X)$ (resp. $\mathbf{l}(X)$). The Jacobson radical, left singular ideal, right singular ideal, left socle and the right socle of R are denoted by $J(R)$, $Z(_RR)$, $Z(R_R)$, $Soc(_RR)$ and $Soc(R_R)$, respectively. Using $I \subseteq^{ess} {}_RR$ to show that I is an essential left ideal of R. And we write $U(R)$ for the group of all units of R.

By the well known theorem of Erlich [5], a map α in the endomorphism ring of a module M is unit regular if and only if it is regular and $M/\mathrm{Im}(\alpha) \cong \mathrm{Ker}(\alpha)$. In [11], Nicholson and Sánchez Campós introduced and studied left morphic rings (which were further studied in [3, 9, 10]). An element a in a ring R is said to be left morphic if $R/Ra \cong \mathbf{l}(a)$, equivalently, there exists $b \in R$ such that $\mathbf{l}(a) = Rb$ and $\mathbf{l}(b) = Ra$. A ring R is called left morphic if every element is left morphic. Right morphic rings are defined analogously.

In this paper, the notion of left morphic rings is extended to left π-morphic (resp.

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left G-morphic) rings. An element a in a ring R is called left π-morphic (resp. left G-morphic) if there exists $n > 0$ such that a^n (resp. $a^n \neq 0$) is left morphic. The ring itself is called left π-morphic (resp. left G-morphic) if every element is left π-morphic (resp. left G-morphic). Right π-morphic (resp. right G-morphic) rings are defined analogously. A ring R is called morphic if it is left and right morphic. The π-morphic (resp. G-morphic) rings are defined similarly. Some examples show that left π-morphic rings need not be left morphic. Some known results of left morphic rings are extended to left π-morphic (resp. left G-morphic) rings. The Morita invariance of these notions is discussed. Several relevant properties are proved. In particular, it is shown that a left Noetherian ring R with $M_4(R)$ left G-morphic or $M_2(R)$ left morphic is QF. Left π-morphic (resp. left G-morphic) rings are directly finite. If R is left π-morphic, then the same is true of the corner ring eRe for any idempotent $e \in R$. Furthermore, it is shown that if R is a local and left G-morphic ring, then the following conditions are equivalent: (1) $J(R)$ is nilpotent; (2) $J(R)$ is nil and $Soc(_RR) \neq 0$; (3) R has ACC on principal left ideals and $Soc(_RR) \neq 0$. In this case, R is left morphic.

2. Examples

We start this section with the following.

Definition 2.1. Let R be a ring. An element a in R is called left π-morphic if there exists $n > 0$ such that $R/Ra^n \cong \mathbf{l}(a^n)$. The ring itself is said to be left π-morphic if every element is left π-morphic.

Lemma 2.2. *Let R be a ring. An element a in R is left π-morphic if and only if there exist $n > 0$ and $b \in R$ such that $\mathbf{l}(a^n) = Rb$ and $\mathbf{l}(b) = Ra^n$.*

Proof. By Definition 2.1 and [11, Lemma 1]. □

Proposition 2.3. *Let R be a ring, and let a in R be left π-morphic. Then the following conditions are equivalent:*
(1) $\mathbf{l}(a) = 0$.
(2) $Ra = R$.
(3) $a \in U(R)$.

Proof. By Lemma 2.2, there exist $n > 0$ and $b \in R$ such that $Ra^n = \mathbf{l}(b)$, $Rb = \mathbf{l}(a^n)$.

(1) ⇒ (2) If $\mathbf{l}(a) = 0$, then it is easy to see that $\mathbf{l}(a^n) = 0$. Hence $b = 0$, $Ra^n = \mathbf{l}(0) = R$, and so $Ra = R$.

(2) ⇒ (3) If $Ra = R$, there exists $r \in R$ such that $ra = 1$, so $r^n a^n = 1$ and hence $Ra^n = R$. Thus $b = 0$, and $\mathbf{l}(a) \subseteq \mathbf{l}(a^n) = 0$. And $ar - 1 \in \mathbf{l}(a)$, $ar = 1$. We are done.

(3) ⇒ (1) is clear. □

A ring R is called directly finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$. R is said to be stably finite if $M_n(R)$ is directly finite for all $n > 0$.

Corollary 2.4. *Let R be a left π-morphic ring. Then it is directly finite.*

Recall that a ring R is called unit π-regular [1], if for any $a \in R$, there exists $n > 0$ such that a^n is unit regular.

Lemma 2.5. *Let R be a regular ring. Then it is left π-morphic if and only if it is unit π-regular.*

Proof. Since R is a regular ring, a^n is unit regular if and only if a^n is left morphic for any $a \in R$ and $n > 0$ (by [11, Example 4] and [11, Proposition 5]). The result follows. □

Lemma 2.6. *Let R be a unit π-regular ring. Then $S = R/I$ is a unit π-regular ring for any ideal I of R.*

Proof. For any $a \in R$, there exist $n > 0$ and $u \in U(R)$ such that $a^n = a^n u a^n$. So $(a+I)^n = a^n + I = a^n u a^n + I = (a^n + I)(u + I)(a^n + I)$. This means that $S = R/I$ is a unit π-regular ring. □

Now, an example is given to show that the converse of Corollary 2.3 is false, even if R is regular and stably finite.

Example 2.7 [7]. Choose a field F, let $T = F[[t]]$ be the ring of formal power series over F in an indeterminate t, and let K denote the quotient field of T. Let $S = \{x \in End_F(T) | (x - a)(t^n T) = 0$, for some $a \in K$ and $n > 0\}$. By [7, Example 4.26], for each $x \in S$ there is an unique element $\varphi x \in K$ such that $(x - \varphi x)(t^n T) = 0$ for some $n > 0$. Since K is commutative, the map $\varphi : S \to K$ also defines a ring map $\varphi : S^{op} \to K$, where S^{op} denotes the opposite ring of S. Consequently, the set $R = \{(x, y) \in S \times S^{op} | \varphi x = \varphi y\}$ is a subring of $S \times S^{op}$. Inasmuch R is a subdirect product of S and S^{op}, then R is regular, stably finite but not left π-morphic.

Proof. From [7, Example 5.10], we know that R is a regular and stably finite ring. Assume that R is left π-morphic, by Lemma 2.5, R is unit π-regular. In fact, as an element of $End_F(T)$, t^n is injective but not surjective. If there exists $u \in S$ such that $t^n u t^n = t^n$, then $u t^n = 1$, and so $t^n = u^{-1}$, a contradiction. Hence, S is not unit π-regular. By Lemma 2.6, we conclude that R is not left π-morphic. □

Recall that R is called right π-P-injective if there exists $n > 0$ such that $Ra^n = \mathbf{lr}(a^n)$, which is also called right GP-injective in [15]. But in this paper, we call R a right GP-injective ring [2], if for any $0 \neq a \in R$, there exists $n > 0$ with $a^n \neq 0$ such that $Ra^n = \mathbf{lr}(a^n)$. An element a in R is said to be left G-morphic if there exists $n > 0$ with $a^n \neq 0$ such that a^n is left morphic, equivalently, there exist $n > 0$ with $a^n \neq 0$ and $b \in R$ such that $\mathbf{l}(a^n) = Rb$ and $\mathbf{l}(b) = Ra^n$. The ring itself is called left G-morphic if every element is left G-morphic.

Lemma 2.8. *Let R be a left π-morphic (resp. left G-morphic) ring. Then it is right π-P-injective (resp. right GP-injective).*

Proof. If R is a left G-morphic ring, then for any a in R, there exist a positive number n with $a^n \neq 0$ and $b \in R$ such that $Ra^n = \mathbf{l}(b)$ and $\mathbf{l}(a^n) = Rb$. So

$\mathbf{lr}(a^n) = \mathbf{lr}(Ra^n) = \mathbf{lrl}(b) = \mathbf{l}(b) = Ra^n$, equivalently, R is right GP-injective. Similarly, if R is a left π-morphic ring, then it is right π-P-injective. □

Theorem 2.9. *Let R be a local and left G-morphic ring. Then the following conditions are equivalent:*
(1) *$J(R)$ is nilpotent.*
(2) *$J(R)$ is nil and $Soc(_RR) \neq 0$.*
(3) *R has ACC on principal left ideal and $Soc(_RR) \neq 0$.*
In this case, R is left morphic.

Proof. (1) ⇒ (2) and (3). If $J(R)$ is nilpotent, then it is clear that $J(R)$ is nil and R is left perfect. Hence R has ACC on principal left ideals (by Jonah's Theorem [8]), and $Soc(R_R) \subseteq^{ess} R_R$. Moreover, by Lemma 2.8, R is right GP-injective. So $Soc(R_R) \subseteq Soc(_RR)$ by [2, Lemma 2.2], and hence $Soc(_RR) \neq 0$.

(2) ⇒ (1). By hypothesis, there exists a nonzero minimal left ideal Ra of R. Since R is left G-morphic, there exist $n > 0$ with $a^n \neq 0$ and $c \in R$ such that $Ra^n = \mathbf{l}(c)$ and $Rc = \mathbf{l}(a^n)$. Hence $R/Rc = R/\mathbf{l}(a^n) \cong Ra^n = Ra$, Rc is a maximal left ideal of R and so $J(R) = Rc$.

Assume that $J(R)^k = Rc^k$ for some positive number $k \geq 1$. $J(R)^{k+1} = J(R) \cdot Rc^k = J(R) \cdot c^k = Rc \cdot c^k = Rc^{k+1}$. By induction, $J(R)^k = Rc^k$ for every $k \geq 1$. Since $J(R)$ is nil, there exists $k > 0$ such that $c^k = 0$ and so $J(R)^k = Rc^k = 0$.

(3) ⇒ (1). By the proof of (2) ⇒ (1), $J(R) = Rc$. If c is nilpotent, then $J(R)$ is nilpotent. If c is not nilpotent, there exist positive numbers $n_0 > 0$ and $n_1 > 0$ such that $0 \neq c^{n_0}$ and $0 \neq c^{2n_0n_1}$ are left morphic. By induction, there exists $\{c^{2^in_0n_1\cdots n_i}\}_{i=0}^{\infty}$ such that $0 \neq c^{2^in_0n_1\cdots n_i}$ is left morphic for any $i \geq 0$, and denotes $2^in_0n_1\cdots n_i$ by t_i. Then $\mathbf{l}(c^{t_i}) \subseteq \mathbf{l}(c^{t_{i+1}})$ dues to $t_i < t_{i+1}$. Because R has ACC on principal left ideals, there exists $n > 0$ such that $\mathbf{l}(c^{t_0}) \subseteq \mathbf{l}(c^{t_1}) \subseteq \cdots \subseteq \mathbf{l}(c^{t_n}) = \mathbf{l}(c^{t_{n+1}})$. We can choose $a, b \in R$ such that $Ra = \mathbf{l}(c^{t_n})$, $\mathbf{l}(a) = Rc^{t_n} = J(R)^{t_n}$ and $Rb = \mathbf{l}(c^{t_{n+1}})$, $\mathbf{l}(b) = Rc^{t_{n+1}} = J(R)^{t_{n+1}}$. Then $Ra = Rb$, there exist $u,\ v \in R$ such that $a = ub$ and $b = va$. Thus $a = uva$, $(1 - uv)a = 0$. $uv \notin J(R)$ because $a \neq 0$. This implies $uv \in U(R)$ (as R is a local ring) and so $u \in U(R)$ by Proposition 2.3. But $0 = c^{t_n}a = c^{t_n}ub$, $c^{t_n}u \in \mathbf{l}(b) = Rc^{t_{n+1}} = J(R)^{t_{n+1}}$, it follows that $c^{t_n} \in J(R)^{t_{n+1}} = Rc^{t_{n+1}}$. Let $c^{t_n} = rc^{t_{n+1}}$, $r \in R$, it implies that $(1 - rc^{t_{n+1}-t_n})c^{t_n} = 0$. Thus $rc^{t_{n+1}-t_n} \notin J(R)$ because $c^{t_n} \neq 0$, a contradiction. Hence c is nilpotent and so $J(R)$ is nilpotent.

Now, we show the last part. Suppose that $J(R)^n = 0$ for some $n > 0$. If $n = 1$, then R is division and so it is left morphic. If $n > 1$, we can assume that $J(R)^{n-1} \neq 0$. Let $0 \neq b \in J(R)^{n-1}$, there exists $m \geq 1$ such that $b^m \neq 0$ is left morphic. Moreover, for any $0 \neq a \in J(R)^{n-1}$, $a^2 = 0$, so $m = 1$. Thus b is left morphic, there exists $c \in R$ such that $\mathbf{l}(b) = Rc$ and $J(R) \subseteq \mathbf{l}(b) = Rc \neq R$. Hence $J(R) = Rc$ and $c^n = 0$. Therefore, by [11, Theorem 9], R is a left morphic ring. □

The following example shows that left π-morphic rings need not be left G-morphic, and hence the notion of left π-morphic rings is a proper generalization of left morphic.

Example 2.10. Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then R is left and right π-morphic but neither left nor right G-morphic.

Proof. Clearly, every element of R is either nilpotent or idempotent or invertible, so R is left and right π-morphic. But $\lambda = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is neither left nor right G-morphic in R. □

Lemma 2.11. *Let R be a local ring with $J(R)$ nil. Then R is left and right π-morphic.*

Proof. By hypothesis, every element of R is either nilpotent or invertible. So R is left and right π-morphic. □

Next example shows that the left G-morphic condition can not be replaced by left π-morphic in Theorem 2.9.

Example 2.12. Let $R = \mathbb{Z}_4C_2$. Then R is a commutative, π-morphic ring but not G-morphic.

Proof. By [11, Example 36], we know that R is a commutative, local and QF ring but not morphic. This implies that $J(R)$ is nilpotent, so R is π-morphic by Lemma 2.11. But from Theorem 2.9, we know that R is not G-morphic. □

Remark. Unfortunately, we can not find a ring R which is left G-morphic but not left morphic. But we have an example which shows that left G-morphic elements need not be left morphic.

Example 2.13. Let $R = F[x,\ \sigma]/(x^2) = \{a + xb \mid a,\ b \in F\}$, where F is a field with an isomorphism σ from F to a subfield $\overline{F} \neq F$ and $cx = x\sigma(c)$ for all $c \in F$. $S = R \oplus R$, then $\lambda = (1,\ xb) \in S$ (where $b \in F$ but $b \notin \overline{F}$) is left G-morphic but not left morphic.

Proof. By symmetry and [11, Example 8], xb is not left morphic in R, but $\lambda^2 = (1, 0)$ is left morphic in S. It is trivial to see that λ is left G-morphic but not left morphic in S. □

Now we give an example to show that " left G-morphic " is not a Morita invariant property.

Example 2.14. Let R be the ring in Example 2.13 and $S = M_2(R)$. Then R is a right morphic, left π-morphic ring and S is a right π-morphic ring, but R is not a left G-morphic ring and S is not a right G-morphic ring.

Proof. By [11, Example 11] and Example 2.13, we know that R is a local, right morphic ring but not left morphic, and $J(R)$ is nilpotent. Hence R is left π-morphic by Lemma 2.11. xb ($b \in F$ but $b \notin \overline{F}$) is not left G-morphic because $(xb)^2 = 0$ and xb is not left morphic.

For any $\lambda = \begin{bmatrix} a_{11} + xb_{11} & a_{12} + xb_{12} \\ a_{21} + xb_{21} & a_{22} + xb_{22} \end{bmatrix}$, where $a_{ij}, b_{ij} \in F$ for $i = 1, 2$ and $j =$

$1, 2$. If $a_{11} = a_{12} = a_{21} = a_{22} = 0$ then $\lambda^2 = 0$, whence λ is right π-morphic. If there exists $a_{ij} \neq 0$, then there exist $\mu_1,\ \mu_2 \in U(S)$ such that $\mu_1 \lambda \mu_2 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \in S$, $r_1,\ r_2 \in R$. Following from [11, Lemma 3] and the fact that $\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ is right morphic, λ is right morphic. Therefore, S is a right π-morphic ring.

By [4, Example 6], S is not left GP-injective. From Lemma 2.8, S is not right G-morphic. □

3. Corner Rings and Kasch Condition

We are going to prove that if R is left π-morphic, then the same is true for the corner ring eRe, in which $e^2 = e \in R$.

Lemma 3.1. *Let R be a left π-morphic ring. Then so is eRe for any idempotent e of R.*

Proof. We need only to show that for any $a \in eRe$, it is left π-morphic in eRe. Set $f = 1 - e$, then $(a+x)^n = a^n + x^n$ for any $a \in eRe,\ x \in fRf$ and $n \geq 1$. Since $a + f$ is left π-morphic in R, there exist $c \in R$ and $n > 0$ such that $\mathbf{l}_R((a+f)^n) = Rc$ and $\mathbf{l}_R(c) = R(a+f)^n$, so $\mathbf{l}_R(a^n + f) = Rc$ and $\mathbf{l}_R(c) = R(a^n + f)$. Hence $0 = (a^n + f)c = a^n c + fc$, $0 = fa^n c + fc = fc$ and $ec = c$. Similarly $ce = c$, $c \in eRe$. Let $b \in \mathbf{l}_{eRe}(a^n)$. Then $b \in eRe \cap Rc = (eRe)c$. Conversely, let $b \in (eRe)c$. Clearly, $b \in \mathbf{l}_R(a^n + f)$, and hence $b \in \mathbf{l}_{eRe}(a^n)$. This implies that $\mathbf{l}_{eRe}(a^n) = (eRe)c$. Thus $(eRe)a^n \subseteq \mathbf{l}_{eRe}(c)$ because $(eRe)a^n \subseteq R(a^n + f) = \mathbf{l}_R(c)$. If $b \in \mathbf{l}_{eRe}(c)$, then $b \in \mathbf{l}_R(c) = R(a^n + f)$, and so $b = ebe \in (eRe)a^n$. It follows that $\mathbf{l}_{eRe}(c) = (eRe)a^n$, hence a is left π-mophic in eRe. This completes the proof. □

Recall that in [14], R is called a left GC2-ring if for any left ideal I with $I \cong R$, I is a summand of ${}_RR$. Equivalently, for any element a in R, if $\mathbf{l}(a) = 0$ then Ra is a summand of ${}_RR$ by [14, Proposition 2.2].

Theorem 3.2. *Let R be a left Noetherian ring with $M_4(R)$ left G-morphic. Then R is QF.*

Proof. Since $M_4(R)$ is left G-morphic, $M_4(R)$ is right GP-injective by Lemma 2.8. Hence $M_2(R)$ is right P-injective by [4, Lemma 3], and so R is right 2-injective. Following from Lemma 3.1, R is left π-morphic. Therefore, by Proposition 2.3, $\mathbf{l}(a) = 0$ for any $a \in R$ if and only if $Ra = R$. This implies that R is left GC2, and so that R is semilocal by [14, Corollary 2.5]. From [6, Theorem 2.7], we have $J(R)$ nilpotent. Hence R is semiprimary, it follows that R is left Artinian. So R is QF by [13, Corollary 3]. □

Similar to the proof of above theorem, we have the following theorem.

Theorem 3.3. *Let R be a left Noetherian ring with $M_2(R)$ left morphic. Then R is QF.*

Remark. The assumption that $M_2(R)$ can not be replaced by R in Theorem 3.3. In fact, the ring R in [11, Example 8] is left Artinian and left morphic but not QF.

Recall that, a Morita context is a four-tuple $(R,\ V,\ W,\ S)$ in which $V = {}_RV_S$ and $W = {}_SW_R$ are bimodules and there exist multiplications $V \times W \to R$ and $W \times V \to S$ such that $C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is an associated ring with the usual matrix operations (called the context ring).

Proposition 3.4. *Let* $C = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ *be a left G-morphic context ring. If either* $VW \subseteq J(R)$ *or* $WV \subseteq J(S)$*, then* $V = 0$ *and* $W = 0$.

Proof. Since C is a left π morphic ring, $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in C$ and $S \cong \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & V \\ W & S \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, S is left π-morphic by Lemma 3.1. So S is directly finite by Corollary 2.4. Assume that $WV \in J(S)$, the argument is similar if $VW \in J(R)$. Let $v \in V$, and write $\lambda = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in C$. Then $\lambda^2 = 0$, it follows that λ is left morphic because C is left G-morphic. Therefore, following from the proof of [11, Proposition 18], we have $V = 0$ and $W = 0$. □

Remark. The G-morphic condition can not be replaced by π-morphic condition in Proposition 3.4. In Example 2.10, $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ is a left π-morphic context ring but $\mathbb{Z}_2 \neq 0$.

A ring R is said to be left Kasch if every simple left R-module embeds in ${}_RR$, equivalently, if $\mathbf{r}(I) \neq 0$ for every proper (maximal) left ideal I of R.

Proposition 3.5. *The following conditions are equivalent for a left G-morphic ring* R*:*
(1) R *is left Kasch.*
(2) *Every maximal left ideal of* R *is an annihilator.*
(3) *Every maximal left ideal of* R *is principal.*

Proof. (2) can be deduced by (1) without the left G-morphic condition.

(2) ⇒ (3) Let I be a maximal left ideal of R. Then there exists a nonzero right ideal A such that $I = \mathbf{l}(A)$. Since R is left G-morphic, for any $0 \neq a \in A$, there exists $n > 0$ such that $a^n \neq 0$ is left morphic. Hence $\mathbf{l}(a^n) \neq R$, and so $I \subseteq \mathbf{l}(a^n) \neq R$. Therefore, $I = \mathbf{l}(a^n)$. It follows that I is principal.

(3) ⇒ (1) Let $I = Ra$ be a maximal left ideal of R. As R is left G-morphic, there exist $n > 0,\ b \in R$ with $a^n \neq 0$ such that $Ra^n = \mathbf{l}(b)$. Since $Ra^n \subseteq Ra = I \neq R$, $b \neq 0$. But $a^n b = 0$, $a(a^{n-1}b) = 0$. If $a^{n-1}b \neq 0$, then $0 \neq a^{n-1}b \in \mathbf{r}(Ra) = \mathbf{r}(I)$, otherwise, by induction, we have $\mathbf{r}(I) \neq 0$ because $b \neq 0$. So R is left Kasch. □

Proposition 3.6. *Let* R *be a left* π*-morphic ring and every maximal right ideal be* *principal. Then R is right Kasch.*

Proof. Given any maximal right ideal I of R, there exists $a \in R$ such that $I = aR$. Since R is left π-morphic, $a \notin U(R)$ and so $Ra \neq R$ by Proposition 2.3. Hence, we can choose $0 \neq b \in R$, $n > 0$ such that $R \neq Ra^n = \mathbf{l}(b)$ and $\mathbf{l}(a^n) = Rb$, so $\mathbf{l}(a^n) = Rb \neq 0$. Therefore, $\mathbf{l}(I) = \mathbf{l}(a) \neq 0$, this implies that R is right Kasch. □

4. Singular ideals and trivial extensions

In this section, we study some properties about singular ideals and trivial extensions of rings under left π-morphic or left G-morphic condition.

Theorem 4.1. *Let R be a left π-morphic ring. Then the following conditions hold.*
(1) *If $\mathbf{r}(a) = 0$ then $a \in U(R)$.*
(2) *$Z({}_RR) \subseteq J(R)$ and $Z(R_R) \subseteq J(R)$.*
(3) *If ${}_RR$ is uniform then R is local and $J(R) = Z({}_RR) = \{a \in R \mid a \notin U(R)\}$.*
(4) *If R_R is uniform then R is local and $J(R) = Z(R_R) = \{a \in R \mid a \notin U(R)\}$.*
But the converses of (3) and (4) are false.

Proof. (1) Since $\mathbf{r}(a) = 0$, $\mathbf{r}(a^n) = 0$ for any $n \geq 1$. By Lemma 2.7, R is right π-P-injective. Thus there exists $n > 0$ such that $Ra^n = \mathbf{lr}(a^n) = \mathbf{l}(0) = R$, which implies that $Ra = R$. Therefore, $a \in U(R)$ by Proposition 2.2.

(2) For any $a \in Z({}_RR)$ and $r \in R$, $ar \in Z({}_RR)$. Thus $\mathbf{l}(ar) \subseteq^{ess} {}_RR$, $\mathbf{l}(ar) \cap \mathbf{l}(1 - ar) = 0$, and so that $\mathbf{l}(1 - ar) = 0$. Again by Proposition 2.3, $1 - ar \in U(R)$, so $a \in J(R)$. Therefore, $Z({}_RR) \subseteq J(R)$. Similarly, for any $a \in Z(R_R)$ and $r \in R$, $\mathbf{r}(1 - ra) = 0$. Hence $1 - ra \in U(R)$ and so $a \in J(R)$. Therefore, $Z(R_R) \subseteq J(R)$.

(3) Assume that ${}_RR$ is uniform. Let $a \notin U(R)$. $\mathbf{l}(a) \neq 0$ by Proposition 2.3, it follows that $\mathbf{l}(a) \subseteq^{ess} {}_RR$, and so $a \in Z({}_RR)$. Conversely, if $a \in Z({}_RR)$, $\mathbf{l}(a) \subseteq^{ess} {}_RR$, then $a \notin U(R)$. This shows that $Z({}_RR) = \{a \in R | a \notin U(R)\}$. Given $a \in J(R)$, then $a \notin U(R)$, and so $a \in Z({}_RR)$. Hence $J(R) \subseteq Z({}_RR)$. Therefore, by (2), R is a local ring with $J(R) = Z({}_RR) = \{a \in R | a \notin U(R)\}$.

(4) Assume that R_R is uniform. If there exists a proper left ideal I such that $Z(R_R) \subset I$, then for any element a which is in I but not in $Z(R_R)$, we have $\mathbf{r}(a) = 0$, it follows that $a \in U(R)$ by (1). It is impossible. Therefore, $Z(R_R)$ is the unique maximal left ideal, whence R is a local ring and $Z(R_R) = J(R) = \{a \in R | a \notin U(R)\}$.

In Example 2.13, R is a left and right π-morphic, local ring with $Z(R_R) = J(R) = Z({}_RR) = \{a \in R | a \notin U(R)\}$ but R_R is not uniform. By symmetry, the last part holds. □

Corollary 4.2. *Let R be a ring with ${}_RR$ (or R_R) uniform and $R/Soc({}_RR)$ (or $R/Soc(R_R)$) having ACC on left (or right) annihilators. Then the following conditions are equivalent:*
(1) *R is a local ring with $J(R)$ nilpotent.*
(2) *R is a local ring with $J(R)$ nil.*
(3) *R is a unit π-regular ring.*

(4) *R is a left π-morphic ring.*
(5) *R is a right π-morphic ring.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are obvious.

(4) or (5) $\Rightarrow$ (1). Given R is left (or right) π-morphic and ${}_RR$ (or R_R) is uniform. Then $J(R) = Z({}_RR)$ (or $J(R) = Z(R_R)$) and R is local by Proposition 4.1. Since $R/Soc({}_RR)$ (or $R/Soc(R_R)$) has ACC on left (or right) annihilators, $Z({}_RR)$ (or $Z(R_R)$) is nilpotent by [12, Lemma 4.20]. Therefore, $J(R)$ is nilpotent. □

For a ring R, the trivial extension $R \propto R = \{(a,\ b)|a,\ b \in R\}$ is a ring with addition defined componentwise and multiplication defined by $(a,\ b)(c,\ d) = (ac,\ ad + bc)$.

Proposition 4.3. *Let $S = R \propto R$ and $a \in R$. Then the following conditions are equivalent:*
(1) *$a \in R$ is left π-morphic in R.*
(2) *$(a,\ 0) \in R \propto R$ is left π-morphic in S.*
(3) *$(a,\ a) \in R \propto R$ is left π-morphic in S.*

Proof. (1) $\Rightarrow$ (2) Since $a \in R$ is left π-morphic in R, there exists $n > 0$ such that a^n is left morphic, so $(a^n,\ 0) = (a,\ 0)^n \in S$ is left morphic by [3, Theorem 19]. Hence $(a,\ 0)$ is left π-morphic in S.

(2) $\Rightarrow$ (1) Assume that $(a,\ 0)$ is left π-morphic. There exists $n > 0$ such that $(a,\ 0)^n = (a^n,\ 0) \in S$ is left morphic, so a^n is left morphic in R again by [3, Theorem 19]. Therefore, a is left π-morphic in R.

(2) $\Leftrightarrow$ (3) For any $n > 0$, $(a,\ a)^n = (a^n,\ na^n)$ and $(a^n,\ na^n)(1,\ -n) = (a^n,\ 0)$. $(a,\ 0)$ is left π-morphic if and only if there exists $n > 0$ such that $(a^n,\ 0)$ is left morphic if and only if $(a^n,\ na^n) = (a,\ a)^n$ is left morphic (by [11, Lemma 3]) if and only if $(a,\ a)$ is left π-morphic. We have done. □

Note that if replace left π-morphic condition by left G-morphic, Proposition 4.3 is also right.

Proposition 4.4. *Let $S = R \propto R$. Then the following conditions hold.*
(1) *If S is a left π-morphic ring, then so is R.*
(2) *If S is a left G-morphic ring, then R is left morphic.*

Proof. (1) By Proposition 4.3.

(2) For any $a \in R$, $(0,\ a) \in S$ is left morphic in S because that $(0,\ a)^2 = 0$. Therefore, a is left morphic in R by [3, Proposition 20]. Thus the result follows. □

During we consider the question when $R \propto R$ is left π(resp. G)-morphic, we have the following example.

Example 4.5. Let p be any prime number and k be any positive integer. Then $S = \mathbb{Z}_{p^k} \propto \mathbb{Z}_{p^k}$ is a π-morphic ring. And it is G-morphic if and only if $k = 1$.

Proof. It is trivial to see that S is a commutative, local and Artinian ring. So it is π-morphic by Lemma 2.11, and if it is G-morphic, then it is morphic by Theorem

2.9. From [3, Theorem 8], S is morphic if and only if $k = 1$. □

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References

[1] H. Y. Chen, *Power-substitution, exchange rings and unit π-regularity*, Comm. Algebra, **28(11)**(2000), 5223-5233.

[2] J. L. Chen and N. Q. Ding, *On general principally injective rings*, Comm. Algebra, **27(5)**(1999), 2097-2116.

[3] J. L. Chen and Y. Zhou, *Morphic rings as trivial extentions*, Glasgow Math. J., **47**(2005), 139-148.

[4] J. L. Chen, Y. Zhou and Z. M. Zhu, *GP-injective rings need not be P-injective*, Comm. Algebra, **33(7)**(2005), 2395-2402.

[5] G. Erlich, *Unit and one-sided units in regular rings*, Trans. Amer. Math. Soc., **216**(1976), 81-90.

[6] J. L. Gomez Pardo and P. A. Guil Asensio, *Torsionless modules and rings with finite essential socle*, Lecture Notes in Pure and Appl. Math., **201**(1998), 261-278.

[7] K. R. Goodearl, "Von Neumann Regular Rings", Second Edition. Krieger, Malabar, Florida, 1991.

[8] D. Jonah, *Rings with the minimum condition for principal right ideals have the maximal condition for principal left ideal*, Math. Z., **113**(1970), 106-112.

[9] W. K. Nicholson and E. Sánchez Campós, *Morphic modules*, Comm. Algebra, **33(8)**(2005), 2629-2647.

[10] W. K. Nicholson and E. Sánchez Campós, *Principal rings with the dual of the isomorphism theorem*, Glasgow Math. J., **46**(2004), 181-191.

[11] W. K. Nicholson and E. Sánchez Campós, *Rings with the dual of the isomorphism theorem*, J. Algebra, **271**(2004), 391-406.

[12] W. K. Nichoson and M. F. Yousif, "Quasi-Frobenius Ring", Cambridge University Press, 2003.

[13] E. A. Rutter, *Rings with the principal extension property*, Comm. Algebra, **3(2)**(1975), 203-212.

[14] M. F. Yousif and Y. Zhou, *Rings for which certain elements have the principal extensions property*, Algebra Colloq., **10(4)**(2003), 501-512.

[15] C. M. Yue, *On annihilator ideals IV*, Riv. Mat. Univ. Parma., **13(4)**(1987), 19-27.