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A Note on c-Separative Modules

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Abstract. A right R-module P is c-separative provided that $P \oplus P \overset{c}{\cong} P \oplus Q \Longrightarrow P \cong Q$ for any right R-module Q. We get, in this paper, two sufficient conditions under which a right module is c-separative. A ring R is a hereditary ring provided that every ideal of R is projective. As an application, we prove that every projective right R-module over a hereditary ring is c-separative.

Let P and Q be right R-modules, and let $p : P \oplus Q \to P$ be the projection on P, and let $q_i : P \to P \oplus P$ be the injections from $P (i = 1, 2)$. If $\varphi : P \oplus P \cong P \oplus Q$ and $(p\varphi q_1)(p\varphi q_2) = (p\varphi q_2)(p\varphi q_1)$, then we say that $P \oplus P \overset{c}{\cong} P \oplus Q$. For example, if R is a commutative ring, then $R \oplus R \overset{c}{\cong} R \oplus Q$ if and only if $R \oplus R \cong R \oplus Q$. A right R-module P is c-separative provided that $P \oplus P \overset{c}{\cong} P \oplus Q \Longrightarrow P \cong Q$ for any right R-module Q. A right R-module P is strongly separative provided that $P \oplus P \cong P \oplus Q \Longrightarrow P \cong Q$ for any right R-module Q. Clearly, the concept of c-separative modules is an extension of that of strongly separative modules. Many authors studied strong separativity for exchange rings (cf. [2], [4] and [9]-[10]). This inspires us to investigate c-separative modules. We get, in this paper, two sufficient conditions under which a right module is c-separative. A ring R is a hereditary ring provided that every ideal of R is projective. As an application, we prove that every projective right R-module over a hereditary ring is c-separative.

Throughout this paper, all rings are associative with identity and all modules are unital right R-modules. $A \lesssim^{\oplus} B$ means that A is isomorphic to a direct summand of B. Let P be a right R-module, and let $E = End_R(P)$. If $aE + bE = E$ with $a, b \in E$, then we denote the submodule $\{p \in P \mid a(p) \in b(P)\}$ of P by $P_{(a,b)}$.

Theorem 1. *Let P be a right R-module, and let $E = End_R(P)$. If $aE + bE = E$ with $ab = ba$ implies that there exists a right R-morphism $\tau : P_{(a,b)} \to P$ such that $a\mid_{P_{(a,b)}} + b\tau : P_{(a,b)} \to b(P)$ is a R-isomorphism, then P is c-separative.*

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Proof. Assume that $P\oplus P\overset{c}{\cong}P\oplus Q$. Then there exists an isomorphism $\theta: P\oplus P\cong P\oplus Q$ such that $(p\theta q_1)(p\theta q_2)=(p\theta q_2)(p\theta q_1)$, where $q_i: P\to P\oplus P$ are natural injections and $p: P\oplus Q\to P$ is the natural projection. Let $p_i: P\oplus P\to P$ and $p': P\oplus Q\to Q$ be natural projections and $q: Q\to P\oplus Q$ be the natural injection. Construct a map $\varphi: P\oplus P\to P$ given by $\varphi(r,s)=(p\theta q_1)(r)+(p\theta q_2)(s)$ for any $(r,s)\in P\oplus P$. For any $r\in P$, it is easy to check that $\varphi\big(p_1\theta^{-1}(r),p_2\theta^{-1}(r)\big)=r$; hence, φ is a R-epimorphism. Construct a map $i: Q\to P\oplus P$ given by $i(r)=\theta^{-1}(r)$ for any $r\in Q$. Clearly, i is a R-monomorphism. For any $r\in Q$, we see that

$$\begin{aligned}\varphi i(r) &= \varphi\big(\theta^{-1}(r)\big)\\ &= \varphi\big(p_1\theta^{-1}(r),p_2\theta^{-1}(r)\big)\\ &= (p\theta q_1)p_1\theta^{-1}(r)+(p\theta q_2)p_2\theta^{-1}(r)\\ &= p\theta(q_1p_1+q_2p_2)\theta^{-1}(r)\\ &= p(r)\\ &= 0,\end{aligned}$$

and thus $Imi\subseteq Ker\varphi$. If $\varphi(r,s)=0$ for some $(r,s)\in P\oplus P$, we get $(p\theta q_1)(r)+(p\theta q_2)(s)=0$; hence, $p\theta(r,s)=0$. Obviously, $p'\theta(r,s)\in Q$. Furthermore,

$$\begin{aligned}i\big(p'\theta(r,s)\big) &= \theta^{-1}\big(p'\theta(r,s)\big)\\ &= \theta^{-1}\big(p'\theta(r,s)+p\theta(r,s)\big)\\ &= \theta^{-1}\big(\theta(r,s)\big)\\ &= (r,s).\end{aligned}$$

That is, $Ker\varphi\subseteq Imi$. Therefore we have an exact sequence

$$0\to Q\overset{i}{\hookrightarrow}P\oplus P\overset{\varphi}{\to}P\to 0,$$

where $\varphi=(a,b), ab=ba, a,b\in E$. Construct a map $\phi: P\to P\oplus P$ given by $\phi(r)=\theta^{-1}(r,0)$ for any $r\in P$. It is easy to verify that $\varphi\phi=1_P$, so we have $\phi=\begin{pmatrix}c\\ d\end{pmatrix}: P\to P\oplus P$ such that $\varphi\phi=1_P, c,d\in E$. Hence, $ac+bd=1_P$. By assumption, there exists a $\tau: P_{(a,b)}\to P$ such that $a\mid_{P_{(a,b)}}+b\tau: P_{(a,b)}\to b(P)$ is a R-isomorphism. Let $u=a\mid_{P_{(a,b)}}+b\tau$ and $M=P_{(a,b)}\oplus P$. Construct a right R-morphism $\psi=\begin{pmatrix}u^{-1}\\ \tau u^{-1}\end{pmatrix}: b(P)\to M$. As $a\mid_{P_{(a,b)}}+b\tau=u: P_{(a,b)}\to b(P)$ is an isomorphism, we get $a\mid_{P_{(a,b)}}u^{-1}+b\tau u^{-1}=1_{b(P)}$. This implies that $(\varphi\mid_M)\psi=1_{b(P)}$. Thus, $M=Ker(\varphi\mid_M)\oplus Im\psi$. Clearly, $Ker(\varphi\mid_M)\subseteq Ker\varphi$. If $(p_1,p_2)\in Ker\varphi$, then $a(p_1)+b(p_2)=0$ with $p_1,p_2\in P$. Then $a(p_1)\in b(P)$; hence, $p_1\in P_{(a,b)}$. As a result, $(p_1,p_2)\in M$, and so $(p_1,p_2)\in Ker(\varphi\mid_M)$. This implies that $Ker\varphi=Ker(\varphi\mid_M)$. So $M=Ker\varphi\oplus Im\psi$. On the other hand, we have $\sigma\psi=1_M$, where $\sigma=(u,0): M=P_{(a,b)}\oplus P\to b(P)$. Consequently, we deduce that $M = Ker\sigma \oplus Im\psi = P \oplus Im\psi$, and so $P \cong Ker\varphi \cong Q$. Therefore we complete the proof. □

Corollary 2. *Let P be a right R-module. If every submodule of P is projective, then P is c-separative.*

Proof. Let $E = End_R(P)$ and $aE + bE = E$ with $ab = ba$. Let $P_{(a,b)} = \{p \in P \mid a(p) \in b(P)\}$. By assumption, $P_{(a,b)}$ is a projective right R-module. Using the Dual Basis Theorem, there exist $\{x_i\} \subseteq P_{(a,b)}$ and $f_i \in Hom_R\big(P_{(a,b)}, R\big)$ such that for any $x \in P_{(a,b)}$, $x = \sum\limits_i x_i f_i(x)$, where only finite $f_i(x)$ are not zero. As $P_{(a,b)}$ is projective, there exists a $\alpha : P_{(a,b)} \to P$ such that the following diagram

$$\begin{array}{ccccc} & & P_{(a,b)} & & \\ \alpha & \swarrow & \downarrow a & & \\ P & \xrightarrow{b} & bP & \to & 0 \end{array}$$

commutates, i.e, $a = b\alpha$. Since $x_i \in P_{(a,b)}$, there exists $p_i \in P$ such that $x_i = b(p_i)$ for each i. Hence $x = \sum\limits_i b(p_i)f_i(x) = b\big(\sum\limits_i p_i f_i(x)\big)$. Define a map $h : P_{(a,b)} \to P$ given by $h(p) = \sum\limits_i p_i f_i(p)$ for any $p \in P_{(a,b)}$. If $p = 0$, then each $f_i(p) = 0$; hence, $h(p) = 0$. That is, h is well defined, In addition, we get $1_{P_{(a,b)}} = bh$. One easily checks that $a\,|_{P_{(a,b)}} + b(1_P - a)h : P_{(a,b)} \to bP$ is a R-morphism. If $\big(a\,|_{P_{(a,b)}} + b(1_P - a)h\big)(p) = 0$ for a $p \in P_{(a,b)}$, then $a(p) + b(1_P - a)h(p) = 0$, and so $p = 0$. Thus, $a\,|_{P_{(a,b)}} + b(1_P - a)h : P_{(a,b)} \to bP$ is a R-monomorphism. Given any $bp \in bP$, we see that $bp \in P_{(a,b)}$. Furthermore, we have $\big(a\,|_{P_{(a,b)}} + b(1_P - a)h\big)(bp) = bp$, i.e., $a\,|_{P_{(a,b)}} + b(1_P - a)h : P_{(a,b)} \to bP$ is a R-epimorphism. It follows that $a\,|_{P_{(a,b)}} + b(1_P - a)h : P_{(a,b)} \to bP$ is a R-isomorphism. Therefore we complete the proof by Theorem 1. □

Corollary 3. *Every projective right module over a hereditary ring is c-separative.*

Proof. Let P be a projective right module over a hereditary ring R, and let $E = End_R(P)$. Since R is a hereditary ring, every submodule of P is projective. In view of Corollary 2, P is c-separative, as asserted. □

Let R be a hereditary ring and $aR + bR = R$ with $ab = ba$. Then $R \cong \{(\xi, \eta) | a\xi + b\eta = 0\}$. We construct a map $\varphi : R \oplus R \to R$ given by $\varphi(\xi, \eta) = a\xi + b\eta$ for any $(\xi, \eta) \in R \oplus R$. As $ab = ba$, we have that $R \oplus R \overset{c}{\cong} R \oplus Ker\varphi$. By virtue of Corollary 3, we see that $R \cong Ker\varphi$, and we are done.

Recall that a domain R is said to be a Dedekind domain in case it is hereditary. It is well known that every finitely generated module over a commutative Dedekind domain is cancellable (cf. [7, Theorem 5.8] and [10, Theorem 4.3.7].

Corollary 4. *Let P be a projective right module over a Dedekind domain R. Then P is c-separative.*

Proof. Since R is a Dedekind domain, it is a hereditary ring. Therefore the proof is true by Corollary 3. $\square$

A ring R is a semihereditary ring provided that every finitely generated right ideal of R is projective. Let P be a finitely generated projective right module over a semihereditary ring. Analogously to Corollary 3, we conclude that P is c-separative.

A ring R is said to have stable rank one provided that $aR+bR=R$ implies that there exists a $y\in R$ such that $a+by\in R$ is invertible. This condition plays an important role in algebraic K-theory (cf. [9]). It is well known that every strongly π-regular ring has stable rank one. Now we observe the following result on c-separativity.

Theorem 5. *Let P be a right R-module, and let $E=End_R(P)$. If $aE+bE=E$ with $ab=ba$ implies that there exists a $y\in E$ such that $a+by:P\to P$ is an isomorphism, then P is c-separative.*

Proof. Given $aE+bE=E$ with $ab=ba$, then there exists a $y\in E$ such that $a+by:P\to P$ is an isomorphism, Let $\tau=y|_{P_{a,b}}$. Then $a|_{P(a,b)}+b\tau:P_{(a,b)}\to bP$. Let $u=(a+by)^{-1}|_{bP}$. For any $bp\in bP$, we let $x=(a+by)^{-1}(bp)$. Then $(a+by)x=bp$, and so $ax=b(p-yx)\in bP$. This implies that $u\in P_{(a,b)}$, i.e., we get $u:bP\to P_{(a,b)}$. It is easy to check that

$$\big(a|_{P(a,b)}+b\tau\big)u=1_{P_{(a,b)}}\quad\text{and}\quad u\big(a|_{P(a,b)}+b\tau\big)=1_{bP}.$$

Therefore $a|_{P(a,b)}+b\tau:P_{(a,b)}\to bP$ is an isomorphism. According to Theorem 1, P is c-separative. $\square$

Corollary 6. *If $aR+bR=R$ with $ab=ba$ implies that there exists a $y\in R$ such that $a+by\in R$ is invertible, then $R\oplus R\overset{c}{\cong}R\oplus P$ implies that $R\cong P$.*

Proof. In view of Theorem 5, R is c-separative, and therefore we complete the proof. $\square$

Theorem 7. *Let P be a right R-module, $E=End_R(P)$, and let $Q\lesssim^{\oplus}P$. Suppose that $aE+bE=E$ with $ab=ba$ implies that there exists a right R-morphism $\tau:P_{(a,b)}\to P$ such that $a\,|_{P_{(a,b)}}+b\tau:P_{(a,b)}\to b(P)$ is a R-isomorphism. Then Q is c-separative.*

Proof. Let $S=End_R(Q)$. Then we can find an idempotent $e\in E$ such that $Q\cong eP$. It will suffice to prove that eP is c-separative. Let $S=eEe$. Given $cS+dS=S$ with $cd=dc$, then $cx+dy=e$ for some $x,y\in S$. Hence, $(c+1_P-e)(x+1_P-e)+dy=1_P$. Clearly, $(c+1_P-e)d=cd=dc=d(c+1_P-e)$. By assumption, we can find a right R-morphism $\tau:P_{(c+1_P-e,d)}\to P$ such that $(c+1_P-e)\,|_{P_{(c+1_P-e,d)}}+d\tau:P_{(c+1_P-e,d)}\to d(P)$ is a R-isomorphism. Clearly, $P_{(c+1_P-e,d)}=\{p\in P\mid (c+1_P-e)p\in d(P)\}=\{p\in P\mid cp\in d(P),p=ep\}$. Clearly, $(eP)_{(c,d)}\subseteq P_{(c+1_P-e,d)}$. Thus, we have $e\tau\,|_{(eP)_{(c,d)}}:(eP)_{(c,d)}\to eP$. On the other hand, $c\,|_{(eP)_{(c,d)}}+d(e\tau)\,|_{(eP)_{(c,d)}}:(eP)_{(c,d)}\to d(eP)$. If $\big(c\,|_{(eP)_{(c,d)}}+d(e\tau\,|_{(eP)_{(c,d)}})\big)(ep)=0$ with $ep\in(eP)_{(c,d)}$, then $ce(p)=d(eq)$ for some $q\in R$. Hence $(c+1-e)(ep)=d(eq)$, and so $ep\in P_{(c+1-e,d)}$. In addition, $\big((c+1_P-e)\,|_{P_{(c+1_P-e,d)}}+d\tau\big)(ep)=0$. This implies that $ep=0$, and then $c\,|_{(eP)_{(c,d)}}+d(e\tau\,|_{(eP)_{(c,d)}}):(eP)_{(c,d)}\to d(eR)$ is a right R-monomorphism. Given any $d(ep) \in d(eP)$, we can find a $t \in P_{(c+1_P-e,d)}$ such that $\big((c+1_P-e)\mid_{P_{(c+1_P-e,d)}} + d\tau\big)(t) = d(ep)$. Clearly, there exists some $s \in P$ such that $(c+1_P-e)(t) = ds$; hence, $c\big(e(t)\big) = d(es)$. It follows that $(1_P - e)(t) = 0$, and so $t = e(t)$. Thus, we see that $e(t) \in (eP)_{(c,d)}$. As a result, we deduce that $\big(c\mid_{(eP)_{(c,d)}} + d(e\tau\mid_{(eP)_{(c,d)}})\big)(e(t)) = d(ep)$. This means that $c\mid_{(eP)_{(c,d)}} + d(e\tau\mid_{(eP)_{(c,d)}}) : (eP)_{(c,d)} \to d(eP)$ is a R-epimorphism. Therefore $c\mid_{(eP)_{(c,d)}} + d(e\tau\mid_{(eP)_{(c,d)}}) : (eP)_{(c,d)} \to d(eP)$ is a R-isomorphism. In view of Theorem 1, Q is c-separative. $\square$

Corollary 8. *Let P be a right R-module,and let $Q \lesssim^{\oplus} P$. If every submodule of P is projective, then Q is c-separative.*

Proof. Let $E = End_R(P)$. As in proof of Corollary 2, $aE + bE = E$ with $ab = ba$ implies that there exists a right R-morphism $\tau : P_{(a,b)} \to P$ such that $a\mid_{P_{(a,b)}} + b\tau : P_{(a,b)} \to bP$ is a R-isomorphism. According to Theorem 7, we complete the proof. $\square$

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