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Oscillation Criteria of Hyperbolic Equations with Continuous Deviating Arguments

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Abstract. In this paper, we shall consider a class of hyperbolic nonlinear differential equations with continuous deviating arguments. Some new sufficient conditions for oscillation of all solutions with two kinds of boundary conditions are obtained.

1. Introduction

The study of oscillatory behavior of solutions of partial differential equations with deviating arguments, besides its theoretical interest, is important from the viewpoint of applications. Examples of applications can be found in [10]. But only a few results on the oscillatory behavior of hyperbolic equations with deviating arguments were recently obtained in [1]-[5] and the references cited therein. In this paper, we shall consider the nonlinear hyperbolic equation with continuous arguments

$$(E) \qquad \frac{\partial}{\partial t}[p(t)\frac{\partial}{\partial t}u(x,t)] = \alpha(t)\Delta u(x,t) + \int_a^b \beta(t,\xi)\Delta u[x,h(t,\xi)]d\sigma(\xi) - \int_a^b q(x,t,\xi)f(u[x,g(t,\xi)])d\sigma(\xi), \qquad (x,t) \in \Omega \times R_+ \equiv G,$$

where Ω is a bounded domain in R^n, $n \geq 1$, with a piecewise smooth bounded $\partial\Omega$, and Δu is the Laplacian in R^n, $R_+ = (0,\infty)$.

Throughout, we will assume that the following conditions hold:

- (H_1) $\alpha, p \in C(R_+, R_+)$, $\int^\infty \frac{1}{p(t)}dt = \infty$, $\sigma \in ([a,b], R)$ is nondecreasing, the integrals of the equation (E) are stieltjes integral.

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- (H_2) $\beta \in C(R_+ \times [a,b], R_+)$, $q \in C(\overline{G} \times [a,b], R_+)$, $f \in C(R,R)$ is convex in R_+, $uf(u) > 0$ and $\frac{f(u)}{u} \geq K > 0$ for $u \neq 0$.
- (H_3) $g,\ h \in C(R_+ \times [a,b], R)$, $\frac{d}{dt}g(t,a) \equiv g'(t,a)$ exists and $g(t,\xi) \leq t$, $h(t,\xi) \leq t$, for $\xi \in [a,b]$, g and h are nondecreasing with t and ξ, respectively, and $\lim_{t\to\infty} \min_{\xi\in[a,b]} g(t,\xi) = +\infty$, $\lim_{t\to\infty} \min_{\xi\in[a,b]} h(t,\xi) = +\infty$

we consider two kinds of boundary conditions:

$$(B_1) \qquad \frac{\partial u(x,t)}{\partial N} + \mu(x,t)u = 0 \qquad on \qquad (x,t) \in \partial\Omega \times R_+,$$

$$(B_2) \qquad u(x,t) = 0 \qquad on \qquad (x,t) \in \partial\Omega \times R_+,$$

where N is the unit exterior normal vector to $\partial\Omega$ and μ is a nonnegative continuous function on $\partial\Omega \times R_+$.

Definition 1. A function $u(x,t) \in C^2(\Omega \times [t_1,\infty), R) \bigcap C^1(\overline{\Omega} \times [t_{-1},\infty), R)$ is called a solution of the problem (E), (B), if it satisfies (E) in the domain G along with the corresponding boundary condition, where

$$t_{-1} = \min\{\inf_{\xi\in[a,b]}\{\inf_{t\geq 0} g(t,\xi)\}, \inf_{\xi\in[a,b]}\{\inf_{t\geq 0} h(t,\xi)\}\}.$$

Definition 2. A solution $u(x,t)$ of the problem (E), (B) is said to be oscillatory in the domain G, if for each positive number γ there exists a point $(x_1,t_1) \in \Omega\times[\gamma,\infty)$, where $u(x_1,t_1) = 0$.

Definition 3. A function $v(t)$ is called eventually positive (negative) if there exists a number $t_1 \geq t_0 > 0$ such that $v(t) > 0 (< 0)$ holds for all $t_1 \geq t_0$.

It is easy to see that equation (E) includes the following delay hyperbolic differential equations:

$$(E_1) \qquad \frac{\partial}{\partial t}[p(t)\frac{\partial}{\partial t}u(x,t)] = \alpha(t)\Delta u(x,t) + \sum_{j=1}^{m}\beta_j(t)\Delta u[x,h_j(t)]$$

$$-q(x,t)u(x,t) - \sum_{k=1}^{s} q_k(x,t)f(u[x,g_k(t)]), (x,t) \in G.$$

and we can see that the hyperbolic equation in [1-5] are special cases of equations (E) and (E_1). Our aim in this paper is to give some new oscillation criteria, Philos-type [6] oscillation criteria for equation (E) with the boundary conditions (B_1) and (B_2). Our results in this paper extend and improve the results in [1]-[5].

2. Main results

In this section we will give some oscillation criteria of (E) with the boundary conditions (B_1) and (B_2).

First, we consider the oscillation of the problem (E), (B_1).

Theorem 1. *Suppose that the condition $(H_1)-(H_3)$ hold. and let the differential inequality*

$$(p(t)v'(t))' + \int_a^b Q(t,\xi)f(v[g(t,\xi)])d\sigma(\xi) \le 0. \tag{1}$$

have no eventually positive solutions. Then each solution $u(x,t)$ of problem (E), (B_1) oscillates in the domain G, where

$$Q(t,\xi) = \min\{q(x,t,\xi) : x \in \overline{\Omega}\}, \tag{2}$$

$$v(t) = \frac{\int_\Omega u(x,t)dx}{\int_\Omega dx}. \tag{3}$$

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B_1). Without loss of generality, we assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0,\infty), t_0 > 0$. By condition (H_3) there exists $t_1 \ge t_0$ such that $g(t,\xi) \ge t_0$, $h(t,\xi) \ge t_0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$. then

$$u[x,h(t,\xi)] > 0 \;\; and \;\; u[x,g(t,\xi)] > 0 \;\; for \;\; (x,t,\xi) \in \Omega \times [t_1,\infty) \times [a,b]. \tag{4}$$

Integrating equation (E) with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}[p(t)\frac{d}{dt}\int_\Omega u(x,t)dx] \\ = \;& \alpha(t)\int_\Omega \Delta u(x,t)dx + \int_\Omega\int_a^b \beta(t,\xi)\Delta u[x,h(t,\xi)]d\sigma(\xi)dx \\ & - \int_\Omega\int_a^b q(x,t,\xi)f(u[x,g(t,\xi)])d\sigma(\xi)dx. \end{aligned} \tag{5}$$

Using Green′s formula and (B_1), we obtain

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N}ds = -\int_{\partial\Omega} \mu(x,t)u(x,t)ds \le 0, \quad t \ge t_1, \tag{6}$$

$$\begin{aligned} \int_\Omega \Delta u[x,h(t,\xi)]dx &= \int_{\partial\Omega} \frac{\partial u[x,h(t,\xi)]}{\partial N}ds \\ &= -\int_{\partial\Omega} \mu[x,h(t,\xi)]u[x,h(t,\xi)]ds \le 0, \quad t \ge t_1, \end{aligned} \tag{7}$$

where ds is the surface element on $\partial\Omega$, and

$$\begin{aligned} &\int_\Omega\int_a^b \beta(t,\xi)\Delta u[x,h(t,\xi)]d\sigma(\xi)dx \\ = \;& \int_a^b \beta(t,\xi)\Big(\int_\Omega \Delta u[x,h(t,\xi)]dx\Big)d\sigma(\xi) \le 0, \quad t \ge t_1. \end{aligned} \tag{8}$$

From Jensen ′ s inequality and (H_2) ,we have

$$\begin{aligned} (9) \quad & \int_\Omega \int_a^b q(x,t,\xi) f(u[x,g(t,\xi)]) d\sigma(\xi) dx \\ = & \int_a^b \int_\Omega q(x,t,\xi) f(u[x,g(t,\xi)]) dx d\sigma(\xi) \\ \geq & \int_a^b Q(t,\xi) \Big[\int_\Omega f(u[x,g(t,\xi)]) dx \Big] d\sigma(\xi) \\ \geq & \int_a^b Q(t,\xi) \Big[\int_\Omega dx f(\frac{\int_\Omega u[x,g(t,\xi)] dx}{\int_\Omega dx}) \Big] d\sigma(\xi), \quad t \geq t_1. \end{aligned}$$

Therefore, combining (5)-(9), we obtain

$$(10) \qquad (p(t)v'(t))' + \int_a^b Q(t,\xi) f(v[g(t,\xi)]) d\sigma(\xi) \leq 0, \quad t \geq t_1.$$

It is easy to see that $v(t)$ is a eventually positive solution of (10), which contradicts the condition of the theorem. □

Next, we present some new oscillation criteria for (E) and (B_1) by using integral averages condition of Philos-type. Following Philos [6], we introduce a class of functions P. Let

$$(11) \qquad D_0 = \{(t,s) : t > s \geq t_0\}, \quad \textit{and} \quad D = \{(t,s) : t \geq s \geq t_0\}.$$

The function $H \in C(D,R)$ is said to belong to the class P if

(T_1) $H(t,t) = 0$ for $t \geq t_0, H(t,s) > 0$ on D_0;

(T_2) H has a continuous and nonpositive partial deviative on D_0 with respect to the second variable and there exist functions $h \in C(D_0,R)$ and $\rho \in C^1([t_0,\infty),R_+)$ such that

$$(12) \qquad -\frac{\partial H(t,s)}{\partial s} - \frac{\rho'(s)}{\rho(s)} H(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in D_0.$$

Theorem 2. *Suppose that $(H_1) - (H_3)$ hold. If there exists a function $\rho \in C^1([t_0,\infty),R_+)$ and let H belong to the class P such that*

$$(13) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t [H(t,s)\rho(s)Q(s) - \frac{\rho(s)p[g(s,a)]}{4g'(s,a)} h^2(t,s)] ds = \infty,$$

where

$$(14) \qquad Q(s) = \int_a^b K Q(s,\xi) d\sigma(\xi).$$

Then each solution of (E), (B_1) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B_1). Without loss of generality, we assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$. By condition (H_3) there exists a $t_1 \geq t_0$ such that (4) holds. From the proof of Theorem 1, we have the inequality (10), and by condition (H_2), we obtain

$$(p(t)v'(t))' + \int_a^b KQ(t,\xi)v[g(t,\xi)]d\sigma(\xi) \leq 0, \quad t \geq t_1, \tag{15}$$

where $Q(t,\xi)$ and $v(t)$ are defined by (2) and (3). It is easy to know that $v(t) > 0$, $v'(t) > 0$ for $t \geq t_1$, and $g(t,\xi)$ is nondecreasing in ξ, we have

$$(p(t)v'(t))' + Q(t)v[g(t,a)] \leq 0, \quad for \quad t \geq t_1, \tag{16}$$

where

$$Q(t) = \int_a^b KQ(t,\xi)d\sigma(\xi).$$

Set

$$W(t) = \rho(t)\frac{p(t)v'(t)}{v[g(t,a)]} \quad for \quad t \geq t_1, \tag{17}$$

then $W(t) > 0$ for $t \geq t_1$. From (17), (16) and (H_3), we obtain

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)}W(t) - \rho(t)Q(t) - \frac{g'(t,a)}{\rho(t)p[g(t,a)]}W^2(t), \quad t \geq t_1. \tag{18}$$

In order to simplify notations we denote by

$$R(t) = \frac{g'(t,a)}{\rho(t)p[g(t,a)]}.$$

Then from (18) for all $t \geq t_1$ we have

$$\begin{aligned}
(19) \quad & \int_{t_1}^t H(t,s)\rho(s)Q(s)ds \\
\leq & \int_{t_1}^t H(t,s)\frac{\rho'(s)}{\rho(s)}W(s)ds - \int_{t_1}^t H(t,s)W'(s)ds - \int_{t_1}^t H(t,s)R(s)W^2(s)ds \\
= & \; H(t,t_1)W(t_1) + \int_{t_1}^t [\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)}H(t,s)]W(s)ds - \int_{t_1}^t H(t,s)R(s)W^2(s)ds \\
= & \; H(t,t_1)W(t_1) - \int_{t_1}^t h(t,s)\sqrt{H(t,s)}W(s)ds - \int_{t_1}^t H(t,s)R(s)W^2(s)ds \\
= & \; H(t,t_1)W(t_1) - \int_{t_1}^t [\sqrt{H(t,s)R(S)}W(s) + \frac{h(t,s)}{2\sqrt{R(s)}}]^2 ds + \int_{t_1}^t \frac{h^2(t,s)}{4R(s)}ds.
\end{aligned}$$

Thereby, we conclude that

$$\begin{aligned}(20)\qquad & \int_{t_1}^{t}[H(t,s)\rho(s)Q(s)-\frac{h^2(t,s)}{4R(s)}]ds\\ \leq\ & H(t,t_1)W(t_1)-\int_{t_1}^{t}[\sqrt{H(t,s)R(s)}W(s)+\frac{h(t,s)}{2R(s)}]^2ds\\ \leq\ & H(t,t_1)|W(t_1)|.\end{aligned}$$

Then by (21) and (T_2), we have

$$(21)\qquad \frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\rho(s)Q(s)-\frac{h^2(t,s)}{4R(s)}]ds \leq \int_{t_0}^{t_1}\rho(s)Q(s)ds+|W(t_1)|.$$

Inequality (21) yields

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\rho(s)Q(s)-\frac{h^2(t,s)}{4R(s)}]ds<\infty,$$

and the latter inequality contradicts assumption (13). If $u(x,t)<0$ for $\Omega\times[t_0,\infty)$, then $-u(x,t)$ is a positive solution of (E), (B_1) and the proof is similar. This completes the proof. □

The following oscillation criterion treats the case when it is not possible to verify easily condition (13).

Theorem 3. *Suppose that $(H_1)-(H_3)$ hold. Let the differentiable function ρ as in Theorem 2 and let H belong to the class P such that*

$$(22)\qquad 0<\inf_{s\geq t_0}[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}]\leq\infty.$$

Let $\varphi\in C\Big([t_0,\infty),R\Big)$ such that for $t\geq t_1$

$$(23)\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{h^2(t,s)}{R(s)}ds<\infty,$$

$$(24)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}\varphi_+^2(s)R(s)ds=\infty,$$

and

$$(25)\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\rho(s)Q(s)-\frac{h^2(t,s)}{4R(s)}]ds\geq\sup_{t\geq t_0}\varphi(t),$$

where $Q(s)$ and $R(s)$ as in Theorem 2, $\varphi_+(t)=\max\{\varphi(t),0\}$, then each solution of (E), (B_1) is oscillatory in G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B_1). Without loss of generality, we assume that $u(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$. By condition (H_3) there exists a $t_1 \geq t_0$ such that the inequalities (4) hold. By Theorem 2 we have (20). The inequality (20) yields

$$\begin{aligned} & \frac{1}{H(t,t_1)} \int_{t_1}^{t} [H(t,s)\rho(s)Q(s) - \frac{h^2(t,s)}{4R(s)}] ds \\ \leq \; & W(t_1) - \frac{1}{H(t,t_1)} \int_{t_1}^{t} [\sqrt{H(t,s)R(s)}W(s) + \frac{h(t,s)}{4R(s)}]^2 ds, \quad t \geq t_1. \end{aligned}$$

Hence, for $t \geq t_1$

$$\begin{aligned} & \limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} [H(t,s)\rho(s)Q(s) - \frac{h^2(t,s)}{4R(s)}] ds \\ \leq \; & W(t_1) - \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} [\sqrt{H(t,s)R(s)}W(s) + \frac{h(t,s)}{2\sqrt{R(s)}}]^2 ds. \end{aligned}$$

By (25) and the last inequality, we obtain for $t \geq t_1$

$$W(t_1) \geq \varphi(t_1) + \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} [\sqrt{H(t,s)R(s)}W(s) + \frac{h(t,s)}{2\sqrt{R(s)}}]^2 ds, \tag{26}$$

and hence

$$\begin{aligned} 0 \; & \leq \; \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} [\sqrt{H(t,s)R(s)}W(s) + \frac{h(t,s)}{2\sqrt{R(s)}}]^2 ds \\ & \leq \; W(t_1) - \varphi(t_1) < \infty. \end{aligned} \tag{27}$$

Define the functions $M(t)$ and $N(t)$ as follows

$$M(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)R(s)W^2(s) ds,$$

$$N(t) = \frac{1}{H(t,t_1)} \int_{t_1}^{t} [\sqrt{H(t,s)}h(t,s)W(s) ds.$$

The remainder of the proof is similar to that the proof of Theorem 2.6 in [7] and hence is omitted. □

Now, we consider the oscillation of the problem (E), (B_2). consider the Dirichlet Problem in the domain Ω

$$\Delta u + \lambda u = 0 \quad \text{in} \quad (x,t) \in \Omega \times R_+, \tag{28}$$

$$u = 0 \quad \text{on} \quad (x,t) \in \partial\Omega \times R_+, \tag{29}$$

in which λ is a constant. It is well know [8] that the smallest eigenvalue λ_1 of problem (28)-(29) is positive and the corresponding eigenfunction $\Psi(x)$ is also positive for $x \in \Omega$.

With each solution $u(x,t)$ of the problem (E), (B_2), we associate a function $U(t)$ defined by

$$U(t) = \frac{\int_\Omega u(x,t)\Psi(x)dx}{\int_\Omega \Psi(x)dx}, \quad t \geq t_1. \tag{30}$$

Theorem 4. *If all conditions of Theorem 2 hold, then each solution of the problem (E), (B_2) is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (E), (B_2). Without loss of generality, we assume that $u(x,t) > 0$ for $(x,t) \in \Omega \times [t_0, \infty)$. By the condition (H_3) there exists a $t_1 \geq t_0$ such that (4) holds. Multiplying both sides of equation (E) by $\Psi(x)$, and integrating equation (E) with respect to x over the domain Ω, we have

$$\begin{aligned} &\frac{d}{dt}[p(t)\frac{d}{dt}\int_\Omega u(x,t)\Psi(x)dx] \\ = \ &\alpha(t)\int_\Omega \Delta u(x,t)\Psi(x)dx + \int_\Omega\int_a^b \beta(t,\xi)\Delta u[x,h(t,\xi)]\Psi(x)d\sigma(\xi)dx \\ &-\int_\Omega\int_a^b q(x,t,\xi)f(u[x,g(t,\xi)])\Psi(x)d\sigma(\xi)dx. \end{aligned} \tag{31}$$

Using Green′s formula and (B_2), we obtain

$$\begin{aligned} \int_\Omega \Delta u(x,t)\Psi(x)dx &= \int_{\partial\Omega}(\Psi(x)\frac{\partial u}{\partial N} - u\frac{\partial\Psi(x)}{\partial N})ds + \int_\Omega u\Delta\Psi(x)dx \\ &= -\lambda_1\int_\Omega u(x,t)\Psi(x)dx, \qquad t \geq t_1, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\int_\Omega\int_a^b \beta(t,\xi)\Delta u[x,h(t,\xi)]\Psi(x)d\sigma(\xi)dx \\ = \ &\int_a^b \beta(t,\xi)\int_\Omega \Delta u[x,h(t,\xi)]\Psi(x)dxd\sigma(\xi) \\ = \ &-\lambda_1\int_a^b \beta(t,\xi)\int_\Omega u[x,h(t,\xi)]\Psi(x)dxd\sigma(\xi), \quad t \geq t_1, \end{aligned} \tag{33}$$

where λ_1 is the smallest eigenvalue of problem (28)-(29). Using Jensen′s inequality and (H_2), we have

$$\begin{aligned} (34) \qquad & \int_\Omega \int_a^b q(x,t,\xi) f(u[x,g(t,\xi)])\Psi(x)d\sigma(\xi)dx \\ = \ & \int_a^b \int_\Omega q(x,t,\xi) f(u[x,g(t,\xi)])\Psi(x)dx d\sigma(\xi) \\ \geq \ & \int_a^b Q(t,\xi) \int_\Omega f(u[x,g(t,\xi)])\Psi(x)dx d\sigma(\xi) \\ \geq \ & \int_a^b Q(t,\xi) \Big[\int_\Omega \Psi(x)dx \cdot f\Big(\frac{\int_\Omega u[x,g(t,\xi)]\Psi(x)dx}{\int_\Omega \Psi(x)dx}\Big)\Big] d\sigma(\xi), \quad t \geq t_1. \end{aligned}$$

Therefore, from (31)-(34), we obtain for $t \geq t_1$

$$\begin{aligned} (35) \qquad & (p(t)U'(t))' + \lambda_1 \alpha(t)U(t) + \lambda_1 \int_a^b \beta(t,\xi)U[h(t,\xi)]d\sigma(\xi) \\ & + \int_a^b Q(t,\xi) f(U[g(t,\xi)])d\sigma(\xi) \leq 0. \end{aligned}$$

In view of (H_2) and (4), inequality (35) yields

$$(10) \qquad (p(t)U'(t))' + \int_a^b Q(t,\xi) f(U[g(t,\xi)])d\sigma(\xi) \leq 0.$$

The remainder of the proof is similar to that of Theorem 2. □

The following theorem is immediate from Theorem 3 and 4.

Theorem 5. *If all conditions of Theorem 3 hold, then every solution of the problem* (E), (B_2) *is oscillatory in* G.

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