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Asymptotic Behaviour of Another Second Order Mock Theta Function

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Abstract. In this paper we show that the second order mock theta function, given by Hikami, is bounded and satisfies the second condition for the mock theta functions.

1. Introduction

Ramanujan's last gift to mathematics was Mock Theta Functions. Ramanujan introduced them in his last letter to Hardy, dated January, 1920. Photocopy of this letter can be found in [5, pp 127-131]. In the letter Ramanujan gave a list of 17 mock theta functions and identities they satisfy. He divided these mock theta functions into "third order", "fifth order" and "seventh order" but did not say what he meant. He did not give any formal definition of "order", but the identities for these mock theta functions are related to the numbers 3, 5, 7. Andrews and Hickerson [1] discovered seven mock theta functions in Ramanujan's "lost" notebook and called them of order six. They called them of "sixth order", considering the combinatorial interpretation of the coefficients $\varphi(q)$ and $\psi(q)$(two of the seven functions found in the "lost" notebook).

Recently Gordon and McIntosh [4] constructed eight mock theta functions and called them of "eighth order". They also gave a formal definition of the order.

In his letter Ramanujan explained what he meant by a mock theta function. In [1] we find a formal definition, slightly rephrased, it is : A mock theta function is a function f of the complex variable q, defined by a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

(i) infinitely many roots of unity are exponential singularities,

(ii) for every root of unity ξ there is a theta function $\theta_\xi(q)$ such that the difference $f(q)- \theta_\xi(q)$ is bounded as $q \to \xi$ radially (presumably with only finitely many of the θ_ξ being different),

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(iii) there is no theta function that works for all ξ, i.e., f is not the sum of two functions one of which is a theta function and the other a function which is bounded in all roots of unity.

The mock theta functions did satisfy conditions (i) and (ii). However, no proof has ever been given that they also satisfy condition (iii). Watson [6] proved a very weak form of condition (iii) for "third order" mock theta function, that they are not theta functions. Recently Hikami [2] gave a mock theta function and called it of second order. Obviously it satisfies the condition (i). In this paper we show that this second order mock theta function satisfies condition (ii) also. In a subsequent paper we will show that the second order mock theta functions and sixth order mock theta functions will not satisfy condition (iii) if we make the condition stronger.

2. Notations

We shall use the following usual basic hypergeometric notations:

For $\left|q^k\right| < 1$,

$$\begin{aligned} (a;q^k)_n &= (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)}),\ n \geq 1, \\ (a;q^k)_0 &= 1, \\ (a;q^k)_\infty &= \prod_{j=0}^{\infty}(1-aq^{kj}), \\ (a;q)_n &= (a)_n. \end{aligned}$$

3. Definition

The second order mock theta function considered by Hikami [2] is

$$\mathfrak{D}_5(q) = \sum_{n=0}^{\infty} \frac{(-q;q)_n}{(q;q^2)_{n+1}} q^n. \tag{1}$$

There are several ways to prove that the mock theta functions satisfy condition (ii). We shall follow the method given by Watson for the fifth order mock theta functions [7, sec. 6]. Andrews and Hickerson [1] closely follows Watson's method. We also follow their method. For showing that $\mathfrak{D}_5(q)$ satisfies condition (ii), we will require the following lemmas of Andrews and Hickerson [1].

Lemma 1. *Suppose that, for each $n \geq 0$, $U_n(r)$ is a bounded function for $a \leq r \leq b$. Suppose further that there exist integers $N \geq 1$ and $K \geq 0$ and a positive real number $\alpha < 1$ such that $|U_{n+N}(r)| \leq \alpha\, |U_n(r)|$, for all $n \geq K$ and $a \leq r \leq b$. Then $\sum_{n=0}^{\infty} |U_n(r)|$ converges and is bounded for $a \leq r \leq b$.*

Lemma 2. *Let $0 < R' \leq R \leq 1$ and $|z| = 1$, then $|1 + Rz| \leq \sqrt{R/R'}\, |1 + R'z|$.*

Lemma 3. *If* $a > 0$, $b > 0$ *and* $0 \le r \le 1$, *then* $r^a(1-r^b) < \frac{b}{a+b}$.

4. The Theorem

To show that $\mathfrak{D}_5(q)$ satisfies condition (ii), we first show $\mathfrak{D}_5(q)$ is bounded and we state this as a theorem.

Theorem. *Let* $\zeta = e^{2\pi i \frac{h}{k}}$ *where* h, k *are integers expressed in their lowest form,* $(h,k)=1$ *and* h *an odd integer and* k *an even integer and* $q = \rho e^{2\pi i \frac{h}{k}}$, *then* $\mathfrak{D}_5(q)$ *is bounded for* $0 \le \rho \le 1$.

Proof. Let

$$U_n(q) = \frac{(-q;q)_n}{(q;q^2)_{n+1}} q^n. \tag{2}$$

Then

$$\begin{aligned} |U_{n+k}(q)| &= \frac{|q|^{n+k} \left|(-q;q)_{n+k}\right|}{\left|(q;q^2)_{n+k+1}\right|} \\ &= \frac{q^k \left|(-q^{n+1};q)_N\right|}{\left|(q^{2n+3};q^2)_k\right|} \left|U_n(q)\right|. \end{aligned}$$

Let $q = e^{2\pi i \frac{h}{k}}$, h is odd and k is even, $(h,k)=1$. Then

$$|U_{n+k}(q)| = \frac{\left|\prod_{p=0}^{k}(1+\rho^{(n+p)} e^{2\pi i (n+p)\frac{h}{k}}\right| \rho^k}{\left|\prod_{p=0}^{k-1}(1-\rho^{2n+3+2p} e^{2\pi i (\frac{h}{k})(2n+3+2p)})\right|} \left|U_n(q)\right|.$$

We estimate the denominator first.

Taking $R = \rho^{2n+3}$, $R' = \rho^{2n+3+2p}$, $z = -\ e^{2\pi i (\frac{h}{k})(2n+3+2p)}$ in Lemma 2, we have

$$\begin{aligned} \left|\prod_{p=0}^{k-1}(1-\rho^{2n+3+2p} e^{2\pi i (\frac{h}{k})(2n+3+2p)})\right| &\ge \prod_{p=0}^{k-1} \rho^p \left|1-\rho^{2n+3} e^{2\pi i (\frac{h}{k})(2n+3+2p)}\right| \\ &= \rho^{\frac{k(k-1)}{2}} \left|\prod_{p=0}^{k-1}(1-\rho^{2n+3} e^{2\pi i (\frac{h}{k})(2n+3+2p)})\right|. \end{aligned}$$

As p ranges from 0 to $k-1$, $1-\rho^{2n+3}e^{2\pi i (\frac{h}{k})(2n+3+2p)}$ runs twice through the roots of the polynomial $(x-1)^{\frac{k}{2}} + (-1)^{\frac{k}{2}} \rho^{\frac{k(2n+3)}{2}}$, and the product of these roots is $2(1+\rho^{\frac{k(2n+3)}{2}}) > 2$.

Hence

$$\left|\prod_{p=0}^{k-1}(1-\rho^{2n+3+2p} e^{2\pi i (\frac{h}{k})(2n+3+2p)})\right| > 2\rho^{\frac{k(k-1)}{2}}. \tag{3}$$

We now estimate the product in the numerator.

Taking $R=\rho^{n+p}$, $R'=\rho^{n+k}$, $z=e^{2\pi i(\frac{h}{k})(n+p)}$ in Lemma 2, the product in the numerator becomes

$$(4)\qquad \left|\prod_{p=0}^{k}(1+\rho^{n+p}e^{2\pi i(\frac{h}{k})(n+p)})\right| \le \prod_{p=1}^{k}\rho^{\frac{(p-k)}{2}}\left|1+\rho^{n+k}e^{2\pi i(\frac{h}{k})(n+p)}\right| = \rho^{\frac{k(1-k)}{4}}\left|\prod_{p=1}^{k}(1+\rho^{n+k}e^{2\pi i(\frac{h}{k})(n+p)})\right|.$$

As p ranges from 1 to k, $e^{2\pi i(\frac{h}{k})(n+p)}$ runs through the k^{th} roots of 1, therefore $1+\rho^{n+k}e^{2\pi i(\frac{h}{k})(n+p)}$ runs through the roots of the polynomial

$$(5)\qquad (x-1)^k-\rho^{k(n+k)},$$

and the product of these roots is $(-1)^k$ multiplied by the coefficient of x^0 in (5). Since k is even the product will be equal to $1-\rho^{k(n+k)}$. Hence

$$\left|\prod_{p=0}^{k}(1+\rho^{(n+p)}e^{2\pi i(\frac{h}{k})(n+p)})\right| \le \rho^{\frac{k(1-k)}{4}}\ (1-\rho^{k(n+k)}).$$

Hence for $0<\rho\le 1$

$$\begin{aligned}
|U_{n+k}(\rho)| &\le \frac{\rho^{\frac{k(1-k)}{4}}\ (1-\rho^{k(n+k)})\rho^k}{2\rho^{\frac{k(k-1)}{2}}}\,|U_n(\rho)| \\
&= \frac{1}{2}\rho^{\frac{(-3k^2+7k)}{4}}\ (1-\rho^{k(n+k)})\,|U_n(\rho)| \\
&\le \frac{k(n+k)}{2(\frac{-3k^2+7k}{4}+nk+k^2)}\,|U_n(\rho)|\,,\ \text{by lemma 3} \\
&= \frac{4(n+k)}{2(k+4n+7))}\,|U_n(\rho)| \\
&\le \frac{4}{5}\,|U_n(\rho)|\,,
\end{aligned}$$

provided $n\ge k$. The result is true for $\rho=0$ also.

Hence by lemma 1

$$\sum_{n=0}^{\infty}|U_n(\rho)|\ \text{is bounded for}\ 0\le\ \rho\le 1.$$

Moreover

$$\mathfrak{D}_5(q)=\sum_{n=0}^{\infty}U_n(\rho),$$

so $\mathfrak{D}_5(q)$ is bounded. □

We have shown that for k an even integer, $\mathfrak{D}_5(q)$ is bounded so we may take $\theta_k(q) = 0$. Hence the second condition is satisfied.

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