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Integrability and L^1-convergence of Certain Cosine Sums

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Abstract. In this paper, we extend the result of Ram [3] and also study the L^1-convergence of the r^{th} derivative of cosine series.

1. Introduction

Consider cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx. \tag{1.1}$$

Let the partial sum of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$. Further, let $f^r(x) = \lim_{n\to\infty} S_n^r(x)$ where $S_n^r(x)$ represents r^{th} derivative of $S_n(x)$.

Definition ([6]). A null sequence $\{a_k\}$ is said to belong to class S if there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, \quad k \to \infty, \tag{1.2}$$

$$\sum_{k=0}^{\infty} A_k < \infty, \tag{1.3}$$

and

$$|\Delta a_k| \le A_k, \quad \forall\ k. \tag{1.4}$$

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Concerning the L^1-convergence of Rees-Stanojevic sums [4]

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx. \tag{1.5}$$

Ram [3] proved the following result:

Theorem A. *If* (1.1) *belongs to class S. Then* $||f - g_n||_{L^1} = o(1), \quad n \to \infty.$

Recently, Tomovski [7] extended the Sidon class to a new class $S_r, \ r = 1, 2, 3, \cdots$ as follows:

Definition. A null sequence $\{a_k\}$ is said to belong to class S_r if there exists a sequence $\{A_k\}$ such that

$$A_k \downarrow 0, \quad k \to \infty, \tag{1.6}$$

$$\sum_{k=0}^{\infty} k^r A_k < \infty, \tag{1.7}$$

and

$$|\Delta a_k| \leq A_k, \quad \forall\ k. \tag{1.8}$$

Clearly $S_{r+1} \subset S_r, \quad \forall\ r = 1, 2, 3, \cdots$.

Note that by $A_k \downarrow 0, \quad k \to \infty$ and $\sum_{k=0}^{\infty} k^r A_k < \infty$, we have

$$k^{r+1} A_k = o(1), \quad k \to \infty.$$

For $r = 0$, this class reduces to class S.

The aim of this paper is to generalize Theorem A for the cosine series with extended class $S_r, \ r = 1, 2, 3, \cdots$ of coefficient sequences and also to study the L^1-convergence of the r^{th} derivative of cosine series.

2. Lemma

The proofs of our results are based on the following lemmas:

Lemma 2.1 ([2]). *If* $|a_k| \leq 1$, *then*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k\ D_k(x) \right| dx \leq C(n+1),$$

where C is positive absolute constant.

Lemma 2.2. *Let $\{a_k\}$ be a sequence of real numbers such that $|a_k| \leq 1, \ \forall\ k$. Then there exists a constant $C > 0$ such that for any $n \geq 0$ and $r = 0, 1, 2, 3, \cdots$*

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k \ D_k^r(x) \right| \ dx \leq C(n+1)^{r+1}.$$

Proof. We note that $\sum_{k=0}^{n} a_k \ D_k(x)$ is a cosine trigonometric polynomial of order n. Applying first Bernstein's inequality ([8] , vol. II, p. 11) and then using lemma 2.1, we have

$$\int_0^{\pi} \left| \sum_{k=0}^{n} a_k \ D_k^r(x) \right| \ dx \leq (n+1)^r \int_0^{\pi} \left| \sum_{k=0}^{n} a_k \ D_k(x) \right| \ dx \leq C(n+1)^{r+1},$$

where $C > 0$. □

Lemma 2.3 ([5]). $||D_n^r(x)||_{L^1} = O(n^r \log n), \ r = 0, 1, 2, 3, \cdots,$ *where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet-Kernel.*

3. Results

Theorem 3.1. *If* (1.1) *belongs to class S_r, then* $||f - g_n||_{L^1} = o(1), \quad n \to \infty.$

Proof. Consider,

$$\begin{aligned} g_n(x) &= \frac{1}{2} \sum_{k=0}^{n} \ \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle a_j \cos kx \\ &= \sum_{k=1}^{n} a_k \cos kx - a_{n+1} D_n(x) \end{aligned}$$

Thus, $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} S_n(x) = f(x)$ (since, $D_n(x)$ is bounded in $(0, \pi)$ and $\{a_k\} \in S_r$).
Now, we consider

$$f(x) - g_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1} D_n(x)$$

Making use of Abel's transformation and lemma 2.1, we have

$$\begin{aligned}
\int_0^{\pi} |f(x) - g_n(x)|dx &= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \triangle a_k D_k(x) \right| dx \\
&= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\triangle a_k}{A_k} D_k(x) \right| dx \\
&\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| dx \\
&\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \frac{k^r}{k^r} \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{(n+1)^r} \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} k^r \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} D_i(x) \right| dx \\
&\leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \triangle A_k
\end{aligned}$$

(1.6) and (1.7) now imply the conclusion of the Theorem 3.1. □

Corollary. *If* (1.1) *belongs to class* S_r, $r = 1, 2, 3, \cdots$ *then* $||f - S_n||_{L^1} = o(1)$, $n \to \infty$ *if and only if* $a_n \log n = o(1)$, $n \to \infty$.

Proof. Consider,

$$\begin{aligned}
||f - S_n|| &= ||f - g_n + g_n - S_n|| \\
&\leq ||f - g_n|| + ||g_n - S_n|| \\
&= ||f - g_n|| + ||a_{n+1} D_n(x)|| \\
&\leq ||f - g_n|| + |a_{n+1}| \int_0^{\pi} |D_n(x)| \, dx
\end{aligned}$$

Further, $||f - g_n||_{L^1} = o(1)$, $n \to \infty$ (by Theorem 3.1) and $||D_n(x)|| = O(log\ n)$ (by Zygmund's Theorem ([1], p. 458)).
The conclusion of corollary follows. □

Remark. Case $r = 0$ yields the result of B. Ram [3].

Theorem 3.2. *If* (1.1) *belongs to class* S_r, *then* $||f^r - {g_n}^r||_{L^1} = o(1)$, $n \to \infty$.

Proof. Consider,

$$g_n(x) = S_n(x) - a_{n+1} D_n(x)$$

We have then

$$g_n{}^r(x) = S_n{}^r(x) - a_{n+1} D_n{}^r(x) \tag{3.1}$$

Where ${g_n}^r(x)$ represents the r^{th} derivative of $g_n(x)$ and ${D_n}^r(x)$ represents the r^{th} derivative of Dirichlet kernel. Since $\{a_k\}$ is a null sequence and ${D_n}^r(x)$ is bounded in $(0, \pi)$.
Therefore,

$$\lim_{n\to\infty} {g_n}^r(x) = \lim_{n\to\infty} {S_n}^r(x) = f^r(x)$$

For $x \neq 0$, it follows from (3.1) that

$$f^r(x) - {g_n}^r(x) = \sum_{k=n+1}^{\infty} a_k k^r \cos\left(kx + \frac{r\pi}{2}\right) + a_{n+1}{D_n}^r(x)$$

Making use of Abel's transformation, we get

$$f^r(x) - {g_n}^r(x) = \sum_{k=n+1}^{\infty} \triangle a_k {D_k}^r(x)$$

Now consider,

$$\begin{aligned}
\int_0^\pi |f^r(x) - {g_n}^r(x)|\, dx &= \int_0^\pi \left|\sum_{k=n+1}^{\infty} \triangle a_k {D_k}^r(x)\right| dx \\
&= \int_0^\pi \left|\sum_{k=n+1}^{\infty} A_k \frac{\triangle a_k}{A_k} {D_k}^r(x)\right| dx \\
&\leq \int_0^\pi \left|\sum_{k=n+1}^{\infty} \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} {D_i}^r(x)\right| dx \\
&\leq \int_0^\pi \left|\sum_{k=n+1}^{\infty} \triangle A_k \sum_{i=0}^{k} \frac{\triangle a_i}{A_i} {D_i}^r(x)\right| dx \\
&\leq C \sum_{k=n+1}^{\infty} (k+1)^{r+1} \triangle A_k
\end{aligned}$$

(1.6), (1.7) and lemma 2.2 imply the conclusion of the Theorem (3.2). □

Corollary. *If* (1.1) *belongs to class* S_r, $r = 1, 2, 3, \cdots$ *then* $||f^r - {S_n}^r||_{L^1} = o(1)$, $n \to \infty$ *if and only if* $a_n n^r \log n = o(1)$, $n \to \infty$.

Proof. Consider,

$$\begin{aligned}
||f^r - {S_n}^r|| &= ||f^r - {g_n}^r + {g_n}^r - {S_n}^{|}| \\
&\leq ||f^r - {g_n}^r|| + ||{g_n}^r - {S_n}^r|| \\
&= ||f^r - {g_n}^r|| + ||a_{n+1}{D_n}^r(x)|| \\
&\leq ||f^r - {g_n}^r|| + |a_{n+1}| \int_0^\pi |{D_n}^r(x)|
\end{aligned}$$

Since, $||f^r - {g_n}^r||_{L^1} = o(1), \quad n \to \infty$ (by Theorem 3.2) and using lemma 2.3, we get the conclusion of corollary. □

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