KYUNGPOOK Math. J. 47(2007), 311-322

Three Characteristic Beltrami System in Even Dimensions (I): p-Harmonic Equation

Gao Hongya
College of Mathematics and Computer Science, Hebei University, Baoding, 071002, P.R. China
e-mail : hongya-gao@sohu.com

Chu Yuming
Department of Mathematics, Huzhou Normal College, Huzhou, 313000, P.R. China
e-mail : chuyuming@hutc.zj.cn

Sun Lanxiang
Department of Mathematics, Cangzhou Teacher's College, Cangzhou, 061001, P.R. China
e-mail : lanxiangsun@sina.com

Abstract. This paper deals with space Beltrami system with three characteristic matrices in even dimensions, which can be regarded as a generalization of space Beltrami system with one and two characteristic matrices. It is transformed into a nonhomogeneous p-harmonic equation $d^*A(x, df^I) = d^*B(x, Df)$ by using the technique of outer differential forms and exterior algebra of matrices. In the process, we only use the uniformly elliptic condition with respect to the characteristic matrices. The Lipschitz type condition, the monotonicity condition and the homogeneous condition of the operator A and the controlled growth condition of the operator B are derived.

1. Introduction and statement of result

For Ω a bounded domain in $\mathrm{R}^n (n \geq 2)$, we consider a mapping $f = (f^1, f^2, \cdots, f^n) \in W^{1,n}_{loc}(\Omega, \mathrm{R}^n)$. The differential $Df(x) = \left(\frac{\partial f^i}{\partial x_j}\right)_{1\leq i,j\leq n}$ and its determinant $J_f(x) = \det Df(x)$ are, therefore, defined almost everywhere in Ω. Throughout this paper, we assume that $J_f(x)$ is non-negative, that is, f is sense-preserving. For a matrix A, we define the operator norm of it by $|A| = \sup_{|h|=1} |Ah|$.

Received September 20, 2004, and, in revised form, August 16, 2005.
2000 Mathematics Subject Classification: 30C62.
Key words and phrases: Beltrami system with three characteristic matrices, p-harmonic equation.
Research supported by NSFC (10471039), Special Fund of Mathematics Research of Natural Science Foundation of Hebei Province (07M003), and the Doctoral Foundation of the Department of Education of Hebei Province (B2004103).

Definition. A mapping $f = (f^1, f^2, \cdots, f^n) \in W^{1,n}_{loc}(\Omega, \mathrm{R}^n)$ is said to be K-*quasiregular*, $1 \le K < \infty$, if

$$|Df(x)|^n \le KJ_f(x) \tag{1.1}$$

for almost every $x \in \Omega$. If, in addition, f is homeomorphic, then it is called K-*quasiconformal*.

The theory of higher dimensional quasiregular mappings began with Yu. G. Reŝhetnyak's theorem, stating that they are continuous, discrete and open, if they are nonconstant, see [9]. Development of the analytic theory of quasiregular mappings depends upon advances in PDEs, harmonic analysis and (in dimension 2) complex function theory. The first equation of particular relevance to the theory of quasiregular mappings is the n-dimensional Beltrami system

$$D^t f(x) Df(x) = J_f^{2/n}(x) G(x), \tag{1.2}$$

where $D^t f(x)$ is the transpose of $Df(x)$, $G(x) \in \mathrm{GL}(n)$ is a positively-defined, symmetric, and determinant 1 matrix, and satisfies the following uniformly elliptic condition

$$\alpha|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le \beta|\xi|^2, \quad \forall \xi \in \mathrm{R}^n, 0 < \alpha \le \beta < \infty,$$

Yu.G.Reŝhetnyak obtained in [9] the following result: every component function $u = f^i, i = 1, 2, \cdots, n$ of equation (1.2) is a weak solution of the following divergence type elliptic equation (also called A-harmonic equation)

$$\mathrm{div} A(x, \nabla u) = 0, \tag{1.3}$$

where $A(x, \xi) = \langle G^{-1}(x)\xi, \xi\rangle^{\frac{n-2}{2}} G^{-1}(x)\xi$. See also [6] and [7]. The weak solution of (1.3) means that

$$\int_\Omega \langle A(x, \nabla u), \nabla \varphi\rangle dx = 0 \tag{1.4}$$

for arbitrary $\varphi \in W_0^{1,n}(\Omega)$. By this result, we know that there is a close relationship between space quasiregular mappings and the weak solution of the A-harmonic equation (1.3).

Consider the following Beltrami system with two characteristic matrices

$$D^t f(x) H(x) Df(x) = J_f^{2/n}(x) G(x), \tag{1.5}$$

where $H(x), G(x) \in \mathrm{GL}(n)$, and satisfy the following uniformly elliptic condition

$$\alpha_1|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le \beta_1|\xi|^2, \quad \forall \xi \in \mathrm{R}^n, 0 < \alpha_1 \le \beta_1 < \infty \tag{1.6}$$

$$\alpha_2|\eta|^2 \le \langle H(x)\eta,\eta\rangle \le \beta_2|\eta|^2, \quad \forall \eta\in \mathrm{R}^n, 0<\alpha_2\le\beta_2<\infty. \tag{1.7}$$

In [5], the authors have derived the following result: let $f \in W^{1,n}_{loc}(\Omega, \mathrm{R}^n)$ is a generalized solution of (1.5) with the conditions (1.6) and (1.7), then $du = df^{i_1}\wedge \cdots \wedge df^{i_l} (1 \le i_1 < \cdots < i_l \le n, 1 \le l \le n)$ is a weak solution of the p-harmonic equation

$$d^*A(x,du) = d^*B(x,Df),$$

where

$$\begin{aligned} A(x,du) &= \left(\frac{\langle G_\#^{-1}(x)du,du\rangle}{H_\#^I(x)}\right)^{\frac{n-2l}{2l}} G_\#^{-1}(x)du \\ &= \left(\frac{\langle G_\#^{-1}(x)du,du\rangle}{H_\#^I(x)}\right)^{\frac{p-2}{2}} G_\#^{-1}(x)du, \\ B(x,Df) &= J_f(x)D_\#^{-1}f(x)[H_\#^{-1}(x)-H_\#^{-1}(x_0)]dx^I, \end{aligned}$$

where $H_\#^I(x)$ is the element of $H_\#^{-1}(x)$ lies in the diagonal with respect to $I = (i_1,\cdots,i_l)$. The weakly monotonicity result of every component function of (1.5) also been derived. For some related results, see [3], [4] and [12].

Remark 1. Let $n=2$. If we let $z = x_1 + ix_2$ and $w(z) = f^1(x_1,x_2) + if^2(x_1,x_2)$, then equations (1.2) and (1.5) are equivalent separately to the Beltrami equation with characteristic function $\mu(z)$

$$w_{\overline{z}} = \mu_1(z)w_z, \quad |\mu(z)| \le k_1 < 1 \tag{1.8}$$

and the Beltrami equation with two characteristic functions $\mu_1(z)$ and $\mu_2(z)$

$$w_{\overline{z}} = \mu_1(z)w_z + \mu_2(z)\overline{w_z}, \quad |\mu_1(z)|+|\mu_2(z)| \le k_2 < 1 \tag{1.9}$$

L.Bers[1], I.N.Vekua[10] and Wen Guochun [11] have made lots of research on the complex equations (1.8) and (1.9), and have obtained some celebrated results on them.

In this paper, we continue to study a more general Beltrami system, i.e., the following Beltrami system with three characteristic matrices

$$D^tf(x)H(x)Df(x) + K(x)D^tf(x)Df(x) = J_f^{2/n}(x)G(x), \tag{1.10}$$

where $H(x)$ is a diagonal matrix, $G(x)$ and $K(x)$ are positively-defined, symmetric matrices, and satisfy the following uniformly elliptic conditions

$$\alpha_1|\xi|^2 \le \langle H(x)\xi,\xi\rangle \le \beta_1|\xi|^2, \quad \forall \xi\in \mathrm{R}^n, 0<\alpha_1\le\beta_1<\infty, \tag{1.11}$$

$$\alpha_2|\eta|^2 \le \langle G(x)\eta,\eta\rangle \le \beta_2|\eta|^2, \quad \forall \eta\in \mathrm{R}^n, 0<\alpha_2\le\beta_2<\infty, \tag{1.12}$$

$$\alpha_3|\zeta|^2 \le \langle K(x)\zeta, \zeta\rangle \le \beta_3|\zeta|^2, \quad \forall \zeta \in \mathrm{R}^n, 0 \le \alpha_3 \le \beta_3 < \infty. \tag{1.13}$$

Remark 2. It is easy to see that if $f = (f^1, f^2, \cdots, f^n) \in W^{1,n}_{loc}(\Omega, \mathrm{R}^n)$ is a generalized solution of (1.10) , then f is $(\frac{\beta_2}{\alpha_1+\alpha_3})^{n/2}$-quasiregular.

In fact, we can obtain by (1.10) that for arbitrary $\xi \in \mathrm{R}^n$,

$$\langle D^t f(x) H(x) Df(x)\xi, \xi\rangle + \langle K(x) D^t f(x) Df(x)\xi, \xi\rangle = J_f^{2/n}(x)\langle G(x)\xi, \xi\rangle.$$

By (1.11) $\sim$ (1.13) we can derive that

$$\begin{aligned} (\alpha_1+\alpha_3)|Df(x)\xi|^2 &\le \langle H(x)Df(x)\xi, Df(x)\xi\rangle + \alpha_3\langle Df(x)\xi, Df(x)\xi\rangle \\ &\le \langle D^t f(x)H(x)Df(x)\xi, \xi\rangle + \langle K(x)D^t f(x)Df(x)\xi, \xi\rangle \\ &= J_f^{2/n}(x)\langle G(x)\xi, \xi\rangle \le \beta_2 J_f^{2/n}(x)|\xi|^2. \end{aligned}$$

Therefore

$$|Df(x)|^2 \le \frac{\beta_2}{\alpha_1+\alpha_3} J_f^{2/n}(x).$$

Thus the desired result follows.

Remark 3. Combined with the result in Remark 2 and the regularity result for K-quasiregular mappings in [11], we can obtain the regularity result for the generalized solution of (1.10): There exists $p = p(n, K) > n$, such that any generalized solution of (1.10) is in fact in the space $W^{1,p}_{loc}(\Omega, \mathrm{R}^n)$.

The system (1.10) has a strong background when $n = 2$ and has been widely studied. But for the case when $n > 2$, it is very difficult to study it directly since the system (1.10) is overdetermined and fully nonlinear. An effective research method is to transform it to an elliptic equation, and study the elliptic equaiton by using the method of differential geometry and the analytical method of Sobolev spaces. The aim of this paper is to transform system (1.10) to a nonhomogeneous p-equation, and derive the estimates of the operators A and B. This make a bridge between system (1.10) and nonhomogeneous elliptic equations. We obtain

Theorem. *Suppose that the space dimension n is even and $f \in W^{1,n}_{loc}(\Omega, R^n)$ is a generalized solution of* (1.10) *,which satisfy conditions* (1.11) $\sim$ (1.13)*, then $df^I = df^{i_1} \wedge \cdots \wedge df^{i_l}$,$1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = \frac{n}{2}$, is a weak solution of the following p-harmonic equation*

$$d^*A(x, df^I) = d^*B(x, Df), \tag{1.14}$$

where

$$\begin{aligned} A(x, df^I) &= G^{-1}_{\#}(x)df^I \\ B(x, Df) &= G^{-1}_{\#}(x)D^t_{\#}f(x)H_{\#}(x)(H^{-1}_{\#}(x) \\ &\quad - H^{-1}_{\#}(x_0))dx^I - G^{-1}_{\#}(x)K_{\#}(x)D^t_{\#}f(x)H^{-1}_{\#}(x_0)dx^I. \end{aligned}$$

The operator A satisfies the Lipschitz type condition

$$|A(x,h_1) - A(x,h_2)| \le C_1|h_1 - h_2|, \quad C_1 = \frac{1}{\alpha_3^l} \tag{1.15}$$

the monotonicity condition

$$\langle A(x,h_1) - A(x,h_2), h_1 - h_2\rangle \ge C_2|h_1 - h_2|^2, \quad C_2 = \frac{1}{\beta_3^l}, \tag{1.16}$$

and the homogeneous condition

$$A(x,\lambda h) = \lambda A(x,h), \quad \forall \lambda \in R, \tag{1.17}$$

and the operator B satisfies the controlled growth condition

$$|B(x,h)| \le C_3|h|, \quad C_3 = \left(\frac{2(\beta_1^l + \beta_3^l)}{\alpha_2^l}\right)^{\frac{1}{2}} \cdot \frac{1}{\alpha_1^l} + \frac{1}{\alpha_3^l}. \tag{1.18}$$

Remark 4. We can derive some useful properties of the generalized solutions of (1.10) by using the p-harmonic equation (1.14). These results will be published in other papers.

2. Preliminaries

Let $e^1, e^2, \cdots, e^n$ denote the standard unit basis of R^n. For $l = 0, 1, \cdots, n$, the linear space of l differential forms, spanned by the exterior products $e^I = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_l}$ corresponding to all ordered l-tuples $I = (i_1, i_2, \cdots, i_l), 1 \le i_1 < i_2 \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathrm{R}^n)$. Thus, $\wedge^0(\mathrm{R}^n) = \mathrm{R}$ and $\wedge^1 = \mathrm{R}^n$. We define the Hodge star operator

$$* : \wedge^l(\mathrm{R}^n) \to \wedge^{n-l}(\mathrm{R}^n)$$

by the rule

$$*1 = e^1 \wedge e^2 \wedge \cdots \wedge e^n,$$

and

$$\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta\rangle(*1)$$

for all $\alpha, \beta \in \wedge^l, l = 1, 2, \cdots n$. The norm of $\alpha \in \wedge^l$ is then

$$|\alpha|^2 = \langle \alpha, \alpha\rangle = *(\alpha \wedge *\alpha) \in \wedge^0 = \mathrm{R}.$$

The Hodge star is an isometric isomorphism on $\wedge^l$ with

$$* : \wedge^l \to \wedge^{n-l},$$

and

$$**=(-1)^{l(n-l)}:\wedge^l\to\wedge^l.$$

A differential l-form ω on Ω is simply a Schwarz distribution on Ω with values in $\wedge^l=\wedge^l(\mathrm{R}^n)$. We write $\omega\in D'(\Omega,\wedge^l)$. Therefore every l-from ω may be written uniquely as

$$\alpha(x)=\sum_{1\le i_1<\cdots<i_l\le n}\alpha_{i_1,\cdots,i_l}(x)dx_{i_1}\wedge\cdots\wedge dx_{i_l},$$

where the coefficients $\alpha_{i_1,\cdots,i_l}(x)$ are functions or distributions.

Of fundamental concern to us will be the exterior derivative

$$d:D'(\Omega,\wedge^l)\to D'(\Omega,\wedge^{l+1})$$

it is uniquely determined by the following three conditions:

(i) if $l=0$, then df is the differential of f;

(ii) for $\alpha\in D'(\Omega,\wedge^l),\beta\in D'(\Omega,\wedge^k)$, we have

$$d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^l\alpha\wedge d\beta$$

(iii) $d(d\alpha)=0$.

The formal adjoint (also called the Hodge codifferential) of d is the operator

$$d^*=(-1)^{nl+1}*d*:D'(\Omega,\wedge^{l+1})\to D'(\Omega,\wedge^l).$$

For any forms $\alpha\in L^p(\Omega,\wedge^l),\beta\in L^q(\Omega,\wedge^l),1\le p,q\le\infty,\frac{1}{p}+\frac{1}{q}=1$, define

$$(\alpha,\beta)=\int_\Omega\langle\alpha(x),\beta(x)\rangle dx.$$

Definition. Let $\beta\in L^q(\Omega,\wedge^l)$. We call the *codifferential* of β is zero in the weak sense, if for any $\alpha\in L_1^p(\Omega,\wedge^{l-1}),1\le p,q<\infty,\frac{1}{p}+\frac{1}{q}=1$ with compact support in Ω, we have

$$(d\alpha,\beta)=0$$

We can also define *the differential of* α *is zero* in the weak sense in the same manner.

Let G be an $n\times n$ matrix. The l^{th} *exterior power* of G is the linear operator

$$G_\#:\wedge^l\to\wedge^l$$

defined by

$$G_\#(\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_l)=G\alpha_1\wedge G\alpha_1\wedge\cdots\wedge G\alpha_l$$

for $\alpha_1,\alpha_2,\cdots,\alpha_l\in\wedge^1(\mathrm{R}^n)$ and then extend linearly to all of $\wedge^l$. Thus $G_\#$ is a $C_n^l\times C_n^l$ matrix.

For the properties of $G_{\#}$ see [7] and [8].

Let $f:\Omega\to R^n, f=(f^1,f^2,\cdots,f^n)\in W^{1,p}_{loc}(\Omega,R^n), p\geq 1$. Then f induces a homomorphism

$$f^*: C^\infty(R^n,\wedge^{l-1})\to L^{1,p}_{loc}(\Omega,\wedge^{l-1})$$

called the *pull back*. More precisely, let $\alpha\in C^\infty(R^n,\wedge^{l-1}), \alpha=\sum_I \alpha^I dx^I$. Then

$$(f^*\alpha)(x)=\sum_I \alpha^I(f(x))df^{i_1}\wedge df^{i_2}\wedge\cdots\wedge df^{i_l}.$$

For the properties of the pull back operator see [6]~[8].

Other unstated symbol in this paper can be found in [6]~[8].

3. p-harmonic equation for the Beltrami system (1.10)

From the system (1.10) we can obtain

$$D^{-1}f(x)=J_f^{-2/n}(x)G^{-1}(x)D^tf(x)H(x)+J_f^{-2/n}(x)G^{-1}(x)K(x)D^tf(x)$$

take the l^{th} exterior power in the above equality, we derive that

$$D_{\#}^{-1}f(x)=J_f^{-2l/n}(x)G_{\#}^{-1}(x)D_{\#}^tf(x)H_{\#}(x)+J_f^{-2l/n}(x)G_{\#}^{-1}(x)K_{\#}(x)D_{\#}^tf(x):\wedge^l\to\wedge^l.$$

Hence

$$\begin{aligned}(3.1)\qquad J_f(x)D_{\#}^{-1}f(x)=&J_f^{(n-2l)/n}(x)G_{\#}^{-1}(x)D_{\#}^tf(x)H_{\#}(x)\\&+J_f^{(n-2l)/n}(x)G_{\#}^{-1}(x)K_{\#}(x)D_{\#}^tf(x):\wedge^l\to\wedge^l.\end{aligned}$$

Since $f(x)$ is differentiable almost everywhere, it is no loss of generality to assume $J_f(x)\neq 0$, x is a differentiable point of $f(x)$, and a Lebesgue point of $J_f(x)$. Then by [2] we know that there exists a neighborhood U of x, such that f is homeomorphic on U. Let $V=f(U)$ and $dx^I=dx^{i_1}\wedge\cdots\wedge dx^{i_l}$. Fix $x_0\in\Omega$. By (3.1), we have

$$\begin{aligned}(3.2)\qquad &d^*[J_f(x)D_{\#}^{-1}f(x)H_{\#}^{-1}(x_0)dx^I]\\=\ &d^*[J_f^{(n-2l)/n}(x)G_{\#}^{-1}(x)D_{\#}^tf(x)H_{\#}(x)H_{\#}^{-1}(x_0)dx^I]\\&+d^*[J_f^{(n-2l)/n}(x)G_{\#}^{-1}(x)K_{\#}(x)D_{\#}^tf(x)H_{\#}^{-1}(x_0)dx^I].\end{aligned}$$

Our nearest goal is to prove that the left hand side in the above equality is zero in the weak sense. To this end, take an arbitrary

$$\psi(\zeta)=\sum_J\psi^J(\zeta)d\zeta^J=\sum_J\psi^J(\zeta)d\zeta^{j_1}\wedge\cdots\wedge d\zeta^{j_{l-1}}\in C_0^\infty(V,\wedge^{l-1}).$$

Let

$$\varphi(x)=(f^*\psi)(x)=\sum_j\psi^J(f(x))df^{j_1}\wedge\cdots\wedge df^{j_{l-1}}.$$

Then by

$$df^j \wedge df^{j_1} \wedge \cdots \wedge df^{j_{l-1}} = D^t_{\#} f(x) dx^j \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{l-1}}.$$

(For this formula see [7] or [8]) we have

$$\begin{aligned} d\varphi(x) &= \sum_J d\psi^J(f(x)) \wedge df^{j_1} \wedge \cdots \wedge df^{j_{l-1}} \\ &= \sum_J \sum_{j=1}^n \frac{\partial \psi^J(\zeta)}{\partial \zeta^j}|_{\zeta=f(x)} df^j \wedge df^{j_1} \wedge df^{j_{l-1}} \\ &= D^t_{\#} f(x) \left(\sum_J \sum_{j=1}^n \frac{\partial \psi^J(\zeta)}{\partial \zeta_j}|_{\zeta=f(x)} \mathrm{d}x^j \right) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{l-1}}. \end{aligned}$$

Therefore

$$\begin{aligned} & \int_U \langle J_f(x) D^{-1}_{\#} f(x) H^{-1}_{\#}(x_0) dx^I, d\varphi(x)\rangle dx \\ = & \int_U \langle H^{-1}_{\#}(x_0) dx^I, (D^t_{\#} f(x))^{-1} d\varphi(x)\rangle J_f(x) dx \\ = & \int_U \langle H^{-1}_{\#}(x_0) dx^I, \sum_J \sum_{j=1}^n \frac{\partial \psi^J(\zeta)}{\partial \zeta^j}|_{\zeta=f(x)} dx^j \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{l-1}}\rangle J_f(x) dx \\ = & \int_V \langle H^{-1}_{\#}(x_0) d\zeta^I, \sum_J \sum_{j=1}^n \frac{\partial \psi^J(\zeta)}{\partial \zeta^j} d\zeta^j \wedge d\zeta^{j_1} \wedge \cdots \wedge d\zeta^{j_{l-1}}\rangle d\zeta \\ = & \int_V \langle H^{-1}_{\#}(x_0) d\zeta^J, d\psi(\zeta)\rangle d\zeta = \int_V \langle d^*[H^{-1}_{\#}(x_0) d\zeta^J], \psi(\zeta)\rangle d\zeta = 0. \end{aligned}$$

The last equality holds since the differential form $H^{-1}_{\#}(x_0) d\zeta^J$ has constant coefficients. Thus we obtain from (3.2) that

$$\begin{aligned} & d^*[J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x) D^t_{\#} f(x) H_{\#}(x) H^{-1}_{\#}(x_0) dx^I] \\ = & -d^*[J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x) K_{\#}(x) D^t_{\#} f(x) H^{-1}_{\#}(x_0) dx^I] \end{aligned}$$

in the weak sense.

Let $df^I = df^{i_1} \wedge \cdots \wedge df^{i_l}$. By $df^I = D^t_{\#} f(x) dx^I$, we have

$$\begin{aligned} & d^*\{J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x)[df^I + D^t_{\#} f(x)(H_{\#}(x) H^{-1}_{\#}(x_0) - \mathrm{Id}_{\#}) dx^I]\} \\ = & -d^*[J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x) K_{\#}(x) D^t_{\#} f(x) H^{-1}_{\#}(x_0) dx^I]. \end{aligned}$$

That is

$$\begin{aligned} (3.3) \qquad & d^*\{J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x) df^I\} \\ = & d^*\{J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x) D^t_{\#} f(x) H_{\#}(x)(H^{-1}_{\#}(x) - H^{-1}_{\#}(x_0)) dx^I\} \\ & - d^*[J_f^{(n-2l)/n}(x) G^{-1}_{\#}(x) K_{\#}(x) D^t_{\#} f(x) H^{-1}_{\#}(x_0) dx^I]. \end{aligned}$$

If the dimension n is even, then take $l = \frac{n}{2}$, (3.3) becomes

$$\begin{aligned} d^*\{G_\#^{-1}(x)df^I\} &= d^*\{G_\#^{-1}(x)D_\#^t f(x)H_\#(x)(H_\#^{-1}(x) - H_\#^{-1}(x_0))dx^I\} \\ &\quad - d^*[G_\#^{-1}(x)K_\#(x)D_\#^t f(x)H_\#^{-1}(x_0)dx^I]. \end{aligned}$$

This is nothing but (1.14).

4. The estimates for the operators A and B

In this section, we derive the estimates (1.15) (1.18) for the operators A and B. We first derive the estimates for the operator

$$A(x,h) = G_\#^{-1}(x)h.$$

(i) **the Lipschitz type condition**

$$|A(x,h_1) - A(x,h_2)| \le C_1|h_1 - h_2|, \quad C_1 = \frac{1}{\alpha_3^l}.$$

By (1.13), we know that

$$\frac{1}{\beta_3^l}|\zeta|^2 \le \langle G_\#^{-1}(x)\zeta,\zeta\rangle \le \frac{1}{\alpha_3^l}|\zeta|^2. \tag{4.1}$$

Therefore

$$|G_\#^{-1}(x)| \le \frac{1}{\alpha_3^l}.$$

Hence

$$|A(x,h_1) - A(x,h_2)| \le |G_\#^{-1}(x)||h_1 - h_2| \le \frac{1}{\alpha_3^l}|h_1 - h_2|.$$

(ii) **the monotonicity inequality**

$$\langle A(x,h_1) - A(x,h_2), h_1 - h_2\rangle \ge C_2|h_1 - h_2|^2, \quad C_2 = \frac{1}{\beta_3^l}.$$

By (4.1) we can derive that

$$\begin{aligned} &\langle G_\#^{-1}(x)h_1 - G_\#^{-1}(x)h_2, h_1 - h_2\rangle \\ = \ &\langle G_\#^{-1}(x)(h_1 - h_2), h_1 - h_2\rangle \ge \frac{1}{\beta_3^l}|h_1 - h_2|^2 \end{aligned}$$

(iii) **the homogeneous condition**

$$A(x,\lambda h) = \lambda A(x,h), \quad \forall\lambda \in \mathrm{R}.$$

This can be easily derived by the definition of $A(x,h)$.

We next derive the controlled growth condition (1.18) of the operator B. By (3.2), we know that

$$B(x,Df)=A(x,df^I)+J_f(x)D_{\#}^{-1}f(x)H_{\#}^{-1}(x_0)dx^I,$$

and from the Lipschitz type condition (1.15), we know that

$$|A(x,df^I)|\le C_1|df^I|,\quad C_1=\frac{1}{\alpha_3^l}. \tag{4.2}$$

Hence we need only to estimate $|J_f(x)D_{\#}^{-1}f(x)H_{\#}^{-1}(x_0)dx^I|$.

By (1.10) we have

$$H(x)+(D^tf(x))^{-1}K(x)D^tf(x)=J_f^{2/n}(x)(D^tf(x))^{-1}G(x)(Df(x))^{-1}.$$

Take the $l^{th}(l=\frac{n}{2})$ exterior power, we have

$$H_{\#}(x)+(D_{\#}^tf(x))^{-1}K_{\#}(x)D_{\#}^tf(x)=J_f(x)(D_{\#}^tf(x))^{-1}G_{\#}(x)(D_{\#}f(x))^{-1}.$$

Thus for any $\eta\in \mathrm{R}^{C_{2l}^l},\eta\neq 0$, we have

$$\begin{aligned}&\langle H_{\#}(x)\eta,\eta\rangle+\langle K_{\#}(x)D_{\#}^tf(x)\eta,(D_{\#}f(x))^{-1}\eta\rangle\\ =\;&J_f(x)\langle G_{\#}(x)D_{\#}^{-1}f(x)\eta,D_{\#}^{-1}f(x)\eta\rangle.\end{aligned} \tag{4.3}$$

By the uniformly elliptic conditions (1.11) (1.13), we know that

$$\alpha_1^l|\tau|^2\le\langle H_{\#}(x)\tau,\tau\rangle\le\beta_1^l|\tau|^2, \tag{4.4}$$

$$\alpha_2^l|\tau|^2\le\langle G_{\#}(x)\tau,\tau\rangle\le\beta_2^l|\tau|^2, \tag{4.5}$$

$$\alpha_3^l|\tau|^2\le\langle K_{\#}(x)\tau,\tau\rangle\le\beta_3^l|\tau|^2. \tag{4.6}$$

Therefore

$$\langle K_{\#}(x)D_{\#}^tf(x)\eta,(D_{\#}f(x))^{-1}\eta\rangle\le\beta_3^l\langle D_{\#}^tf(x)\eta,(D_{\#}f(x))^{-1}\eta\rangle=\beta_3^l|\eta|^2, \tag{4.7}$$

and

$$\alpha_2^l|D_{\#}^{-1}f(x)\eta|^2\le\langle G_{\#}(x)D_{\#}^{-1}f(x)\eta,D_{\#}^{-1}f(x)\eta\rangle\le\beta_2^l|D_{\#}^{-1}f(x)\eta|^2. \tag{4.8}$$

Thus, by (4.3) (4.8) we can derive that

$$\alpha_2^lJ_f(x)|D_{\#}^{-1}f(x)\eta|^2\le(\beta_1^l+\beta_3^l)|\eta|^2.$$

That is

$$J_f(x)|D_{\#}^{-1}f(x)|^2\le\frac{\beta_1^l+\beta_3^l}{\alpha_2^l}. \tag{4.9}$$

By using the system (1.10) again, we have

$$J_f(x)H_\#^{-1}(x) = D_\# f(x)G_\#^{-1}(x)D_\#^t f(x) + D_\# f(x)G_\#^{-1}(x)K_\#(x)D_\#^t f(x)H_\#^{-1}(x).$$

Thus

$$\begin{aligned} & J_f(x)\langle H_\#^{-1}(x)dx^I, dx^I\rangle \\ = \ & \langle D_\# f(x)G_\#^{-1}(x)D_\#^t f(x)dx^I, dx^I\rangle + \langle D_\# f(x)G_\#^{-1}(x)K_\#(x)D_\#^t f(x)H_\#^{-1}(x)dx^I, dx^I\rangle. \end{aligned}$$

Since the matrix $H(x)$ is diagonal, so is $H^{-1}(x)$ and $H_\#^{-1}(x)$. Therefore

$$\begin{aligned} & J_f(x)(H_\#^{-1}(x))^I \\ = \ & \langle G_\#^{-1}(x)D_\#^t f(x)dx^I, D_\#^t f(x)dx^I\rangle \\ & + (H_\#^{-1}(x))^I\langle G_\#^{-1}(x)K_\#(x)D_\#^t f(x)dx^I, D_\#^t f(x)dx^I\rangle \\ = \ & \langle G_\#^{-1}(x)df^I, df^I\rangle + (H_\#^{-1}(x))^I\langle G_\#^{-1}(x)K_\#(x)df^I, df^I\rangle, \end{aligned} \tag{4.10}$$

where $(H_\#^{-1}(x))^I$ is the element lies in the diagonal of $H_\#^{-1}(x)$ with respect to $I = (i_1, \cdots, i_l)$. By the condition (1.11) we know that

$$0 < \frac{1}{\beta_1^l} \le |(H_\#^{-1}(x))^I| \le \frac{1}{\alpha_1^l}.$$

Hence from (4.10) we have

$$J_f(x) = \frac{\langle G_\#^{-1}(x)df^I, df^I\rangle}{(H_\#^{-1}(x))^I} + \langle G_\#^{-1}(x)K_\#(x)df^I, df^I\rangle. \tag{4.11}$$

By (4.11) and (4.4) (4.6), we can easily derive that

$$J_f(x) \le \frac{\beta_1^l + \beta_3^l}{\alpha_2^l}|df^I|^2. \tag{4.12}$$

Combining (4.9) with (4.12), we have

$$J_f^2(x)|D_\#^{-1}f(x)|^2 \le \frac{2(\beta_1^l + \beta_3^l)}{\alpha_2^l}|df^I|^2.$$

It follows that

$$J_f(x)|D_\#^{-1}f(x)| \le \left(\frac{2(\beta_1^l + \beta_3^l)}{\alpha_2^l}\right)^{\frac{1}{2}}|df^I|.$$

Hence

$$|J_f(x)D_\#^{-1}f(x)H_\#^{-1}(x_0)dx^I| \le \left(\frac{2(\beta_1^l + \beta_3^l)}{\alpha_2^l}\right)^{\frac{1}{2}} \cdot \frac{1}{\alpha_1^l}|df^I|. \tag{4.13}$$

The desired controlled growth condition of B is obtained by combining (4.2) with (4.13).

References

[1] Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Trans. Math. Monographs, Amer. Math. Soc., **73**(1989).

[2] T. Iwaniec and G.Martin, *Quasiregular mappings in even dimensions*, Acta Math., **170**(1993), 29-81.

[3] T. Iwaniec, *p-harmonic tensors and quasiregular mappings*, Ann. of Math., **136**(1992), 586-624.

[4] Gao Hongya and Wu Zemin, *On Beltrami system with two characteristic matrices*, Acta Math. Sci., **4(22)**(2002), 433-440. (In Chinese)

[5] Cheng Jinfa and Fang Ainong, *Generalized Beltrami system in even dimensions and Beltrami system in high dimensions*, Chn. Ann. of Math., **18A(6)**(1997) 789-798. (In Chinese)

[6] Cheng Jinfa and Fang Ainong, (G, H)*-quasiregular mappings and* B*-harmonic equation*, Acta Math. Sin, **42(5)**(1999), 883-888. (In Chinese)

[7] Zheng Shenzhou, *Two characteristic Beltrami equation and quasiregular mappings*, Acta Math. Sin., **40(5)**(1997), 745-750. (In Chinese)

[8] L. Bers, Psedueanalytic function theory, Science Press, 1964. (In Chinese)

[9] I. N. Vekua, Generalized analytic functions, Pergamon, Oxford, 1962.

[10] Wen Guochun, Linear and nonlinear elliptic complex equations, Shanghai Science and technology press, 1986. (In Chinese)

[11] T. Iwaniec and G. Martin, Geometric function theory and nonlinear analysis, Claredon Press, 2000.

[12] B. Bojarski and T. Iwaniec, *Analytic foundations of the theory of quasiconformal mappings in* R^n, Ann. Acad. Sci. Fenn. Ser. A I Math., **8**(1983), 257-324.