

Commuting Pair Preservers of Matrices

SEOK-ZUN SONG AND JIN-YOUNG OH

Department of Mathematics and Research Institute for Basic Sciences, Cheju National University, Jeju 690-756, Korea
e-mail : szsong@cheju.ac.kr

ABSTRACT. There are many papers on linear operators that preserve commuting pairs of matrices over fields or semirings. From these research works, we have a motivation to the research on the linear operators that preserve commuting pairs of matrices over nonnegative integers. We characterize the surjective linear operators that preserve commuting pairs of matrices over nonnegative integers.

1. Introduction and preliminaries

Let \mathbb{Z}^+ be the nonnegative part of the ring of integers \mathbb{Z} and let $\mathcal{M}_n(\mathbb{Z}^+)$ denote the set of all $n \times n$ matrices over \mathbb{Z}^+ . Similarly let $\mathbb{B} = \{0, 1\}$ be the binary Boolean algebra and let $\mathcal{M}_n(\mathbb{B})$ denote the set of all $n \times n$ matrices over \mathbb{B} . We denote the $n \times n$ identity matrix by I_n and the $n \times n$ zero matrix by O_n . The $n \times n$ matrix all of whose entries are zero except its (i, j) th, which is 1, is denoted $E_{i,j}$. We call $E_{i,j}$ a *cell*. We denote the $n \times n$ matrix all of whose entries are 1 by J_n . We omit the subscripts on I, O , and J when they are implied by the context. If A and B are matrices in $\mathcal{M}_n(\mathbb{Z}^+)$, we say B *dominates* A (written $B \geq A$ or $A \leq B$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i, j . This provides a reflexive, transitive relation on $\mathcal{M}_n(\mathbb{Z}^+)$.

A mapping $T: \mathcal{M}_n(\mathbb{Z}^+) \rightarrow \mathcal{M}_n(\mathbb{Z}^+)$ is called a *linear operator* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{Z}^+$. For a linear operator T on $\mathcal{M}_n(\mathbb{Z}^+)$ and $A, B \in \mathcal{M}_n(\mathbb{Z}^+)$ with $A \leq B$, we can easily show that $T(A) \leq T(B)$. Let $\Delta_n = \{(i, j) \mid 1 \leq i, j \leq n\}$. The set \mathcal{C} of commuting pairs of matrices is the set of (unordered) pairs of matrices (X, Y) such that $XY = YX$. We say that a linear operator T *preserves* \mathcal{C} when $T(X)T(Y) = T(Y)T(X)$ if $XY = YX$. We also say that a linear operator T *strongly preserves* \mathcal{C} when $T(X)T(Y) = T(Y)T(X)$ if and only if $XY = YX$. Commuting pairs of matrices over fields or semirings have been the subject of research by many authors ([1]-[8]). In 1976 Watkins [8] proved that if $n \geq 4$, $\mathcal{M}(\mathbb{F})$ is the set of $n \times n$ matrices over an algebraically closed field of characteristic 0, and L is a nonsingular linear operator on $\mathcal{M}(\mathbb{F})$ which preserves commuting pairs, then there exists an invertible matrix S in $\mathcal{M}(\mathbb{F})$, a nonzero scalar c , and a linear functional f such that either $L(X) = cSXS^{-1} + f(X)I$ or $L(X) = cSX^tS^{-1} + f(X)I$, for all X in $\mathcal{M}(\mathbb{F})$. In 1978, Beasley [1] extended this to the case $n = 3$. Also in [1], Beasley showed that the same characterization holds if $n \geq 3$ and L strongly preserves commuting pairs. The real symmetric and complex Hermitian cases were first investigated by Chan and Lim [3] in 1982; the same results were established as

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in the general case, with the exception that the invertible matrix must be orthogonal or unitary. Further extensions and generalizations to more general fields were obtained by Radjavi [5] and Choi, Jafarian, and Radjavi [4]. Beasley and Pullman [2] characterize the linear operators that preserve commuting pairs of Boolean matrices. Boolean matrices are important subject to the combinatorial matrix researchers. Often, parallels are sought for results known for certain semirings. Song and et al obtained characterizations of the linear operators that preserve the commuting pairs of matrices over nonnegative reals [6] and general Boolean algebras [7]. Here we investigate the set of linear operators on $\mathcal{M}_n(\mathbb{Z}^+)$ which preserve the set of pairs of commuting matrices, where \mathbb{Z}^+ is the nonnegative part of the ring of integers \mathbb{Z} .

We obtain characterizations of surjective linear operators that preserve commuting pairs of matrices over nonnegative integers.

2. Commuting pairs preservers of nonnegative integer matrices

Evidently, the following operations strongly preserve the set of commuting pairs of matrices;

- (a) transposition ($X \rightarrow X^t$) ;
- (b) similarity ($X \rightarrow SXS^{-1}$ for some fixed invertible matrix S).

In this section, we characterize the surjective linear operators that preserve commuting pairs of matrices over nonnegative integers. We show that these linear operators are the compositions of the transposition and similarity operators.

Lemma 2.1. *Let $T: \mathcal{M}_n(\mathbb{Z}^+) \rightarrow \mathcal{M}_n(\mathbb{Z}^+)$ be a linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$. Then the following are equivalent:*

- (1) *T is bijective.*
- (2) *T is surjective.*
- (3) *There exists a permutation σ on Δ_n such that $T(E_{i,j}) = E_{\sigma(i,j)}$*

Proof. That (1) implies (2) and (3) implies (1) is straight forward. We now show that (2) implies (3). We assume that T is surjective. Then, for any pair $(i, j) \in \Delta_n$, there exists a matrix $X \in \mathcal{M}_n(\mathbb{Z}^+)$ such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T . Thus there is $(r, s) \in \Delta_n$ such that $X = x_{r,s}E_{r,s} + X'$ where (r, s) entry of X' is zero and the following two conditions are satisfied: $x_{r,s} \neq 0$ and $T(E_{r,s}) \neq O$. Since \mathbb{Z}^+ has no zero divisors it follows that

$$T(x_{r,s}E_{r,s}) \leq T(x_{r,s}E_{r,s}) + T(X \setminus (x_{r,s}E_{r,s})) = T(X) = E_{i,j},$$

equivalently,

$$T(x_{r,s}E_{r,s}) = x_{r,s}T(E_{r,s}) \leq E_{i,j},$$

and so $T(E_{r,s}) \leq E_{i,j}$. It follows from $x_{r,s} \neq 0$ that $T(E_{r,s}) = b_{r,s}E_{i,j}$ for some nonzero scalar $b_{r,s}$. Let $P_{i,j} = \{E_{r,s} \mid T(E_{r,s}) \leq E_{i,j}\}$. By the above $P_{i,j} \neq \emptyset$ for all $(i, j) \in \Delta_n$. By its definition, $P_{i,j} \cap P_{u,v} = \emptyset$ whenever $(i, j) \neq (u, v)$. That is, $\{P_{i,j}\}$ is the set of n^2 nonempty sets which partition the set of cells. By the pigeonhole principle, we must have that $|P_{i,j}| = 1$ for all $(i, j) \in \Delta_n$. Necessarily, for each pair (r, s) there is the unique pair (i, j) such that $T(E_{r,s}) = b_{r,s}E_{i,j}$. Thus, there is some permutation σ on $\{(i, j) \mid i, j = 1, 2, \dots, n\}$ such that $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$, for scalars $b_{i,j}$. We now only

need to show that $b_{i,j} = 1$, for all i, j . Since T is surjective and $T(E_{r,s}) \not\leq E_{\sigma(i,j)}$ for $(r, s) \neq (i, j)$, there is some α such that $T(\alpha E_{i,j}) = E_{\sigma(i,j)}$. Since T is linear,

$$E_{\sigma(i,j)} = T(\alpha E_{i,j}) = \alpha T(E_{i,j}) = \alpha b_{i,j} E_{\sigma(i,j)}.$$

That is, $\alpha b_{i,j} = 1$, or $b_{i,j}$ is unit. Since 1 is the only unit element in \mathbb{Z}^+ , $b_{i,j} = 1$ for all $(i, j) \in \Delta_n$ □

We denote $\mathcal{C}_n(\mathbb{Z}^+)$ as the set of commuting pairs of matrices over \mathbb{Z}^+ ; that is, $\mathcal{C}_n(\mathbb{Z}^+) = \{(A, B) \in \mathcal{M}^2(\mathbb{Z}^+) \mid AB = BA\}$.

Example 2.2. For $A \in \mathcal{M}_n(\mathbb{Z}^+)$, define a map T on $\mathcal{M}_n(\mathbb{Z}^+)$ by

$$T(X) = \left(\sum_{i,j=1}^n x_{i,j} \right) A$$

for all $X = [x_{i,j}] \in \mathcal{M}_n(\mathbb{Z}^+)$. Then T is a linear operator and that $T(E_{r,s}) = A$ for all $(r, s) \in \Delta_n$. Thus T is not surjective by Lemma 2.1. And we can easily show that T preserves commuting pairs of matrices, while it does not preserve non-commuting pairs of matrices.

Thus, we are interested in the surjective linear operators that

$$(T(A), T(B)) \in \mathcal{C}_n(\mathbb{Z}^+) \text{ if and only if } (A, B) \in \mathcal{C}_n(\mathbb{Z}^+).$$

For a matrix $A \in \mathcal{M}_n(\mathbb{Z}^+)$, A is called *invertible* in $\mathcal{M}_n(\mathbb{Z}^+)$ if there exists a matrix $B \in \mathcal{M}_n(\mathbb{Z}^+)$ such that $AB = BA = I_n$. It is well known [7] that all permutation matrices are the only invertible matrices in $\mathcal{M}_n(\mathbb{B})$. Using this fact, we can easily show that all permutation matrices are the only invertible matrices in $\mathcal{M}_n(\mathbb{Z}^+)$.

Theorem 2.3. *Let T be a linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$. Then T is a surjective linear operator which preserves commuting pairs of matrices over nonnegative integers if and only if there exists an invertible matrix $U \in \mathcal{M}_n(\mathbb{Z}^+)$ such that either*

- (1) $T(X) = UXU^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$, or
- (2) $T(X) = UX^tU^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

Proof. Let T be a surjective linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$ that preserves pairs of commuting matrices. By Lemma 2.1, T is bijective and there exists a permutation σ on Δ_n such that $T(E_{i,j}) = E_{\sigma(i,j)}$. Note that if $AX = XA$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$, then we have $A = \alpha I_n$ for some $\alpha \in \mathbb{Z}^+$. Thus we have $T(I_n) = \beta I_n$ for some $\beta \in \mathbb{Z}^+$ because T is bijective. Since T maps a cell onto a cell, $T(I_n) = I_n$. It follows that there is a permutation γ of $\{1, \dots, n\}$ such that $T(E_{i,i}) = E_{\gamma(i)\gamma(i)}$ for each $i = 1, \dots, n$. Define $L: \mathcal{M}_n(\mathbb{Z}^+) \rightarrow \mathcal{M}_n(\mathbb{Z}^+)$ by $L(X) = PT(X)P^t$, where P is the permutation matrix corresponding to γ so that $L(E_{i,i}) = E_{i,i}$ for each $i = 1, \dots, n$. Then we can easily show that L is a bijective linear operator on $\mathcal{M}_n(\mathbb{Z}^+)$ which preserves pairs of commuting matrices. By Lemma 2.1, L maps a cell onto a cell. Therefore, there exists $(p, q) \in \Delta_n$ such that $L(E_{r,s}) = E_{p,q}$ for any $(r, s) \in \Delta_n$.

Suppose that $r \neq s$. Since L is bijective, we have $p \neq q$ because $L(E_{i,i}) = E_{i,i}$ for each $i = 1, \dots, n$. Assume that $p \neq r$ and $p \neq s$. Then

$$E_{r,s}(E_{r,r} + E_{s,s} + E_{p,p}) = (E_{r,r} + E_{s,s} + E_{p,p})E_{r,s}$$

so that

$$L(E_{r,s})L(E_{r,r} + E_{s,s} + E_{p,p}) = L(E_{r,r} + E_{s,s} + E_{p,p})L(E_{r,s}),$$

equivalently,

$$E_{p,q}(E_{r,r} + E_{s,s} + E_{p,p}) = (E_{r,r} + E_{s,s} + E_{p,p})E_{p,q}.$$

It follows that $q = r$ or $q = s$. Since $E_{r,s}(E_{r,r} + E_{s,s}) = (E_{r,r} + E_{s,s})E_{r,s}$, we have

$$L(E_{r,s})L(E_{r,r} + E_{s,s}) = L(E_{r,r} + E_{s,s})L(E_{r,s}),$$

equivalently,

$$E_{p,q}(E_{r,r} + E_{s,s}) = (E_{r,r} + E_{s,s})E_{p,q}.$$

Since $q = r$ or $q = s$, we have $E_{p,q}(E_{r,r} + E_{s,s}) = E_{p,r}$ or $E_{p,s}$, but $(E_{r,r} + E_{s,s})E_{p,q} = 0$, a contradiction. Hence we have $p = r$ or $p = s$. Similarly we obtain $q = r$ or $q = s$. Therefore we have $L(E_{r,s}) = E_{r,s}$ or $L(E_{r,s}) = E_{s,r}$ for each $(r, s) \in \Delta_n$. Suppose that $L(E_{r,s}) = E_{r,s}$ with $r \neq s$ and $L(E_{r,t}) = E_{t,r}$ for some $t \neq r, s$. Then we have $L(E_{s,t} + E_{t,s}) = E_{s,t} + E_{t,s}$. Let $A = E_{r,r} + E_{s,t} + E_{t,s}$ so that $L(A) = E_{r,r} + E_{s,t} + E_{t,s}$. Then $(E_{r,s} + E_{r,t})A = A(E_{r,s} + E_{r,t})$, and hence

$$L(E_{r,s} + E_{r,t})L(A) = L(A)L(E_{r,s} + E_{r,t}).$$

But

$$L(E_{r,s} + E_{r,t})L(A) = E_{r,t} + E_{t,r},$$

while

$$L(A)L(E_{r,s} + E_{r,t}) = E_{r,s} + E_{s,r}.$$

Thus we have $t = s$, a contradiction. It follows that if $L(E_{i,j}) = E_{i,j}$ for some pair $(i, j) \in \Delta_n$ with $i \neq j$, then $L(E_{r,s}) = E_{r,s}$ for all pairs $(r, s) \in \Delta_n$. Similarly, if $L(E_{i,j}) = E_{j,i}$ for some pair $(i, j) \in \Delta_n$ with $i \neq j$, then $L(E_{r,s}) = E_{s,r}$ for all pairs $(r, s) \in \Delta_n$. We have established that either $L(X) = X$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$ or $L(X) = X^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$. Therefore $T(X) = P^t X P$ or $T(X) = P^t X^t P$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$. If $U = P^t$, then we have $T(X) = U X U^t$ or $T(X) = U X^t U^t$ for all $X \in \mathcal{M}_n(\mathbb{Z}^+)$.

The converse is immediate. \square

Thus we have characterized the surjective linear operators that preserve commuting pairs of matrices over nonnegative integers.

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