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Injective JW-algebras

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ABSTRACT. Injective JW-algebras are defined and are characterized by the existence of projections of norm 1 onto them. The relationship between the injectivity of a JW-algebra and the injectivity of its universal enveloping von Neumann algebra is established. The Jordan analgue of Theorem 3 of [3] is proved, that is, a JC-algebra A is nuclear if and only if its second dual A^{**} is injective.

0. Introduction

The recognition of the importance of completely positive maps in the tensor products of C*-algebras was due to C. Lance [13] and E. Effors and C. Lance [5]. It has been used to characterize the so called injective von Neumann algebras [4] and nuclear C*-algebras [13]. In fact, C. Lance have shown that a C*-algebra $\mathfrak{A}$ with completely positive approximation prperty is nuclear [13, Theorem 3.6]. Choi and Effors then showed that a C*-algebra $\mathfrak{A}$ is nuclear if and only if its second dual $\mathfrak{A}^{**}$ is injective [3, Theorem 3.].

It was shown in [9] that a JC-algebra A is nuclear if and only if its universal enveloping C*-algebra $C^*(A)$ is nuclear. In analogue with injective C*-algebra (von Neumann algebra) we introduce the notion of injective JC-algebra (JW-algebra) which generalises this concept to Jordan algebras. We then establish the relationship between the injectivity of them and the injectivity of their enveloping algebras, and the relationship between the nuclearity of a JC-algebra A and the injectivity of its second dual A^{**}.

A JC-algebra A is a norm (uniformly) closed Jordan subalgebra of the Jordan algebra $B(H)_{s.a}$ of all bounded self adjoint operators on a Hilbert space H. The Jordan product is given by $a \circ b = (ab+ba)/2$, where juxtaposition denotes the ordinary operator multiplication. *A JW-algebra* $M \subseteq B(H)_{s.a}$ is a weakly closed JC-algebra. The second dual A^{**} of a JC-algebra A is a JW-algebra whose product extends the original product in A [11, 4.7.3]. An element $a \in A$ is called positive, written $a \geq 0$, if a is of a square [7, 3.3.3]. The set of all positive elements of A is denoted by

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A^+. A linear map $\varphi : A \to B$ between JC-algebras A and B is called *a (Jordan) homomorphism* if it preserve the Jordan product. A Jordan homomorphism which is one to one is called *a Jordan isomorphism*. *A representation* of a JC-algebra A is a (Jordan) homomorphism $\pi : A \to B(H)_{s.a}$, for some complex Hilbert space H. It is known that a (Jordan) homomorphism φ between JC-algebras A and B is continuous, and $\varphi(A)$ is a JC-subalgebra B [7, 3.4.3]. A JC-algebra A is said to be *reversible* if $a_1a_2\cdots a_n + a_na_{n-1}\cdots a_1 \in A$ whenever $a_1, a_2, \cdots, a_n \in A$, and is said to be *universally reversible* if $\pi(A)$ is reversible for every representation π of A [6, p. 5].

A projection e of a JW-algebra M is said to be *abelian* if eMe is associative. A JW-algebra M is said to be of *Type I_n* if there is a family of abelian projections $(e_\alpha)_{\alpha\in J}$ such that $c_M(e_\alpha) = 1_M$, $\sum\limits_{\alpha\in J} e_\alpha = 1_M$ and card $J = n$. *A JW-factor* is a JW-algebra with trivial centre; It is known that (see [7, section 5.3]) each Type I JW-factor is of Type I_n, for some n, probably infinite. *A spin factor V_k* is a $(k+1)$-dimensional JW-factor of Type I_2. A spin factor V is universally reversible when $\dim V = 3$ or 4, non-reversible when $\dim V \neq 3, 4$ or 6, and it can be either reversible or non-reversible if $\dim V = 6$ [1, p. 280], [7, 6.2.5]. It was shown in [18, Theorems 6.4 and 6.6] that in every JW-algebra M there is a unique central projection e such that $M = eM \oplus (1-e)M$, where eM is a universally reversible JW-algebra and $(1-e)M$ is a JW-algebra of Type I_2. Every JW-algebra with out a direct summand of Type I_2 is universally reversible, and a JC-algebra is universally reversible if and only if it has no factor representation onto a spin factor other than V_2 and V_3 [6, Theorem 2.2.] and [7, 6.2.3].

If A is a JC-algebra (resp. a JW-algebra), let $C^*(A)$ (resp. $W^*(A)$) be its universal enveloping C*-algebra (resp. von Neumann algebra), and let θ (resp. Φ) be the canonical involutory *-antiautomorphism of $C^*(A)$ (resp. $W^*(A)$). Usually we will regard A as generating Jordan subalgebra of $C^*(A)$ and $W^*(A)$) so that θ and Φ fix each point of A [7, 7.1.8, 7.1.9]. It is known that $C^*(A)^{**} \cong W^*(A^{**})$ when A is a JC-algebra [7, 7.1.11], and that a JC-algebra A is universally reversible if and only if it is reversible in $C^*(A)$. The reader is refered to [1], [2], [6], [7], [17], [18] for a detailed account of the theory of JC-algebras and JW-algebras. The relevant background on the theory of C*-algebras and von Neumann algebras can be found in [12], [19].

Let A and B be a pair of JC-algebras canonically embedded in their respective universal enveloping C*-algebras $C^*(A)$ and $C^*(B)$, respictively, and let λ be a C*-norm on $C^*(A) \otimes C^*(B)$, the algebraic tenor product of $C^*(A)$ and $C^*(B)$. Then the *JC-tensor product of A and B with respect to* λ is the completion $JC(A \underset{\lambda}{\otimes} B)$ of the real Jordan algebra $J(A \otimes B)$ generated by $A \otimes B$ in $C^*(A) \underset{\lambda}{\otimes} C^*(B)$.

Given a pair of JW-algebras, M and N, canonically embedded in their respective universal enveloping von Neumann algebras $W^*(M)$ and $W^*(N)$), respictively, the *JW-tensor product $JW(M \bar{\otimes} N)$ of M and N* is defined to be the JW-algebra generated by $M \otimes N$ in the von Neumann tensor product $W^*(M) \bar{\otimes} W^*(N)$ of

$W^*(M)$ and $W^*(N)$.

The reader is referred to [10] for the properties of the JC-tensor product of JC-algebras and to [11] for the properties of the JW-tensor product of JW-algebras.

Theorem [10, Corollary 2.3]**.** *Let A and B be JC-algebras. Then*

$$C^*(JC(A \underset{\lambda}{\otimes} B) = C^*(A) \underset{\lambda}{\otimes} C^*(B)$$

where λ is the minimum (min) or the maximum (max) C^*-norm.

Theorem [11, Corollary 2.3]**.** *Let M be a JW-algebra and N be a JW-algebra with out type $I_{2,k}$ part. Then*

$$W^*(M \bar{\otimes} N) = W^*(M) \bar{\otimes} W^*(N).$$

1. Nuclearity and injectivity

Let $\mathfrak{A}$, $\mathfrak{B}$ be C*-algebras, then the identity map $x \mapsto x : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B} \hookrightarrow \mathfrak{A} \underset{\min}{\otimes} \mathfrak{B}$ extends to a *-homomorphism from $\mathfrak{A} \underset{\max}{\otimes} \mathfrak{B}$ onto $\mathfrak{A} \underset{\min}{\otimes} \mathfrak{B}$ [19, p. 208]. A C*-algebra $\mathfrak{A}$ is said to be *nuclear* if the maximal and the minimal C*-norms on $\mathfrak{A} \otimes \mathfrak{B}$ coincide for any C*-algebra $\mathfrak{B}$. Equivalently, if the canonical *-homomorphism from $\mathfrak{A} \underset{\max}{\otimes} \mathfrak{B}$ onto $\mathfrak{A} \underset{\min}{\otimes} \mathfrak{B}$ is an isomorphism. The class of nuclear C*-algebras includes all finite dimensional C*-algebras and all abelian C*-algebras [12, 11.3.7, 11.3.11]. If $\mathfrak{A}$ is a C*-algebra, let $M_n(\mathfrak{A})$ denote the algebra of $n \times n$ matrices $x = [x_{ij}]$, $x_{ij} \in \mathfrak{A}$ with the usual matrix product and the *-operation $[x_{ij}]^* = [x_{ji}^*]$. If $\mathfrak{A}$ acts as an algebra of operators on a Hilbert space H then $M_n(\mathfrak{A})$ acts as a C*-algebra of operators on $H_n = H \oplus ... \oplus H$ (n copies of H) if we define

$$[x_{ij}](\xi_1, \cdots, \xi_n) = (\sum_{,j} x_{1j}\xi_j, \cdots, \sum_{,j} x_{nj}\xi_j),\ (\xi_1, \cdots, \xi_n) \in H_n.$$

The norm thus defined on $M_n(\mathfrak{A})$ does not depend on the particular space H on which $\mathfrak{A}$ acts since an isomorphism of C*-algebras is an isometry. An element $x = [x_{ij}]$, $x_{ij} \in \mathfrak{A}$ in $M_n(\mathfrak{A})$ is positive if and only if

$$\sum_{i,,j} < x_{ij}\xi_j, \xi_i > \geq 0, \qquad \xi_1, \cdots, \xi_n \in H.$$

Let $\mathfrak{A}, \mathfrak{B}$ be C*-algebras (or JC-algebras), and let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a linear mapping. Define $\varphi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$. We say that φ is called *positive* if $\varphi(\mathfrak{A}^+) \subseteq \mathfrak{B}^+$, it is called *n-positive* if $\varphi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ is positive, and it is called completely positive if φ_n is positive for all n. If φ is *completely positive* with $\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$, then it is called a *morphism*. It is easy to see that the composition of completely positive maps is completely positive.

A C*-algebra (respectively a von Neumann algebra) $\mathfrak{A}$ is called *injective* if given any C*-algebras $\mathfrak{B}, \mathfrak{C}$ with $\mathfrak{B} \subseteq \mathfrak{C}$, and any morphism $\varphi : \mathfrak{B} \to \mathfrak{A}$, there is a morphism $\hat{\varphi} : \mathfrak{C} \to \mathfrak{A}$ which extends φ [14, p.164]. The well known example of an injective von Neumann algebra is $B(H)$, the von Neumann algebra of all bounded linear operators on a Hilbert space H [14, Theorem 3.2.]. The connection between the nuclearity of a C*-algebras $\mathfrak{A}$ and the injectivity of its second dual $\mathfrak{A}^{**}$ is given in [3, Theorem 3.], where it is shown that $\mathfrak{A}$ is nuclear if and only if $\mathfrak{A}^{**}$ is injective. A characterisation of injective von Neumann algebras in terms of projections is give in [16, Theorem p.46] where it is proved that a von Neumann algebra $\mathfrak{M} \subseteq B(H)$ is injective.if and only if there is a Banach space projection $P : B(H) \to \mathfrak{M}$ of norm 1 (that is $P^2 = P$ and $\|P\| = 1$).

Our aim in this paper is to establish the Jordan analogue of these results.

Recall that a JC-algebra A is said to be nuclear if for any JC-algebra B all the restrictions of C*-norms on $C^*(A) \otimes C^*(B)$ coincide on $J(A \otimes B)$. Equivalently, the natural surjective map $JC(A \underset{\max}{\otimes} B) \to JC(A \underset{\min}{\otimes} B)$ is an isomorphism for any JC-algebra B [9, Definition 1.1.]. The connection between the nuclearity of a JC-algebra and the nuclearity of its universal enveloping is given in Theorem 1.2 of [9] which asserts that any JC-algebra A is nuclear if and only if $C^*(A)$ is nuclear.

Definition 1.1. Let $M \subseteq B(H)_{s.a}$ be a JW-algebra (respectively JC-algebra). We say that M is injective if for any C*-algebras $\mathfrak{B}, \mathfrak{C}$ with $\mathfrak{B} \subseteq \mathfrak{C} \subseteq B(K)$ and any morphism $\varphi : \mathfrak{B}_{s.a} \to M$ there is a morphism $\hat{\varphi} : \mathfrak{C}_{s.a} \to M$ such that $\hat{\varphi} |_{\mathfrak{B}_{s.a}} = \varphi$. Throughout, whenever $\mathfrak{C}$ is a C*-algebra, we let $\mu_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}_{s.a}$ be the projection defined by $\mu_{\mathfrak{C}}(x) = \dfrac{x + x^*}{2}$, $x \in \mathfrak{C}$.

Theorem 1.2. *Let $M \subseteq B(H)_{s.a}$ be a JW-algebra. Then M is injective if and only if there is a Banach space projection $P : B(H) \to M$ of norm 1.*

Proof. Suppose that M is injective, then considering M as a norm closed selfadjoint subspace of $B(H)$, the identity map $id : M \to M$, being a homomorphism is completely positive (see [13, Lemma 2.4]), and hence it extends to a morphism $P : B(H)_{s.a} \to M$ such that $P |_M = id$. Let $\widetilde{P} : B(H) \to M$ be defined by $\widetilde{P}(x) = P(\frac{x+x^*}{2})$. Clearly $\widetilde{P}$ is the desired projection.

Conversely, let $P : B(H) \to M$ be a norm 1 projection onto M, and let $\mathfrak{B}, \mathfrak{C}$ be C*-algebras with $\mathfrak{B} \subseteq \mathfrak{C}$. If $\varphi : \mathfrak{B}_{s.a} \to M$ is a morphism and $i_1 : M \hookrightarrow B(H)_{s.a} \hookrightarrow B(H)$ is the natural injection, then $i_1 \circ \varphi : \mathfrak{B}_{s.a} \to B(H)$is completely positive, since i_1 is completely positive. Since $\mu_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}_{s.a}$ is a projection, it is completely positive, and hence the coposition map $\psi = i_1 \circ \varphi \circ \mu_{\mathfrak{B}} : \mathfrak{B} \xrightarrow{\mu_{\mathfrak{B}}} \mathfrak{B}_{s.a} \xrightarrow{\varphi} M \xhookrightarrow{i_1} B(H)$ is completely positive. By [14, Theorem 3.2.], ψ has a completely positive extension $\hat{\psi} : \mathfrak{C} \to B(H)$. But then $\hat{\varphi} = P \circ \hat{\psi} \circ i_2 : \mathfrak{C}_{s.a} \xhookrightarrow{i_2} \mathfrak{C} \xrightarrow{\hat{\psi}} B(H) \xrightarrow{P} M$ is completely positive which extends φ, and obviously is a morphism, completing the proof. $\square$

Theorem 1.3. *Let M be a JW-algebra with no Type I_2 part. Then M is injective if and only if $W^{*(M)}$ is injective.*

Proof. Since M has no Type I_2 part, $M = W^*(M)_{s.a}^{\Phi_M} = \{x \in W^*(M) : x = x^* = \Phi_M(x)\}$, where Φ_M is the canonical antiautomorphism of $W^*(M)$ [7, 7.3.3]. Let $W^*(M)^{\Phi_M} = \{x \in W^*(M) : x = \Phi_M(x)\}$, and let $\eta : W^*(M) \xrightarrow{\eta} W^*(M)^{\Phi_M}$ be defined by $\eta(x) = \dfrac{x + \Phi_M(x)}{2}$. It is easy to see that η is a projection of $W^*(M)$ onto $W^*(M)^{\Phi_M}$, since Φ_M is of order 2. Now, let $\theta = \eta \circ \mu_{W^*(M)} : W^*(M) \xrightarrow{\mu_{W^*(M)}} W^*(M)_{s.a} \xrightarrow{\eta} W^*(M)_{s.a}^{\Phi_M} = M$, since $\mu_{W^*(M)} \circ \eta = \eta \circ \mu_{W^*(M)}$, θ is a projection.
Suppose that M is injective, and let $\mathfrak{B}, \mathfrak{C}$ be C*-algebras such that $\mathfrak{B} \subseteq \mathfrak{C}$. Let $\psi : \mathfrak{B} \to W^*(M)$ be a morphism, then $\varphi = \theta \circ \psi \circ i_{\mathfrak{B}} : \mathfrak{B}_{s.a} \overset{i_{\mathfrak{B}}}{\hookrightarrow} \mathfrak{B} \xrightarrow{\psi} W^*(M) \xrightarrow{\theta} W^*(M)_{s.a}^{\Phi_M} = M$ is a morphism, where $\mathfrak{B}_{s.a} \overset{i_{\mathfrak{B}}}{\hookrightarrow} \mathfrak{B}$ is the natural inclusion map. Hence, it extends to a morphism $\hat{\varphi} : \mathfrak{C}_{s.a} \to M$. It is clear now that the composition map $\hat{\psi} = i_{W^*(M)} \circ \hat{\varphi} \circ \mu_{\mathfrak{C}} : \mathfrak{C} \xrightarrow{\mu_{\mathfrak{C}}} \mathfrak{C}_{s.a} \xrightarrow{\hat{\varphi}} M = W^*(M)_{s.a} \overset{i_{W^*(M)}}{\hookrightarrow} W^*(M)$ is a morphism which extends ψ.

Conversely, suppose that $W^*(M)$ is injective. We may suppose that $W^*(M) \subseteq B(H)$ [19, Theorem 1.9.18], then there is a norm 1 projection P of $B(H)$ onto $W^*(M)$ by [16, Theorem], and so, $\theta \circ P : B(H) \xrightarrow{P} W^*(M) \xrightarrow{\theta} W^*(M)_{s.a}^{\Phi_M} = M$ is a norm 1 projection of $B(H)$ onto M. Therefore, M is injective by Theorem 1.2. $\square$

Lemma 1.4. *Let $M \subseteq B(H)_{s.a}$ be a JW-algebra such that $M = \mathfrak{M}_{s.a}$ for some von Neumann algebra $\mathfrak{M}$. Then M is injective if and only if $\mathfrak{M}$ is injective.*

Proof. Suppose M is injective. Since $M = \mathfrak{M}_{s.a}$, M is universally reversible, by [7, 7.4.6] and hence it has no Type I_2 part [18, Theorems 6.4 and 6.6]. Thus $W^*(M)$ is injective, by Theorem 1.3. The identity map $id : M = \mathfrak{M}_{s.a} \to \mathfrak{M}$ extends to a normal *-homomorphism $\hat{id} : W^*(M) \to \mathfrak{M}$, by the universal property of $W^*(M)$. Since $\mathfrak{M}$ is the von Neumann algebra generated by $M = \mathfrak{M}_{s.a}$, $\hat{id}(W^*(M)) = \mathfrak{M}$. Since $\ker \hat{id}$ is a weakly closed ideal of $W^*(M)$, $\ker \hat{id} = eW^*(M)$ for some central projection e of $W^*(M)$ [12, 6.8.8]. Hence, $\mathfrak{M} \cong (1-e)W^*(M)$. That is, $W^*(M) = eW^*(M) \oplus (1-e)W^*(M) \cong \ker \hat{i} \oplus \mathfrak{M}$, which implies that $\mathfrak{M}$ is injective, by [5, Corollary 3.2 and Theorem 5.1]. The converse is immediate, since by [16, Theorem], there is a norm 1 projection P of $B(H)$ onto $\mathfrak{M}$, and so $\mu_{\mathfrak{M}} \circ P : B(H) \xrightarrow{P} \mathfrak{M} \xrightarrow{\mu_{\mathfrak{M}}} \mathfrak{M}_{s.a} = M$ is a norm 1 projection of $B(H)$ onto M. $\square$

Let X be a compact hypertsonean space, and A a JC-algebra. Let $C(X, A)$ denote the set of all continuous functions on X with values in A. We shall denote by $C_{\mathbb{C}}(X)$ (resp. $C_{\mathbb{R}}(X)$) the algebra of all continuous complex-valued (resp. real-valued) functions on X.

Corollary 1.5. *Let M be an associative JW-algebra, then M is injective.*

Proof. Since M is associative, $M \cong C_{\mathbb{R}}(X)$ for some compact hypertsonean space X [7, 3.2.2]. Since $C_{\mathbb{C}}(X)$ is an abelian von Neumann algebra, it is of Type I_1 and hence is injective by [5, Proposition 3.5 and Theorem 5.1.]. Therefore M is injective by Lemma 1.4, since $C_{\mathbb{R}}(X) = (C_{\mathbb{C}}(X))_{s.a}$. □

Recall that [7, 6.3.10] a JW-algebra M is said to be of Type $I_{2,k}$ if every factor representation of M is onto the spin factor V_k. If $k < \infty$, this means that $M = C(X, V_k)$ for some compact hypertsonean space X; and when k is an infinite cardinal, it is equivalent to the existence of a weakly dense JC-subalgebra in M of the form $C(X, V_k)$ for some compact hypertsonean space X (Stacey [17]).

Theorem 1.6. *M is a JW-algebra of Type I_2, then $W^*(M)$ is injective.*

Proof. Since M is of Type I_2, we may write

$$M = \sum_{k \in K}^{\oplus} M_k,$$

where K is a set of cardinal numbers and where, for each $k \in K$, M_k is a JW-algebra of Type $I_{2,k}$ (see Stacey [17], or [7, 6.3.14]). Thus, for each $k \in K$, there is a compact hypertsonean space X_k and a normal surjective homomorphism

$$\pi_k : C(X_k, V_k)^{**} \to M_k$$

which extends the the identity map $x \mapsto x : C(X_k, V_k) \to M_k$, since $C(X, V_k)^- = M_k$. By the universal property, this extends to a normal homomorphism

$$\overset{\wedge}{\pi}_k : W^*(C(X_k, V_k)^{**}) \to W^*(M_k)$$

Note that since $C(X_k, V_k)$ is the JC-algebra $C_{\mathbb{R}}(X_k) \underset{\min}{\otimes} V_k$ generated by $C_{\mathbb{R}}(X_k) \otimes V_k$ in $C_{\mathbb{C}}(X_k) \underset{\min}{\otimes} C^*(V_k)$, we have

$$C_{\mathbb{C}}(X_k) \underset{\min}{\otimes} C^*(V_k) = C^*(C(X_k, V_k)) = C(X_k, C^*(V_k)),$$

by Grothendieck's result [19, 4.4.14, 4.7.3]. Therefore,

$$W^*(C(X_k, V_k)^{**}) = C^*(C(X_k, V_k))^{**} = C(X_k, C^*(V_k))^{**},$$

by [7, 7.1.11]. Since $C^*(V_k)$ can be realised as an inductive limit of finite dimensional C*-algebras [7, 6.2.1, 6.2.2], it is nuclear, by [12, 11.3.12]. Consequently, $C^*(C(X_k, V_k)) = C_{\mathbb{C}}(X_k) \underset{\min}{\otimes} C^*(V_k)$ is nuclear, by [12, 11.3.7] and [15, p. 389], and hence $C(X_k, C^*(V_k))^{**}$ is injective, by [5, Theorem 6.2.] or [15, Theorem A.]. But since $W^*(M_k)$ is isomorphic to a W*-closed ideal of $C(X_k, C^*(V_k))^{**}$, it is injective by [5, Proposition 3.1.]. Therefore,

$$W^*(M) = \sum_{k \in K}^{\oplus} W^*(M_k)$$

is injective, by [5, Proposition 5.4.], proving the Theorem. □

Lemma 1.7. *Let $M_i \subseteq B(H_i)$, $i = 1, 2$ be a JW-algebra. Then $M = M_1 \oplus M_2$ is injective if and only if each M_i is injective.*

Proof. Identify H_1 with $H_1 \oplus 0 \hookrightarrow H_1 \oplus H_2$ and H_2 with $0 \oplus H_2 \hookrightarrow H_1 \oplus H_2$, and let f_i be the projection from $H_1 \oplus H_2$ onto H_i. Also identify $B(H_1)$ with $B(H_1) \oplus 0 \hookrightarrow B(H_1) \oplus B(H_2) \hookrightarrow B(H_1 \oplus H_2)$ and $B(H_2)$ with $0 \oplus B(H_2) \hookrightarrow B(H_1) \oplus B(H_2) \hookrightarrow B(H_1 \oplus H_2)$ and note that f_i is the identity of $B(H_i)$ so that $M_1 \oplus M_2 \subseteq B(H_1 \oplus H_2)$. Define the map $\theta_{f_i} : B(H_1 \oplus H_2) \to B(H_i)$ by $\theta_{f_i}(x) = f_i x f_i$, $x \in B(H_1 \oplus H_2)$, it is not hard to see that θ_{f_i} is a norm one projection of $B(H_1 \oplus H_2)$ onto $B(H_i)$.
Suppose that each M_i is injective. Then there is a norm one projection $P_i : B(H_i) \to M_i$, by Theorem 1.2. If $\varphi_i = P_i \circ \theta_{f_i}$ then, for each $x \in B(H_1 \oplus H_2)$

$$\varphi_i^2(x) = \varphi_i(P_i(f_i x f_i) = P_i(P_i(f_i x f_i)) = P_i(f_i x f_i) = \varphi_i(x),$$

since $P_i \mid_{M_i}$ is the identity map. It is easy to see that $\|\varphi_i\| = 1$, that is, φ_i is a norm one projection of $B(H_1 \oplus H_2)$ onto M_i. Thus the map $P : B(H_1 \oplus H_2) \to M_1 \oplus M_2$ defined by $P(x) = \varphi_1(x) \oplus \varphi_2(x)$ is a norm one projection of $B(H_1 \oplus H_2)$ onto $M_1 \oplus M_2$. Hence $M_1 \oplus M_2$ is injective, by Theorem 1.2.
Conversely, suppose that $M = M_1 \oplus M_2 \subseteq B(H_1 \oplus H_2)$ is injective. Then there is a norm one projection $P : B(H_1 \oplus H_2) \to M_1 \oplus M_2$ of $B(H_1 \oplus H_2)$ onto $M_1 \oplus M_2$, by Theorem 1.2. Let $q_1 : M_1 \oplus M_2 \to M_1 \oplus 0 \cong M_1$ and $q_2 : M_1 \oplus M_2 \to 0 \oplus M_2 \cong M_2$ be the Banach space projections of $M_1 \oplus M_2$ onto M_1 and M_2, respectively. Define $P_i : B(H_i) \to M_i$ by $P_i(x) = q_i \circ P(x)$, $x \in B(H_i)$, then

$$P_i^2(x) = P_i(q_i(P(x))) = q_i(P(q_i(P(x))) = q_i(q_i(P(x))) = q_i(P(x)) = P_i(x),$$

since $q_i(P(x)) \in M_i$, and $P \mid_{M_1 \oplus M_2}$is the identity map. Note that

$$\begin{aligned} \|P_i\| &= \sup\{\|q_i(P(x))\| : x \in B(H_i), \|x\| \leq 1\} \\ &= \sup\{\|q_i(x)\| : x \in B(H_i), \|x\| \leq 1\} \\ &\geq \|q_i(id \mid_{M_i})\| = \|id \mid_{M_i}\| = 1. \end{aligned}$$

Since the reverse inequality is obvious, P_i is a norm one projection of $B(H_i)$ onto M_i, and so, M_i is injective, by Theorem 1.2. □

Corollary 1.8. *Let M be a JW-algebra. Then*

(i) *If M is injective so is $W^*(M)$.*

(ii) *If M has no Type I_2 part and $W^*(M)$ is injective, then M is injective.*

Proof. (i) Let $M = M_1 \oplus M_2$, where M_1 is a JW-algebra with no Type I_2 part and M_2 is a JW-algebra of Type I_2 [17]. Then M_2 is injective, by Theorem 1.6. If M is injective, then M_1 is injective, by Lemma 1.7, and hence $W^*(M_1)$ is injective, by Theorem 1.3. Since $W^*(M) = W^*(M_1 \oplus M_2) = W^*(M_1) \oplus W^*(M_2)$ [8, Lemma

2.6.], $W^*(M)$ is injective, by [5, Proposition 5.4.].
(ii) Follows from Theorem 1.3. $\square$

Proposition 1.9. *If $\{M_i\}_{i\in I}$ is a family of JW-algebras, then $M = \sum_{i\in I}^{\oplus} M_i$ is injective if and only if each M_i is injective.*

Proof. Suppose that $M_i \subseteq B(H_i)$, for each $i \in I$. Let $H = \sum_{i\in I}^{\oplus} H_i$. For each finite subset $J \subseteq I$, let $H_J = \sum_{i\in J}^{\oplus} H_i$ (considered as a subspace of H), $M_J = \sum_{i\in J}^{\oplus} M_J$ and let f_J be the projection from H onto H_J. Suppose that each M_i is injective, then M_J is injective for each finite subset $J \subseteq I$, by Lemma 1.7. So there is a norm 1 projection P_J of $B(H)$ onto M_J. With the set of finite subsets of I directed by inclusion, and since $M_J = \sum_{i\in J}^{\oplus} M_J \hookrightarrow \sum_{i\in I}^{\oplus} B(H_i) \hookrightarrow B(H_J) \hookrightarrow B(H)$, $\{P_J\}$ is a directed net of projections in the unit ball of $B(H)$, which is compact in the topology of simple ultraweak convergence. Let P be the limit point of the net, then P is the desired projection. That is M is injective. The converse is immediate. $\square$

Lemma 1.10. *Let $M \subseteq B(H)$ be a JW-algebra. If M is injective so is $M^{'}$, the set of all elements of $B(H)$ which commutes with M.*

Proof. By Corollary 1.8, $W^*(M)$ is injective. Let $[M]$ be the C*-algebra generated by M in $B(H)$, and let $[M]^-$ be the weak closure of $[M]$ in $B(H)$. Then the identity map $i : M \to [M]^-$ extends to a norma homomorphism $\hat{i} : W^*(M) \to [M]^-$, which obviously surjective since both $W^*(M)$ and $[M]^-$ are generated as von Neumann algebras by M. Hence $[M]^-$ is isomorphic to a direct summand of $W^*(M)$, and so is injective, by [5, Proposition 5.4.]. Therefore the commutant $([M]^-)^{'}$ of $[M]^-$ is injective, by [5, Theorem 5.1 and Theorem 5.3.]. Since $M \subseteq [M]^-$, $([M]^-)^{'} \subseteq M^{'}$. But every element of $M^{'}$ commutes with every element of $[M]^-$, being the weak closure of the set of all finite linear combinations of M, and since multiplication is separately weakly continuous, it follows that $M^{'} \subseteq ([M]^-)^{'}$, and so, $([M]^-)^{'} = M^{'}$. That is $M^{'}$ is injective. $\square$

It is known that given von Neumann algebras $\mathfrak{M}$ and $\mathfrak{N}$ then $\mathfrak{M}\bar{\otimes}\mathfrak{N}$ is injective if an only if both $\mathfrak{M}$ and $\mathfrak{N}$ are injective [5, Proposition 5.6.]. The Jordan analogue of this result is given in the following:

Theorem 1.11. *Let M be a JW-algebras, N be a JW-algebra with no type I_2 parts. Then $JW(M \bar{\otimes} N)$ is injective if and only if M and N both are injective.*

Proof. Note that $JW(M \bar{\otimes} N)$ is universally reversible, by [11, Proposition 2.7.], and hence $W^*(M)\bar{\otimes}W^*(N) = W^*(JW(M\bar{\otimes}N))$, by [11, Theorem 2.9.]. Therefore, M and N are injective JW-algebras, if and only if $W^*(M)$ and $W^*(N)$ are injective von Neumann algebras, by Theorem 1.3, if and only if $W^*(M)\bar{\otimes}W^*(N)$ is injective [5, Proposition 5.6.], if and only if $JW(M \bar{\otimes} N)$ is injective by Theorem 1.3. $\square$

Recall that a C*-algebra $\mathfrak{A}$ is said to be *nuclear* if for every C*-algebra $\mathfrak{B}$, there is only one C*-norm on $\mathfrak{A} \otimes \mathfrak{B}$. Equivalently, the max C*-norm coincides with the min C*-norm on $\mathfrak{A} \otimes \mathfrak{B}$ [15]. A JC-algebra A is said to be nuclear if for any JC-algebra B all the restrictions of C*-norms on $C^*(A) \otimes C*(B)$ coincide on $J(A \otimes B)$. Equivalently, the natural surjective map $JC(A \underset{\max}{\otimes} B) \to JC(A \underset{\min}{\otimes} B)$ is an isomorphism [9]. The connection between the nuclearity of a C*-algebra $\mathfrak{A}$ and the injectivity of its second dual $\mathfrak{A}^{**}$ was studied in [15, Theorem A.], where it has been shown that $\mathfrak{A}$ is nuclear if and only if $\mathfrak{A}^{**}$ is injective. Our next result gives the Jordan analogue of this result.

Theorem 1.12. *Let A be a JC-algebra with no spin factor representation, then A is nuclear if and only if its second dual A^{**} is injective.*

Proof. Note first that $C^*(A)^{**} \cong W^*(A^{**})$ [7, 7.1.11]. Since A has no spin factor representation, A^{**} has no type I_2 part. Hence, A is nuclear if and only if $C^*(A)$ is nuclear [9, Theorem 1.2], if and only if $C^*(A)^{**} \cong W^*(A^{**})$ is injective [15, Theorem A, p. 387], if and only if A^{**} is injective, by Theorem 1.3. □

Recall that, given C*-algebras $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ such that $\mathfrak{B} \subseteq \mathfrak{C}$. If $\mathfrak{A}^{**}$ is injective then the natural homomorphism $\mathfrak{A} \underset{\max}{\otimes} \mathfrak{B} \to \mathfrak{A} \underset{\max}{\otimes} \mathfrak{C}$ is an isomorphism [14, Corollary 3.5.].

Our next Theorem is its Jordan analogue.

Theorem 1.13. *Let A be a JC-algebra with no spin factor representation. If A^{**} is injective then given any JC-algebra B, C with $B \subseteq C$ such that B has no spin factor representation into spin factor of the form V_{4n+1}, $n < \infty$, the natural homomorphism $JC(A \underset{\max}{\otimes} B) \to JC(A \underset{\max}{\otimes} C)$ is an isomorphism.*

Proof. Since B has no factors representation into spin factors of the form V_{4n+1}, $n < \infty$, the inclusion map $\mathfrak{i} : B \hookrightarrow C$ extends to an isomorphism $\hat{\mathfrak{i}} : C^*(B) \to C^*(C)$ (see [10, p. 86]), which implies that the natural map of the algebraic tensor product $id \otimes \hat{\mathfrak{i}} : C^*(A) \otimes C^*(B) \to C^*(A) \otimes C^*(C)$ extends to an isomorphism of $C^*(A) \underset{\min}{\otimes} C^*(B)$ into $C^*(A) \underset{\min}{\otimes} C^*(C)$, by [19, Proposition 4.4.22.]. If $A^{**}$is injective, then A is nuclear, by Theorem 1.12, which implies that $C^*(A)$ is nuclear, by [9, Theorem 1.2.]. Hence the restriction of the map

$$C^*(A) \underset{\max}{\otimes} C^*(B) = C^*(A) \underset{\min}{\otimes} C^*(B) \hookrightarrow C^*(A) \underset{\min}{\otimes} C^*(C) = C^*(A) \underset{\max}{\otimes} C^*(C)$$

to $JC(A \underset{\max}{\otimes} B)$ is the required isomorphism, proving the theorem. □

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