

Every Operator Almost Commutes with a Compact Operator

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ABSTRACT. In this note we set forth three possible definitions of the property of “almost commuting with a compact operator” and discuss an old result of W. Arveson that says that every operator on Hilbert space has the weakest of the three properties. Finally, we discuss some recent progress on the hyperinvariant subspace problem (see the bibliography), and relate it to the concept of almost commuting with a compact operator.

1. Introduction

Let \mathcal{H} be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . We will write \mathbb{K} for the (closed) ideal of all compact operators in $\mathcal{L}(\mathcal{H})$, and, as usual, \mathbb{N} for the set of all positive integers. If $T \in \mathcal{L}(\mathcal{H})$, we will denote by $\{T\}'$ the *commutant* of T , i.e.,

$$\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : ST = TS\}.$$

How might one define the concept of “almost commuting with a compact operator?” There would seem to be three fairly reasonable alternatives, as follows.

Definition 1.1. An operator T in $\mathcal{L}(\mathcal{H})$ will be said to have *property (A)* if there exists a sequence $\{K_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ such that $\|TK_n - K_nT\| \rightarrow 0$ and $\{K_n\}_{n \in \mathbb{N}}$ converges to a nonzero operator in the weak operator topology (WOT). Henceforth,

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any sequence of compact operators that converges in the WOT to a nonzero operator will be called a *nontrivial sequence of compact operators*.

Definition 1.2. An operator T in $\mathcal{L}(\mathcal{H})$ will be said to have *property (E)* if there exist sequences $\{B_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ and $\{K_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ such that $\|B_n - T\| \rightarrow 0$, $K_n B_n = B_n K_n$ for each $n \in \mathbb{N}$, and $\{K_n\}_{n \in \mathbb{N}}$ is a nontrivial sequence of compact operators.

Definition 1.3. An operator T in $\mathcal{L}(\mathcal{H})$ will be said to have *property (PS)* if there exist sequences $\{S_n\}_{n \in \mathbb{N}} \subset \{T\}'$ and $\{K_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ such that $\|S_n - K_n\| \rightarrow 0$ and $\{K_n\}_{n \in \mathbb{N}}$ is a nontrivial sequence of compact operators.

We remark first that property (PS) has been in use for some time. It originated in [4], and played a prominent role in the papers [12], [11], [8], and [3], among others. In the terminology of Lomonosov, it is called the ‘‘Percy-Salinas property’’, and in [11] the following beautiful result was proved, which generalized theorems from [4] and [12]. (A different, perhaps simpler, proof of this result was given in [3].)

Theorem 1.4 (Lomonosov). *If T belongs to $\mathcal{L}(\mathcal{H}) \setminus \mathbb{C}1_{\mathcal{H}}$ and has property (PS), then T has a nontrivial hyperinvariant subspace.*

It should also be said the definition of property (PS) was given more generally in [11], with nets in place of sequences and for operators acting on a complex Banach space \mathcal{X} , where Theorem 1.4 remains true if \mathcal{X} is reflexive.

In this note we discuss how the properties (A), (E), and (PS) are related.

2. Main results

We begin our work with the following proposition.

Proposition 2.1. *An operator T in $\mathcal{L}(\mathcal{H})$ with either property (E) or property (PS) has property (A).*

Proof. Suppose first that T has property (E). Then, with the notation as in Definition 1.2, we have

$$\begin{aligned} \|TK_n - K_nT\| &= \|(T - B_n)K_n + K_n(B_n - T)\| \\ &\leq 2\|K_n\|\|B_n - T\|, \end{aligned}$$

and since the sequence $\{K_n\}$ is WOT-convergent, it is bounded, so T has property (A). On the other hand, if T has property (PS), then, with the notation as in Definition 1.3,

$$\begin{aligned} \|TK_n - K_nT\| &= \|T(K_n - S_n) - (K_n - S_n)T\| \\ &\leq 2\|T\|\|K_n - S_n\| \rightarrow 0, \end{aligned}$$

so again T has property (A). □

This proposition shows that property (A) is the weakest of these three properties, so we adopt it as our definition.

Definition 2.2. An operator T in $\mathcal{L}(\mathcal{H})$ will be said to *almost commute with a compact operator* if it has property (A).

We next consider a property that is stronger than property (E), and its relation to the class of quasidiagonal operators.

Definition 2.3. An operator T in $\mathcal{L}(\mathcal{H})$ will be said to have property (E_s) if there exist sequences $\{B_n\}_{n \in \mathbb{N}}$ and $\{E_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}(\mathcal{H})$ with $B_n E_n = E_n B_n$ for all $n \in \mathbb{N}$ such that $\|B_n - T\| \rightarrow 0$ and $\{E_n\}_{n \in \mathbb{N}}$ is a sequence of finite-rank projections WOT-convergent (equivalently, SOT-convergent) to $1_{\mathcal{H}}$.

It is obvious that every operator that has property (E_s) has property (E). Recall next that an operator T in $\mathcal{L}(\mathcal{H})$ is *quasidiagonal* if there exists an increasing sequence $\{P_n\}_{n \in \mathbb{N}}$ of finite rank projections converging in the SOT to $1_{\mathcal{H}}$ such that

$$\|P_n T - T P_n\| \rightarrow 0 \quad (n \rightarrow \infty).$$

The basic structure theorem for quasidiagonal operators is the following, which utilizes the concept of a block-diagonal operator. By definition, an operator T in $\mathcal{L}(\mathcal{H})$ is *block-diagonal* if T can be written as a (countably infinite, orthogonal) direct sum of operators each of which acts on a finite dimensional space.

Theorem 2.4 (Halmos). *If T is a quasidiagonal operator in $\mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$, then there exists a block-diagonal operator B_ε and a $K_\varepsilon \in \mathbb{K}$ such that $T = B_\varepsilon + K_\varepsilon$ and $\|K_\varepsilon\| < \varepsilon$.*

Theorem 2.5. *An operator T in $\mathcal{L}(\mathcal{H})$ has property (E_s) if and only if it is quasidiagonal.*

Proof. Suppose first that T is quasidiagonal. For each $n \in \mathbb{N}$, set $\varepsilon_n = 1/n$ in Theorem 1.8, and thereby obtain a sequence $\{B_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ of block-diagonal operators and a sequence $\{K_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ such that $T = B_n + K_n$ for each n and $\|K_n\| \rightarrow 0$. Since each B_n is block-diagonal, for each $n \in \mathbb{N}$ there exists an increasing sequence $\{E_k^{(n)}\}_{k \in \mathbb{N}}$ of finite-rank projections converging to $1_{\mathcal{H}}$ in the SOT such that

$$E_k^{(n)} B_n = B_n E_k^{(n)}, \quad k \in \mathbb{N}.$$

Now choose a countable dense set $\{x_n\}_{n \in \mathbb{N}}$ on the unit sphere of \mathcal{H} , which will be used to construct the desired implementing sequence of finite-rank projections converging in the WOT (equivalently, SOT) to $1_{\mathcal{H}}$ as follows. Choose $E_{k_1}^{(1)} \in \{E_k^{(1)}\}_{k \in \mathbb{N}}$ such that $(E_{k_1}^{(1)} x_1, x_1) > 1/2$. Then choose $E_{k_2}^{(2)} \in \{E_k^{(2)}\}_{k \in \mathbb{N}}$ such that both $(E_{k_2}^{(2)} x_1, x_1)$ and $(E_{k_2}^{(2)} x_2, x_2)$ are greater than $2/3$, etc. By an obvious definition by induction, we finally arrive at a sequence $\{E_{k_n}^{(n)}\}_{n \in \mathbb{N}}$ with the property that

$$\lim_{n \rightarrow \infty} (E_{k_n}^{(n)} x_j, x_j) = 1, \quad j \in \mathbb{N}.$$

Since the x_j are dense in the unit sphere of \mathcal{H} , we easily obtain that

$$(E_{k_n}^{(n)}x, x) \rightarrow (1_{\mathcal{H}}x, x), \quad x \in \mathcal{H},$$

which, by the polarization identity, gives us that $\{E_{k_n}^{(n)}\}_{n \in \mathbb{N}}$ converges in the WOT and SOT to $1_{\mathcal{H}}$. Since $\|B_n - T\| \rightarrow 0$ and

$$E_{k_n}^{(n)}B_n = B_nE_{k_n}^{(n)}, \quad n \in \mathbb{N},$$

T has property (E_s) .

On the other hand, if T has property (E_s) , then, with the sequences $\{B_n\}_{n \in \mathbb{N}}$ and $\{E_n\}_{n \in \mathbb{N}}$ as in Definition 2.3, we have

$$\begin{aligned} \|E_nT - TE_n\| &\leq \|E_n(T - B_n)\| + \|(T - B_n)E_n\| \\ &\leq 2\|T - B_n\|, \quad n \in \mathbb{N}, \end{aligned}$$

so $\|E_nT - TE_n\| \rightarrow 0$ and T is quasicommuting. □

There doesn't seem to be any obvious relationship between property (E) (or (E_s)) and property (PS) . However, the following is an old theorem of Arveson [1], although the proof given below is based on a theorem of Arveson and Herrero (cf. [7]), which can be established (as is done in [9]) without using the theory of C^* -algebras.

Theorem 2.6 ([1]). *Every operator T in $\mathcal{L}(\mathcal{H})$ has property (A) , and thus almost commutes with a compact operator. Moreover, the implementing sequence of compact operators $\{K_n\}_{n \in \mathbb{N}}$ from Definition 1.1 can be taken to be a sequence of finite rank, positive semidefinite operators converging in the SOT to $1_{\mathcal{H}}$.*

Proof. The nice theorem of Arveson-Herrero, mentioned above, states that given any operator T in $\mathcal{L}(\mathcal{H})$, there exists an operator $R = R_T$ in $\mathcal{L}(\mathcal{H})$ such that $R \oplus T$ is a quasicommuting operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. Thus, by Proposition 2.5, there exist sequences $\{B_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ and $\{E_n\}_{n \in \mathbb{N}} \subset \mathbb{K}(\mathcal{H} \oplus \mathcal{H})$ such that $\|B_n - (R \oplus T)\| \rightarrow 0$ and $\{E_n\}_{n \in \mathbb{N}}$ is a sequence of finite-rank projections converging in the SOT to $1_{\mathcal{H} \oplus \mathcal{H}}$ with $E_nB_n = B_nE_n$ for each $n \in \mathbb{N}$. Since

$$\|E_n(R \oplus T) - (R \oplus T)E_n\| \leq 2\|B_n - (R \oplus T)\|,$$

it is clear that

$$(1) \quad E_n(R \oplus T) - (R \oplus T)E_n \longrightarrow 0.$$

Thus, if we write E_n as a 2×2 matrix with entries from $\mathcal{L}(\mathcal{H})$ —say $E_n = (E_{ij}^{(n)})_{i,j=1,2}$, then we get from (1) that $\|TE_{22}^{(n)} - E_{22}^{(n)}T\| \rightarrow 0$, and since $\{E_n\}_{n \in \mathbb{N}}$ has the properties set forth above, it is clear that $\{E_{22}^{(n)}\}_{n \in \mathbb{N}}$ is a sequence of finite rank, positive semidefinite, operators converging in the SOT to $1_{\mathcal{H}}$. □

Remark 2.7. It is a result of Hadwin [8] that not every operator in $\mathcal{L}(\mathcal{H})$ has property (PS). Thus property (A) is strictly weaker than property (PS).

Remark 2.8. The primary purpose of this note is to bring properties (A) and (E) to the attention of the interested reader, with the hope that one or the other might be used to enlarge the class of operators in $\mathcal{L}(\mathcal{H})$ known to have nontrivial invariant or hyperinvariant subspaces, perhaps using the ideas below.

Problem 2.9. Does every quasihyponormal operator have property (PS)?

This is an important question, because, by virtue of very recent work in [2], [5], [6] and [10], one knows the following.

Theorem 2.10. *Every operator T in $\mathcal{L}(\mathcal{H})$ that is not algebraic (i.e., such that there exists no nonzero polynomial p with $p(T) = 0$) has the property that $\text{Hlat}(T)$ (i.e., the lattice of all hyperinvariant subspaces of T) is lattice isomorphic to $\text{Hlat}(Q)$ for some quasihyponormal operator Q in $\mathcal{L}(\mathcal{H})$.*

An immediate corollary of Theorems 1.4 and 2.10 is the following.

Corollary 2.11. *If every quasihyponormal operator in $\mathcal{L}(\mathcal{H})$ has property (PS), then every nonscalar operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial hyperinvariant subspace.*

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