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Stable Rank of Group C^*-algebras of Some Disconnected Lie Groups

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Abstract. We estimate the stable rank and connected stable rank of group C^*-algebras of certain disconnected solvable Lie groups such as semi-direct products of connected solvable Lie groups by the integers.

Introduction

The stable rank for C^*-algebras was introduced by Rieffel [13] as a noncommutative counterpart to the covering dimension of topological spaces. Indeed, the stable rank of the commutative C^*-algebra of all continuous functions on a compact Hausdorff space is computed by the covering dimension of the space (see (F2) below). For the (full) group C^*-algebras of Lie groups that are noncommutative in general but close to commutative C^*-algebras in some sense such as K-theory, the Rieffel's question [13, Question 4.14] is to describe the stable rank of the group C^*-algebras in terms of the structure of Lie groups. For this interesting question, some partial answers were obtained by Sheu [15], Takai-Sudo [23], [24] and the author [16], [17], [18] and [21] for the connected case, and by [19] and [20] for the disconnected case. On the other hand, in [22] we showed that the group C^*-algebras of some connected Lie groups such as the motion groups have stable rank one.

Our question as the motivation of this paper is whether or not the group C^*-algebras of disconnected solvable Lie groups such as semi-direct products of connected solvable Lie groups by the integers have stable rank one. We have already considered the similar question for the connected solvable case, and obtained some results in [23], [24], [17] and [22]. Since the class of C^*-algebras with stable rank one is quite important in the C^*-algebra theory, our question should be reasonable and interesting in some sense. Indeed, among other things, the stable rank one condition for C^*-algebras implies the cancellation of their projections (cf. [2]). See also [14] for some relations among stable rank, connected stable rank, and K-groups for C^*-algebras.

As the main results we show that the group C^*-algebras of semi-direct products

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of (most) non-compact connected solvable Lie groups by the integers $\mathbb{Z}$ have stable rank more than one, and the group C^*-algebras of semi-direct products of compact connected commutative Lie groups (that are the k-tori $\mathbb{T}^k$), more generally, of compact Lie groups, by $\mathbb{Z}$ have stable rank one and connected stable rank two. In addition, we obtain the stable rank estimates in the case of semi-direct products of compact Lie groups by finite cyclic groups. For the proofs, we consider the structure of those group C^*-algebras, and use some basic formulas for the K-groups of C^*-algebras such as the Pimsner-Voiculescu six term exact sequence and some basic results on the stable rank and connected stable rank of C^*-algebras (see below). Our main interest in this paper is the lower bounds for the stable rank of those group C^*-algebras. This point should be new and interesting. See [17], [18], $\cdots$, [21] for some results on the upper bounds for the stable rank of group C^*-algebras. We also consider the estimates of the stable rank for group C^*-algebras of semi-direct products of amenable or non-amenable locally compact groups by their quotient group C^*-algebras.

Notation and facts. Let $C_0(X)$ be the C^*-algebra of continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space X. When X is compact, we set $C(X) = C_0(X)$. For a locally compact group G, we denote by $C^*(G)$ its full group C^*-algebra (cf. Pedersen [11, Chapter 7]). Let $\mathbb{K}$ be the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space. For a C^*-algebra $\mathfrak{A}$ (or its unitization $\mathfrak{A}^+$), its stable rank and connected stable rank are denoted by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ respectively (cf. Rieffel [13]). By definition, for $n \in \mathbb{N}$, $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, and $\mathrm{csr}(\mathfrak{A}) \leq n$ if and only if $GL_m(\mathfrak{A})_0$ acts transitively on $L_m(\mathfrak{A})$ for all $m \geq n$, and equivalently, $L_m(\mathfrak{A})$ for all $m \geq n$ are connected, where $L_n(\mathfrak{A}) = \{(a_j)_{j=1}^n \in \mathfrak{A}^n \mid \sum_{j=1}^n a_j^* a_j \text{ is invertible in } \mathfrak{A}\}$, and $GL_m(\mathfrak{A})_0$ is the connected component of $GL_m(\mathfrak{A})$ with the identity matrix ([13, Corollary 8.5]). Recall the following formulas:

(F1) : $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ for any C^*-algebra $\mathfrak{A}$,

(F2) : $\mathrm{sr}(C(X)) = [\dim X/2] + 1, \ \mathrm{csr}(C(X)) \leq [(\dim X + 1)/2] + 1,$

(F3) : For an exact sequence of C^*-algebras: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$, $\mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \leq \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}),$

where $\dim X$ is the covering dimension of X, and $[x]$ means the maximum integer $\leq x$, and $\vee$ is the maximum and $\wedge$ is the minimum (Rieffel [13, Proposition 1.7, Theorems 4.3, 4.4, 4.11, Corollary 4.10 and p.328], Nistor [8] and Sheu [15, Theorems 3.9 and 3.10]). Let $\mathfrak{A} \rtimes_\alpha G$ be the (full) crossed product of a C^*-algebra $\mathfrak{A}$ by a locally compact group G with α an action, that is, a homomorphism from G to the automorphism group of $\mathfrak{A}$ (cf. [11]). We often omit the symbol α in what follows. Let $K_0(\mathfrak{A})$, $K_1(\mathfrak{A})$ be the K-groups of a C^*-algebra $\mathfrak{A}$. The following Pimsner-Voiculescu six term exact sequence (P-V sequence, for short) is known (cf.

Blackadar [2, Section 10.2]):

$$\begin{array}{ccccc} K_0(\mathfrak{A}) & \xrightarrow{\mathrm{id}-\alpha_*} & K_0(\mathfrak{A}) & \xrightarrow{i_*} & K_0(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(\mathfrak{A}) & \xleftarrow{\mathrm{id}-\alpha_*} & K_1(\mathfrak{A}), \end{array}$$

where id means the identity map on $\mathfrak{A}$, i is the canonical inclusion from $\mathfrak{A}$ to $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$, and i_*, $\mathrm{id} - \alpha_* = (\mathrm{id} - \alpha)_*$ are the induced maps from i, $\mathrm{id} - \alpha$ on K-groups respectively. Furthermore,

$$(F4): \quad \mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1, \quad \mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1$$

for a (unital) C^*-algebra $\mathfrak{A}$ (in the second) by Rieffel [13, Theorem 7.1 and Corollary 8.6].

1. The main results

Theorem 1.1. *Let G be a simply connected solvable Lie group and $G \rtimes_\alpha \mathbb{Z}$ a semi-direct product of G by $\mathbb{Z}$ with α an action. Then*

$$\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) \geq 2.$$

Moreover, we have $\mathrm{csr}(C^*(G \rtimes_\alpha \mathbb{Z})) \geq 2$.

Proof. First note that $C^*(G \rtimes_\alpha \mathbb{Z}) \cong C^*(G) \rtimes_\alpha \mathbb{Z}$. Since G is a simply connected solvable Lie group, the quotient group $G/[G,G]$ of G by the (Lie) commutator $[G,G]$ of G is isomorphic to $\mathbb{R}^n$ for some $n \geq 1$ since $G/[G,G]$ is a simply connected commutative Lie group (note that there exists the following exact sequence: $0 = \pi_1(G) \to \pi_1(G/[G,G]) \to [G,G]/[G,G]_0 = 0$, where $\pi_1(\cdot)$ means the fundamental group and $[G,G]_0$ is the connected component of the identity). Then we have the quotient: $C^*(G) \rtimes_\alpha \mathbb{Z} \to C^*(G/[G,G]) \rtimes_\alpha \mathbb{Z} \to 0$ since the spectrum of $G/[G,G]$ just corresponds to the space of 1-dimensional representations of G, and this space is invariant under the action α. Furthermore, by the Fourier transform we have

$$\begin{aligned} C^*(G/[G,G]) \rtimes_\alpha \mathbb{Z} &\cong C_0((G/[G,G])^\wedge) \rtimes_{\hat\alpha} \mathbb{Z} \\ &= C_0((\mathbb{R}^n)^\wedge) \rtimes_{\hat\alpha} \mathbb{Z} = C_0(\mathbb{R}^n) \rtimes_{\hat\alpha} \mathbb{Z}, \end{aligned}$$

where $(G/[G,G])^\wedge$ means the dual group of $G/[G,G]$, and $\hat\alpha$ is the dual action of α via the duality on $\mathbb{R}^n$. If the action $\hat\alpha$ on $\mathbb{R}^n$ is trivial, then

$$C_0(\mathbb{R}^n) \rtimes_{\hat\alpha} \mathbb{Z} \cong C_0(\mathbb{R}^n) \otimes C^*(\mathbb{Z}) \cong C_0(\mathbb{R}^n) \otimes C(\mathbb{T}) \cong C_0(\mathbb{R}^n \times \mathbb{T}).$$

Since $\dim(\mathbb{R}^n \times \mathbb{T}) \geq 2$, it follows by (F2) that $\mathrm{sr}(C_0(\mathbb{R}^n \times \mathbb{T})) \geq 2$. Thus, we may assume that $\hat\alpha$ on $\mathbb{R}^n$ is nontrivial in the following. Since the origin 0 of $\mathbb{R}^n$ is fixed under the action $\hat\alpha$, we have

$$(E): \quad 0 \to C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{Z} \to C_0(\mathbb{R}^n) \rtimes_{\hat\alpha} \mathbb{Z} \to \mathbb{C} \rtimes \mathbb{Z} \to 0$$

and $\mathbb{C} \rtimes \mathbb{Z} = C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform.

Applying the P-V sequence to the crossed product $C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}$ in the middle of the above exact sequence (E), we obtain

$$\begin{array}{ccccc}
K_0(C_0(\mathbb{R}^n)) & \xrightarrow{\mathrm{id}-\hat{\alpha}_*} & K_0(C_0(\mathbb{R}^n)) & \xrightarrow{i_*} & K_0(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \xleftarrow{i_*} & K_1(C_0(\mathbb{R}^n)) & \xleftarrow{\mathrm{id}-\hat{\alpha}_*} & K_1(C_0(\mathbb{R}^n)).
\end{array}$$

When n is even, by the Bott periodicity, $K_0(C_0(\mathbb{R}^n)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{R}^n)) \cong K_1(\mathbb{C}) \cong 0$. Thus, the following commutative diagram holds:

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{\mathrm{id}-\hat{\alpha}_*} & \mathbb{Z} & \xrightarrow{i_*} & K_0(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \xleftarrow{i_*} & 0 & \xleftarrow{\mathrm{id}-\hat{\alpha}_*} & 0.
\end{array}$$

Since the map $\mathrm{id} - \hat{\alpha}_*$ in the first line is zero, we deduce that $K_0(C_0(\mathbb{R}^n) \rtimes \mathbb{Z}) \cong \mathbb{Z}$ and $K_1(C_0(\mathbb{R}^n) \rtimes \mathbb{Z}) \cong \mathbb{Z}$. Indeed, since the group for $\hat{\alpha}$ is $\mathbb{Z}$, there exists an implementing unitary U such that $\hat{\alpha}_1 = \mathrm{Ad}U$ (the adjoint action by U). Note also that for the crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ of a (unital) C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$, there exists an implementing unitary U such that $UaU^* = \alpha_1(a)$ for $a \in \mathfrak{A}$, where U is not necessarily contained in $\mathfrak{A}$ or its multiplier algebra (if $\mathfrak{A}$ is non unital, and if so, we can consider its unitization by $\mathbb{C}$ and the trivially extended action on it) (for example, the rotation algebra generated by two unitaries U, V with $VU = e^{2\pi i\theta}UV$ for some real number θ can be written as the crossed product $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ with the action θ by θ-rotation on the torus $\mathbb{T}$ (cf. [1])). Therefore, for $[p]$ the class in $\mathbb{Z}$ at the upper left corner of the diagram, we have

$$\hat{\alpha}_*([p]) = [\mathrm{Ad}U(p)] = [UpU^*] = [p] = \mathrm{id}([p]).$$

Moreover, note that there exists a homotopy path p_t for $0 \leq t \leq 1$ between UpU^* and p defined by

$$\begin{aligned}
p_t &= w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*, \quad \text{and} \\
w_0 &= \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
w_t &= \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} u_t \begin{pmatrix} U^* & 0 \\ 0 & 1 \end{pmatrix} u_t^* \quad \text{for } 0 \leq t \leq 1, \text{ where} \\
u_t &= \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}
\end{aligned}$$

(cf. [25]). When n is odd, we obtain the same conclusion for K-groups of $C_0(\mathbb{R}^n) \rtimes \mathbb{Z}$ by the similar way.

Furthermore, applying the P-V sequence to the crossed product $C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{Z}$ in the left of the above exact sequence (E), we obtain

$$\begin{array}{ccccc} K_0(C_0(\mathbb{R}^n \setminus 0)) & \xrightarrow{\mathrm{id}-\hat{\alpha}_*} & K_0(C_0(\mathbb{R}^n \setminus 0)) & \xrightarrow{i_*} & K_0(C_0(\mathbb{R}^n \setminus 0) \rtimes_{\hat{\alpha}} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C_0(\mathbb{R}^n \setminus 0) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \xleftarrow{i_*} & K_1(C_0(\mathbb{R}^n \setminus 0)) & \xleftarrow{\mathrm{id}-\hat{\alpha}_*} & K_1(C_0(\mathbb{R}^n \setminus 0)). \end{array}$$

Moreover, for $n \geq 2$ we have

$$\begin{aligned} K_0(C_0(\mathbb{R}^n \setminus 0)) &\cong K_0(C_0(\mathbb{R} \times S^{n-1})) \cong K_1(C(S^{n-1})) \\ &\cong K_1(C_0(\mathbb{R}^{n-1})^+) \cong K_1(C_0(\mathbb{R}^{n-1})) \\ &\cong 0 \text{ if } n \text{ is odd, and } \mathbb{Z} \text{ if } n \text{ is even,} \\ K_1(C_0(\mathbb{R}^n \setminus 0)) &\cong K_1(C_0(\mathbb{R} \times S^{n-1})) \cong K_0(C(S^{n-1})) \\ &\cong K_0(C_0(\mathbb{R}^{n-1})^+) \cong K_0(C_0(\mathbb{R}^{n-1})) \oplus \mathbb{Z} \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \text{ if } n \text{ is odd, and } \mathbb{Z} \text{ if } n \text{ is even.} \end{aligned}$$

When $n = 1$, we obtain

$$K_0(C_0(\mathbb{R} \setminus 0)) \cong K_0(C_0(\mathbb{R})) \oplus K_0(C_0(\mathbb{R})) \cong K_1(\mathbb{C}) \oplus K_1(\mathbb{C}) \cong 0$$

and $K_1(C_0(\mathbb{R} \setminus 0)) \cong \mathbb{Z}^2$. Therefore, it follows that $K_0(C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{Z}) \cong \mathbb{Z}^2$ and $K_1(C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{Z}) \cong \mathbb{Z}^2$.

Summing up, we obtain the following six-term exact sequence associated with the above exact sequence (E):

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z}^2, \end{array}$$

where ∂ means the index map. Note that $K_0(C(\mathbb{T})) \cong \mathbb{Z}$ and $K_1(C(\mathbb{T})) \cong \mathbb{Z}$. It follows from this commutative diagram that the map ∂ is nonzero. Therefore, by using [7] or [9], we have $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes \mathbb{Z}) \geq 2$. By (F3), we conclude $\mathrm{sr}(C^*(G) \rtimes_\alpha \mathbb{Z}) \geq 2$.

To estimate the connected stable of $C^*(G) \rtimes_\alpha \mathbb{Z}$, we use the P-V sequence:

$$\begin{array}{ccccc} K_0(C^*(G)) & \xrightarrow{\mathrm{id}-\alpha_*} & K_0(C^*(G)) & \xrightarrow{i_*} & K_0(C^*(G) \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C^*(G) \rtimes_\alpha \mathbb{Z}) & \xleftarrow{i_*} & K_1(C^*(G)) & \xleftarrow{\mathrm{id}-\alpha_*} & K_1(C^*(G)). \end{array}$$

Since G is a simply connected solvable Lie group, it is isomorphic to the k-times successive semi-direct product: $G \cong \mathbb{R} \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$ for $k = \dim G$ (see [6]). By using Connes' Thom isomorphism for crossed products by $\mathbb{R}$ (cf. [2, Section 10.2]) repeatedly,

$$\begin{aligned} K_0(C^*(G)) &\cong K_0(C^*(G_{k-1}) \rtimes \mathbb{R}) \cong K_1(C^*(G_{k-1})) \cong K_1(C^*(G_{k-2}) \rtimes \mathbb{R}) \\ &\cong K_0(C^*(G_{k-2})) \cong \cdots \cong \mathbb{Z} \text{ if } k \text{ even, and } 0 \text{ if } k \text{ odd,} \\ K_1(C^*(G)) &\cong K_1(C^*(G_{k-1}) \rtimes \mathbb{R}) \cong K_0(C^*(G_{k-1})) \cong K_0(C^*(G_{k-2}) \rtimes \mathbb{R}) \\ &\cong K_1(C^*(G_{k-2})) \cong \cdots \cong 0 \text{ if } k \text{ even, and } \mathbb{Z} \text{ if } k \text{ odd,} \end{aligned}$$

where $G = G_k = G_{k-1} \rtimes \mathbb{R}$, $G_1 = \mathbb{R}$ and $G_l = G_{l-1} \rtimes \mathbb{R}$ inductively for $1 \le l \le k$. Therefore, if $\dim G$ is even, we obtain $K_1(C^*(G) \rtimes \mathbb{Z}) \cong \mathbb{Z}$ since the map $\mathrm{id} - \hat{\alpha}_*$ from $K_0(C^*(G))$ in the P-V sequence is zero. If $\dim G$ is odd, we obtain $K_1(C^*(G) \rtimes \mathbb{Z}) \cong \mathbb{Z}$ since the map $\mathrm{id} - \hat{\alpha}_*$ from $K_1(C^*(G)) \cong K_0(\mathbb{C})$ in the P-V sequence is zero. Since $K_1(C^*(G) \rtimes \mathbb{Z})$ is nonzero, we obtain $\mathrm{csr}(C^*(G) \rtimes \mathbb{Z}) \ge 2$ by [4, Corollary 1.6]. □

Remark. When the action α is trivial, we have $C^*(G) \rtimes_\alpha \mathbb{Z} \cong C^*(G) \otimes C^*(\mathbb{Z}) \cong C^*(G) \otimes C(\mathbb{T})$. In addition, $C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong C_0(\mathbb{R}^n \times \mathbb{T})$ so that $\mathrm{sr}(C_0(\mathbb{R}^n \times \mathbb{T})) \ge 2$ by (F2) since $n \ge 1$. Hence, $\mathrm{sr}(C^*(G) \otimes C(\mathbb{T})) \ge 2$. If G is a connected solvable Lie group, then $G/[G,G]$ is isomorphic to the product group $\mathbb{R}^n \times \mathbb{T}^m$ for some $n, m \ge 0$. If $n \ge 1$, then we have $\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) \ge 2$ by considering the quotient from $C_0(\mathbb{R}^n \times \mathbb{Z}^m) \rtimes_{\hat{\alpha}} \mathbb{Z}$ to $C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}$ and by the same proof above.

Using (F4) we obtain $\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) \le \mathrm{sr}(C^*(G)) + 1$ for G a Lie group and α an action. Note that $C^*(G)$ is non-unital in general (if G non-discrete). See [17], [18] and [21] for the estimates of $\mathrm{sr}(C^*(G))$ for G certain (simply connected solvable) Lie groups. See also [19] and [20] for G certain disconnected solvable Lie groups.

As a comparison, we now consider the case of semi-direct products $G \rtimes_\alpha \mathbb{Z}_k$ of simply connected solvable Lie groups G by finite cyclic groups $\mathbb{Z}_k$. By the same analysis as the above proof, the group C^*-algebra $C^*(G \rtimes_\alpha \mathbb{Z}_k)$ has a quotient isomorphic to $C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}_k$ for $n \ge 1$. However, if $\hat{\alpha}$ on $\mathbb{R}^n$ is trivial, then

$$C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}_k \cong C_0(\mathbb{R}^n) \otimes C^*(\mathbb{Z}_k) \cong C_0(\mathbb{R}^n) \otimes C(\mathbb{Z}_k) \cong \oplus^k C_0(\mathbb{R}^n).$$

Therefore, if $n = 1$, then $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}_k) = 1$. Thus, we can not use the same argument as the above proof. Even if $n \ge 2$, we can not use the exact sequence: $0 \to C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{Z}_k \to C_0(\mathbb{R}^n) \rtimes \mathbb{Z}_k \to \mathbb{C} \rtimes \mathbb{Z}_k \to 0$ as (E) in the proof. The reason is that the index map of K-groups associated with this exact sequence vanishes since $K_1(\mathbb{C} \rtimes \mathbb{Z}_k) \cong K_1(\oplus^k \mathbb{C}) \cong 0$. However, the estimate $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes \mathbb{Z}_k) \ge 2$ could be deduced from that $C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{Z}_k \cong C_0(\mathbb{R}_+) \otimes C(S^n) \rtimes \mathbb{Z}_k$ and $\dim S^n/\mathbb{Z}_k \ge 2$, where S^n means the n-dimensional sphere.

We extensively consider the case of semi-direct products of connected solvable Lie groups by $\mathbb{Z}$. By a technical reason, connected solvable Lie groups are restricted to be linearizable as follows:

Theorem 1.2. *Let G be a linearizable connected solvable Lie group and $G \rtimes_\alpha \mathbb{Z}$ a* *semi-direct product of G by $\mathbb{Z}$. If G is noncompact, then*

$$\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) \geq 2.$$

Proof. First recall that a connected solvable Lie group G is linearizable, that is, it has a faithful finite-dimensional representation if and only if G is isomorphic to a semi-direct product $N \rtimes \mathbb{T}^k$ of a simply connected solvable Lie group N by a torus $\mathbb{T}^k$ for $k \geq 0$ ([10, Theorem 7.1 in page 66]). Thus, we have

$$C^*(G \rtimes_\alpha \mathbb{Z}) \cong C^*(N \rtimes \mathbb{T}^k) \rtimes_\alpha \mathbb{Z} \cong (C^*(N) \rtimes \mathbb{T}^k) \rtimes_\alpha \mathbb{Z}.$$

Note the quotient: $(C^*(N) \rtimes \mathbb{T}^k) \rtimes_\alpha \mathbb{Z} \to (C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes_{(\hat{\alpha},\alpha)} \mathbb{Z}$, where $N/[N,N] \cong \mathbb{R}^n$ for some $n \geq 1$ since N is a simply connected solvable Lie group and G is noncompact, and $(\hat{\alpha}, \alpha)$ means that the action by $\mathbb{Z}$ on $C_0(\mathbb{R}^n)$ is $\hat{\alpha}$, and on $\mathbb{T}^k$ is α. Since the origin 0 in $\mathbb{R}^n$ is fixed under $\hat{\alpha}$, we have

$$\begin{aligned} (E_2): \quad & 0 \to (C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z} \equiv \mathfrak{I} \to \\ & (C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes_{(\hat{\alpha},\alpha)} \mathbb{Z} \to C^*(\mathbb{T}^k) \rtimes \mathbb{Z} \to 0. \end{aligned}$$

We first compute the K-groups of the crossed product $C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k$ as follows:

$$\begin{aligned} & K_0(C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \cong K_0^{\mathbb{T}^k}(C_0(\mathbb{R}^n)) \\ & \cong \begin{cases} K_0^{\mathbb{T}^k}(\mathbb{C}) \cong K_0(C^*(\mathbb{T}^k)) \cong K_0(C_0(\mathbb{Z}^k)) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z} & \text{if } n \text{ even,} \\ K_0^{\mathbb{T}^k}(C_0(\mathbb{R})) \cong K_1^{\mathbb{T}^k}(\mathbb{C}) \cong K_1(C^*(\mathbb{T}^k)) \cong K_1(C_0(\mathbb{Z}^k)) = 0 & \text{if } n \text{ odd,} \end{cases} \end{aligned}$$

where $K_*^{\mathbb{T}^k}(\cdot)$ for $* = 0, 1$ means the equivariant K-theory (cf. [2, Sections 11.7 and 11.9] for the basic formula for crossed products by compact groups and for the Bott periodicity). Furthermore,

$$\begin{aligned} & K_1(C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \cong K_0(C_0(\mathbb{R}) \otimes C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \cong K_0^{\mathbb{T}^k}(C_0(\mathbb{R}^{n+1})) \\ & \cong \begin{cases} K_0^{\mathbb{T}^k}(C_0(\mathbb{R})) \cong K_1^{\mathbb{T}^k}(\mathbb{C}) \cong K_1(C^*(\mathbb{T}^k)) \cong K_1(C_0(\mathbb{Z}^k)) = 0 & \text{if } n \text{ even,} \\ K_0^{\mathbb{T}^k}(\mathbb{C}) \cong K_0(C^*(\mathbb{T}^k)) \cong K_0(C_0(\mathbb{Z}^k)) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Applying the P-V sequence to $(C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}$ in the middle of the above exact sequence (E_2), we obtain that if n is even, then

$$\begin{array}{ccccc} \oplus_{\mathbb{Z}^k}\mathbb{Z} & \longrightarrow & \oplus_{\mathbb{Z}^k}\mathbb{Z} & \longrightarrow & K_0((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

which implies $K_*((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z}$ for $* = 0, 1$. If n is odd, then

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & K_0((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) & \longleftarrow & \oplus_{\mathbb{Z}^k}\mathbb{Z} & \longleftarrow & \oplus_{\mathbb{Z}^k}\mathbb{Z} \end{array}$$

which implies $K_*((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z}$ for $* = 0, 1$.

We next compute the K-groups of the crossed product $C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k$ as follows:

$$\begin{aligned}
&K_0(C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \cong K_0^{\mathbb{T}^k}(C_0(\mathbb{R}^n \setminus 0)) \cong K_0^{\mathbb{T}^k}(C_0(\mathbb{R} \times S^{n-1}))\\
&\cong K_1^{\mathbb{T}^k}(C(S^{n-1})) \cong K_1^{\mathbb{T}^k}(C_0(\mathbb{R}^{n-1})^+) \cong K_1^{\mathbb{T}^k}(C_0(\mathbb{R}^{n-1}))\\
&\cong \begin{cases} K_1^{\mathbb{T}^k}(C_0(\mathbb{R})) \cong K_0^{\mathbb{T}^k}(\mathbb{C}) \cong K_0(C^*(\mathbb{T}^k)) \cong K_0(C_0(\mathbb{Z}^k)) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z} & \text{if } n \text{ even},\\ K_1^{\mathbb{T}^k}(\mathbb{C}) \cong K_1(C^*(\mathbb{T}^k)) \cong K_1(C_0(\mathbb{Z}^k)) = 0 & \text{if } n \text{ odd}.\end{cases}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&K_1(C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \cong K_1^{\mathbb{T}^k}(C_0(\mathbb{R}^n \setminus 0))\\
&\cong K_1^{\mathbb{T}^k}(C_0(\mathbb{R} \times S^{n-1})) \cong K_0^{\mathbb{T}^k}(C(S^{n-1})) \cong K_0^{\mathbb{T}^k}(C_0(\mathbb{R}^{n-1})^+)\\
&\cong K_0^{\mathbb{T}^k}(C_0(\mathbb{R}^{n-1})) \oplus K_0(C^*(\mathbb{T}^k))\\
&\cong \begin{cases} K_0^{\mathbb{T}^k}(C_0(\mathbb{R})) \oplus K_0(C_0(\mathbb{Z}^k)) & \text{if } n \text{ is even}\\ K_0^{\mathbb{T}^k}(\mathbb{C}) \oplus K_0(C_0(\mathbb{Z}^k)) & \text{if } n \text{ is odd}\end{cases}\\
&\cong \begin{cases} K_1(C^*(\mathbb{T}^k)) \oplus (\oplus_{\mathbb{Z}_k}\mathbb{Z}) \cong \oplus_{\mathbb{Z}_k}\mathbb{Z} & \text{if } n \text{ is even}\\ K_0(C^*(\mathbb{T}^k)) \oplus (\oplus_{\mathbb{Z}_k}\mathbb{Z}) \cong (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}_k}\mathbb{Z}) & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}$$

Applying the P-V sequence to $(C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z} = \mathfrak{I}$ in the left of the above exact sequence (E_2), we obtain that if n is even, then

$$\begin{CD}
\oplus_{\mathbb{Z}^k}\mathbb{Z} @>>> \oplus_{\mathbb{Z}^k}\mathbb{Z} @>>> K_0((C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z})\\
@AAA @. @VVV\\
K_1((C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) @<<< \oplus_{\mathbb{Z}^k}\mathbb{Z} @<<< \oplus_{\mathbb{Z}^k}\mathbb{Z}
\end{CD}$$

which implies $K_*((C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \cong (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}^k}\mathbb{Z})$ for $* = 0, 1$. Furthermore, if n is odd, then

$$\begin{CD}
0 @>>> 0 @>>> K_0(\mathfrak{I})\\
@AAA @. @VVV\\
K_1(\mathfrak{I}) @<<< (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}^k}\mathbb{Z}) @<<< (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}^k}\mathbb{Z})
\end{CD}$$

which implies $K_*((C_0(\mathbb{R}^n \setminus 0) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \cong (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}^k}\mathbb{Z})$ for $* = 0, 1$.

Furthermore, note that $C^*(\mathbb{T}^k) \cong C_0(\mathbb{Z}^k)$. Then $K_0(C_0(\mathbb{Z}^k)) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z}$ and $K_1(C_0(\mathbb{Z}^k)) \cong 0$. Applying the P-V sequence to $C^*(\mathbb{T}^k) \rtimes \mathbb{Z}$ in the right of the above exact sequence (E_2), we obtain

$$\begin{CD}
\oplus_{\mathbb{Z}^k}\mathbb{Z} @>>> \oplus_{\mathbb{Z}^k}\mathbb{Z} @>>> K_0(C^*(\mathbb{T}^k) \rtimes \mathbb{Z})\\
@AAA @. @VVV\\
K_1(C^*(\mathbb{T}^k) \rtimes \mathbb{Z}) @<<< 0 @<<< 0
\end{CD}$$

Hence, it follows that $K_*(C^*(\mathbb{T}^k) \rtimes \mathbb{Z}) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z}$ for $* = 0, 1$.

Summing up the above argument, we obtain the following six-term exact sequence of K-groups associated with the above exact sequence:

$$\begin{tikzcd}
(\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \arrow[r] & \oplus_{\mathbb{Z}^k}\mathbb{Z} \arrow[r] & \oplus_{\mathbb{Z}^k}\mathbb{Z} \arrow[d] \\
\oplus_{\mathbb{Z}^k}\mathbb{Z} \arrow[u, "\partial"] & \oplus_{\mathbb{Z}^k}\mathbb{Z} \arrow[l] & (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}^k}\mathbb{Z}) \arrow[l]
\end{tikzcd}$$

Therefore, the index map ∂ is nonzero. Hence $\mathrm{sr}((C_0(\mathbb{R}^n) \rtimes \mathbb{T}^k) \rtimes \mathbb{Z}) \geq 2$ (cf. [7], [9]). Thus, $\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) \geq 2$. $\square$

Remark. Let $G \cong N \rtimes \mathbb{T}^k$ be as above. If $K_1(C^*(G \rtimes_\alpha \mathbb{Z}))$ is nonzero, we obtain $\mathrm{csr}(C^*(G \rtimes_\alpha \mathbb{Z})) \geq 2$ by [4, Corollary 1.6]. In fact, by the P-V sequence,

$$\begin{tikzcd}
K_0(C^*(G)) \arrow[r] & K_0(C^*(G)) \arrow[r] & K_0(C^*(G) \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(C^*(G) \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(C^*(G)) \arrow[l] & K_1(C^*(G)) \arrow[l]
\end{tikzcd}$$

and $K_*(C^*(G)) \cong K_*(C^*(N) \rtimes \mathbb{T}^k) \cong K_*^{\mathbb{T}^k}(C^*(N))$ for $* = 0, 1$. Since N is a simply connected solvable Lie group, it is a successive semi-direct product by $\mathbb{R}$. Thus, if the Thom isomorphism for equivalent K-theory is true, we obtain

$$\begin{aligned}
&K_*^{\mathbb{T}^k}(C^*(N)) \cong K_*^{\mathbb{T}^k}(C^*(N_{s-1}) \rtimes \mathbb{R}) \cong K_{*+1}^{\mathbb{T}^k}(C^*(N_{s-1})) \\
&\cong K_{*+1}^{\mathbb{T}^k}(C^*(N_{s-2}) \rtimes \mathbb{R}) \cong \cdots \cong K_{*+s}^{\mathbb{T}^k}(\mathbb{C}) \\
&\cong K_{*+s}(C^*(\mathbb{T}^k)) \cong K_{*+s}(C_0(\mathbb{Z}^k)) \\
&\cong \oplus_{\mathbb{Z}^k}\mathbb{Z} \text{ if } * + s = 0 \ (\mathrm{mod}\ 1), \text{ and } 0 \text{ if } * + s = 1 \ (\mathrm{mod}\ 1),
\end{aligned}$$

where $N = N_s = N_{s-1} \rtimes \mathbb{R}$, $N_1 = \mathbb{R}$, $N_l = N_{l-1} \rtimes \mathbb{R}$ inductively for $1 \leq l \leq s$ and $s = \dim N$. Therefore, we obtain $K_1(C^*(G) \rtimes_\alpha \mathbb{Z}) \cong \oplus^k\mathbb{Z}$. However, the Thom isomorphism for equivalent K-theory seems to be unknown in the literature so far, and it is desirable but might be wrong in general. Also, since N is isomorphic to $\mathbb{R}^f \rtimes \mathbb{R} \rtimes \cdots \rtimes \mathbb{R}$ for some $f \geq 1$, if $\mathbb{T}^k$ is non-trivial only on $\mathbb{R}^f$, then

$$\begin{aligned}
C^*(N) \rtimes \mathbb{T}^k &\cong C^*(\mathbb{R}^f) \rtimes \mathbb{T}^k \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}, \quad \text{and} \\
K_*(C^*(\mathbb{R}^f) \rtimes \mathbb{T}^k \rtimes \mathbb{R} \cdots \rtimes \mathbb{R}) &\cong K_{*+s-f}(C^*(\mathbb{R}^f) \rtimes \mathbb{T}^k) \cong K_{*+s-f}^{\mathbb{T}^k}(C_0(\mathbb{R}^f)) \\
&\cong K_{*+s}^{\mathbb{T}^k}(\mathbb{C}) \cong K_{*+s}(C^*(\mathbb{T}^k))
\end{aligned}$$

for $* = 0, 1$ by using the Connes' Thom isomorphism and the Bott periodisity (cf. [12, Section 6.3]). An action of $\mathbb{T}^k$ on $\mathbb{R}$ is always trivial, but an action of $\mathbb{T}^k$ on $\mathbb{R}^e$ for $e \geq 2$ is nontrivial in general so that an action of $\mathbb{T}^k$ does not necessarily commute with $(s - f)$-actions of $\mathbb{R}$ as above.

If a linearizable connected solvable Lie group is compact, then it is isomorphic to $\mathbb{T}^k$. In this case, we obtain the following:

Theorem 1.3. *For a semi-direct product $\mathbb{T}^k \rtimes_\alpha \mathbb{Z}$, we have*

$$\mathrm{sr}(C^*(\mathbb{T}^k \rtimes_\alpha \mathbb{Z})) = 1.$$

Moreover, we have $\mathrm{csr}(C^*(\mathbb{T}^k \rtimes_\alpha \mathbb{Z})) = 2$.

As a note, the proof below includes the case by case study as examples for the convenience, and the general case is given at the bottom of the proof.

Proof. Note that $C^*(\mathbb{T}^k \rtimes_\alpha \mathbb{Z}) \cong C^*(\mathbb{T}^k) \rtimes_\alpha \mathbb{Z} \cong C_0(\mathbb{Z}^k) \rtimes_{\hat{\alpha}} \mathbb{Z}$. If $\hat{\alpha}$ is trivial on a direct factor $\mathbb{Z}^l$ of $\mathbb{Z}^k$, then

$$C_0(\mathbb{Z}^k) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong C_0(\mathbb{Z}^l) \otimes C_0(\mathbb{Z}^{k-l}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong \oplus_{\mathbb{Z}^l} C_0(\mathbb{Z}^{k-l}) \rtimes_{\hat{\alpha}} \mathbb{Z}.$$

Therefore, we may assume that $\hat{\alpha}$ is nontrivial on each direct factor of $\mathbb{Z}^k$.

Case 1: $k = 1$. Note that an automorphism of $\mathbb{T}$ is either trivial or the reflection. When α_1 is the reflection, the duality $\langle \alpha_1(z), n\rangle = (\bar{z})^n = (z)^{-n} = \langle z, -n\rangle = \langle z, \hat{\alpha}_1(n)\rangle$ for $z \in \mathbb{T}$ and $n \in \mathbb{Z}$ holds. Thus, we consider the decomposition $\mathbb{Z} = \{0\} \sqcup \cup_{n\in\mathbb{Z}_+}\{n, -n\}$ (a disjoint union). Then we have the direct sum decomposition: $C_0(\mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong C^*(\mathbb{Z}) \oplus (\oplus_{\mathbb{Z}_+} C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z})$, where $X_2 = \{\pm n\}$. Moreover, since $\hat{\alpha}_1^2 = 1$ on X_2,

$$0 \to S(C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}_2) \to C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \to 0,$$

where $S(C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}_2)$ means the suspension $C_0(\mathbb{R}) \otimes C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}_2$, and we have $C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \cong M_2(\mathbb{C})$ (cf. [2] for the exact sequence of crossed products by $\mathbb{Z}$ with periods). By the six-term exact sequence of K-groups,

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ \partial \big\uparrow & & & & \big\downarrow \\ 0 & \longleftarrow & K_1(C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \longleftarrow & \mathbb{Z}. \end{array}$$

Therefore, the index map ∂ is zero. Since $\mathrm{sr}(S(M_2(\mathbb{C}))) = 1$ and $\mathrm{sr}(M_2(\mathbb{C})) = 1$ by [13, Theorem 6.1], it follows from [7] or [9] that $\mathrm{sr}(C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$. Hence $\mathrm{sr}(C_0(\mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$.

Case 2: $k \geq 1$ and $\hat{\alpha}$ is the reflection on each direct factor $\mathbb{Z}$ of $\mathbb{Z}^k$. By the duality, $\hat{\alpha}_1(n) = (-n) = ((-n_j))$ for $n = (n_j) \in \mathbb{Z}^k$. Thus, we have the orbit decomposion of $\mathbb{Z}^k$ (a disjoint union): $\mathbb{Z}^k = \{0_k\} \sqcup (\sqcup_{(\mathbb{Z}^k \setminus \{0_k\})/\mathbb{Z}_2}\{n, -n\})$, where $(\mathbb{Z}^k \setminus \{0_k\})/\mathbb{Z}_2 = (\mathbb{Z}^k \setminus \{0_k\})/\mathbb{Z}$ means the orbit space of $\mathbb{Z}^k \setminus \{0_k\}$ by $\mathbb{Z}$. Then we have $C_0(\mathbb{Z}^k) \rtimes_{\hat{\alpha}} \mathbb{Z} = C^*(\mathbb{Z}) \oplus (\oplus_{(\mathbb{Z}^k \setminus \{0_k\})/\mathbb{Z}_2} C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z})$, where $X_2 = \{\pm n\}$. By the same analysis as Case 1, we obtain $\mathrm{sr}(C_0(\mathbb{Z}^k) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$.

Case 3: $k = 2$ and α is the permutation of $\mathbb{T}^2$. By the duality,

$$\langle \alpha_1(z, w), (s, t)\rangle = w^s z^t = \langle (z, w), (t, s)\rangle = \langle (z, w), \hat{\alpha}_1(s, t)\rangle$$

for $z, w \in \mathbb{T}^2$ and $s, t \in \mathbb{Z}$. Then we have the following orbit decomposition:

$$\mathbb{Z}^2 = (\sqcup_{n\in\mathbb{Z}}\{n, n\}) \sqcup (\sqcup_{s\neq t\in\mathbb{Z}}\{s, t\}).$$

Thus, we have $C_0(\mathbb{Z}^2) \rtimes_{\hat{\alpha}} \mathbb{Z} = (\oplus_{\mathbb{Z}} C^*(\mathbb{Z})) \oplus (\oplus_{s\neq t\in\mathbb{Z}} C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z})$, where $X_2 = \{s, t\}$. By the same analysis as Case 1, we obtain $\mathrm{sr}(C(X_2) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$. Hence, $\mathrm{sr}(C_0(\mathbb{Z}^2) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$ follows.

Case 4: $k \geq 2$ and α is a finite composition of the permutations and the reflections of each direct factor $\mathbb{T}$ of $\mathbb{T}^k$. In this case, $\alpha_1^n = 1$ for some $n \geq 2$. For example, if α is given by $\alpha_1(z, w) = (w, \bar{z})$ for $z, w \in \mathbb{T}$, then $\alpha_1^4 = 1$. By the duality, $\langle \alpha_1(z, w), (s, t)\rangle = \langle (w, \bar{z}), (s, t)\rangle = w^s z^{-t} = \langle (z, w), (-t, s)\rangle = \langle (z, w), \hat{\alpha}_1(s, t)\rangle$. Therefore, $\hat{\alpha}_1^4 = 1$ on $\mathbb{Z}^2$. When $\hat{\alpha}_1^n = 1$ on $\mathbb{Z}^k$, we consider the decomposition: $\mathbb{Z}^k = F \sqcup (\mathbb{Z}^k \setminus F)$, where the subset F consists of all fixed points of $\mathbb{Z}^k$ under $\hat{\alpha}$. Then we have $C_0(\mathbb{Z}^k) \rtimes_{\hat{\alpha}} \mathbb{Z} = (\oplus_F C^*(\mathbb{Z})) \oplus (\oplus_{(\mathbb{Z}^k\setminus F)/\mathbb{Z}_n} C(X_n) \rtimes \mathbb{Z})$, where X_n means an orbit by $\hat{\alpha}$ consisting of n points in $\mathbb{Z}^k \setminus F$, and $(\mathbb{Z}^k \setminus F)/\mathbb{Z}_n = (\mathbb{Z}^k \setminus F)/\mathbb{Z}$ means the orbit space of $\mathbb{Z}^k \setminus F$ under $\hat{\alpha}$. Moreover, as in the Case 1, we have the following exact sequence:

$$0 \to S(C(X_n) \rtimes \mathbb{Z}_n) \to C(X_n) \rtimes \mathbb{Z} \to C(X_n) \rtimes \mathbb{Z}_n \to 0$$

and $C(X_n) \rtimes \mathbb{Z}_n \cong M_n(\mathbb{C})$. Furthermore, by the six-term exact sequence,

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(X_n) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(C(X_n) \rtimes_{\hat{\alpha}} \mathbb{Z}) & \longleftarrow & \mathbb{Z}. \end{array}$$

Hence, the index map ∂ is zero. Since $\mathrm{sr}(SM_n(\mathbb{C})) = 1$ and $\mathrm{sr}(M_n(\mathbb{C})) = 1$ by [13, Theorem 6.1], we have $\mathrm{sr}(C(X_n) \rtimes \mathbb{Z}) = 1$ by [7] or [9]. Therefore, we obtain $\mathrm{sr}(C_0(\mathbb{Z}^k) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$.

Case 5: the action α is given by $\alpha_1(z, w) \mapsto (z, zw)$ for $z, w \in \mathbb{T}$. By the duality,

$$\begin{aligned} \langle \alpha_1(z, w), (s, t)\rangle &= \langle (z, zw), (s, t)\rangle = (z)^s (zw)^t \\ &= \langle (z, w), (s + t, t)\rangle = \langle (z, w), \hat{\alpha}_1(s, t)\rangle. \end{aligned}$$

Thus, we consider the decomposition: $\mathbb{Z}^2 = (\mathbb{Z} \times \{0\}) \sqcup (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))$. Then

$$\begin{aligned} C_0(\mathbb{Z} \times \{0\}) \rtimes_{\hat{\alpha}} \mathbb{Z} &\cong C_0(\mathbb{Z}) \otimes C^*(\mathbb{Z}) \cong C_0(\mathbb{Z} \times \mathbb{T}), \\ C_0(\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) \rtimes_{\hat{\alpha}} \mathbb{Z} &\cong \oplus_{t\in\mathbb{Z}\setminus\{0\}} C_0(\mathbb{Z} \times \{t\}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong \oplus_{\mathbb{Z}\setminus\{0\}} \mathbb{K}. \end{aligned}$$

Since $\mathrm{sr}(C_0(\mathbb{Z} \times \mathbb{T})) = 1$ and $\mathrm{sr}(\mathbb{K}) = 1$ ([13]), we obtain $\mathrm{sr}(C_0(\mathbb{Z}^2) \rtimes_{\hat{\alpha}} \mathbb{Z}) = 1$.

Case 6: the action α is given by $\alpha_1(z_1, z_2, z_3) \mapsto (z_1, z_1z_2, z_1z_2z_3)$ for $z_1, z_2, z_3 \in \mathbb{T}$. By the duality,

$$\begin{aligned} \langle \alpha_1(z_1, z_2, z_3), (s_1, s_2, s_3)\rangle &= z_1^{s_1}(z_1z_2)^{s_2}(z_1z_2z_3)^{s_3} \\ &= \langle (z_1, z_2, z_3), (s_1 + s_2 + s_3, s_2 + s_3, s_3)\rangle \\ &= \langle (z_1, z_2, z_3), \hat{\alpha}_1(s_1, s_2, s_3)\rangle. \end{aligned}$$

Thus, $\mathbb{Z}^3 = (\mathbb{Z}\times\{0\}\times\{0\}) \sqcup (\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})\times\{0\}) \sqcup (\mathbb{Z}^2\times(\mathbb{Z}\setminus\{0\}))$, and

$$\begin{aligned} C_0(\mathbb{Z}\times\{0\}\times\{0\}) \rtimes_{\hat\alpha} \mathbb{Z} &\cong C_0(\mathbb{Z})\otimes C^*(\mathbb{Z}) \cong C_0(\mathbb{Z}\times\mathbb{T}), \\ C_0(\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})\times\{0\}) \rtimes_{\hat\alpha} \mathbb{Z} &\cong \oplus_{\mathbb{Z}\setminus\{0\}} C_0(\mathbb{Z}) \rtimes_{\hat\alpha} \mathbb{Z} \cong \oplus_{\mathbb{Z}\setminus\{0\}}\mathbb{K}, \\ C_0(\mathbb{Z}^2\times(\mathbb{Z}\setminus\{0\})) \rtimes_{\hat\alpha} \mathbb{Z} &\cong \oplus_{\mathbb{Z}^2\times(\mathbb{Z}\setminus\{0\})/\mathbb{Z}} C_0(\mathbb{Z}) \rtimes_{\hat\alpha} \mathbb{Z} \cong \oplus_{\mathbb{Z}^2\times(\mathbb{Z}\setminus\{0\})/\mathbb{Z}}\mathbb{K}, \end{aligned}$$

where $\mathbb{Z}^2\times(\mathbb{Z}\setminus\{0\})/\mathbb{Z}$ means the orbit space of $\mathbb{Z}^2\times(\mathbb{Z}\setminus\{0\})$ by $\hat\alpha$. Therefore, we deduce $\mathrm{sr}(C_0(\mathbb{Z}^3)\rtimes_{\hat\alpha}\mathbb{Z}) = 1$.

The general case: any action α of $\mathbb{Z}$ on $\mathbb{T}^k$ is a finite composition of the reflections, the permutations and the similar actions as in Cases 5 and 6, and the direct sum decomposition of $C_0(\mathbb{Z}^k)\rtimes_{\hat\alpha}\mathbb{Z}$ is obtained as the above cases. In fact, we consider the decomposition: $\mathbb{Z}^k = F \sqcup (\sqcup_{n\in\mathbb{Z}_+} P_n) \sqcup S$, where F consists of all fixed points of $\mathbb{Z}^k$ under $\hat\alpha$, and any point of P_n has the period n under $\hat\alpha$, and $\hat\alpha$ on S is free. Note that those subsets F, P_n, S correspond to the cases, where the stabilizers of points of $\mathbb{Z}^k$ are either $\mathbb{Z}$, $\mathbb{Z}_n$ or $\{0\}$ respectively. Then

$$\begin{aligned} C_0(F)\rtimes_{\hat\alpha}\mathbb{Z} &\cong C_0(F)\otimes C^*(\mathbb{Z}) \cong C_0(F\times\mathbb{T}), \\ C_0(P_n)\rtimes_{\hat\alpha}\mathbb{Z} &\cong \oplus_{P_n/\mathbb{Z}_n} C(X_n)\rtimes_{\hat\alpha}\mathbb{Z}, \\ C_0(S)\rtimes_{\hat\alpha}\mathbb{Z} &\cong \oplus_{S/\mathbb{Z}} C_0(\mathbb{Z})\rtimes_{\hat\alpha}\mathbb{Z} \cong \oplus_{S/\mathbb{Z}}\mathbb{K}, \end{aligned}$$

where $P_n/\mathbb{Z}_n = P_n/\mathbb{Z}$, $S/\mathbb{Z}$ are the orbit spaces of P_n, S by $\hat\alpha$ respectively, and X_n is an orbit by $\hat\alpha$ consisting of n points in P_n. As in the Case 4, we obtain $\mathrm{sr}(C(X_n)\rtimes_{\hat\alpha}\mathbb{Z}) = 1$. Therefore, we obtain $\mathrm{sr}(C_0(\mathbb{Z}^n)\rtimes_{\hat\alpha}\mathbb{Z}) = 1$.

To estimate the connected stable rank of $C^*(\mathbb{T}^k\rtimes_\alpha\mathbb{Z})$, we use the P-V sequence:

$$\begin{array}{ccccc} K_0(C^*(\mathbb{T}^k)) & \longrightarrow & K_0(C^*(\mathbb{T}^k)) & \longrightarrow & K_0(C^*(\mathbb{T}^k)\rtimes\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathbb{T}^k)\rtimes\mathbb{Z}) & \longleftarrow & K_1(C^*(\mathbb{T}^k)) & \longleftarrow & K_0(C^*(\mathbb{T}^k)). \end{array}$$

Since $K_0(C^*(\mathbb{T}^k)) \cong K_0(C_0(\mathbb{Z}^k)) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z}$ and $K_1(C^*(\mathbb{T}^k)) \cong K_1(C_0(\mathbb{Z}^k)) \cong 0$, it follows that $K_1(C^*(\mathbb{T}^k)\rtimes\mathbb{Z}) \cong \oplus_{\mathbb{Z}^k}\mathbb{Z}$. By [4, Corollary 1.6], we obtain $\mathrm{csr}(C^*(\mathbb{T}^k\rtimes_\alpha\mathbb{Z})) \geq 2$. On the other hand, since we have proved $\mathrm{sr}(C^*(\mathbb{T}^k\rtimes_\alpha\mathbb{Z})) = 1$ above, the conclusion follows from (F1). □

Corollary 1.4. *For a semi-direct product $\mathbb{T}^n\rtimes_\alpha\mathbb{Z}_k$, we have*

$$\mathrm{sr}(C^*(\mathbb{T}^n\rtimes_\alpha\mathbb{Z}_k)) = 1.$$

Proof. Note that $C^*(\mathbb{T}^n\rtimes_\alpha\mathbb{Z}_k) \cong C_0(\mathbb{Z}^n)\rtimes_{\hat\alpha}\mathbb{Z}_k$ and the following exact sequence:

$$0 \to C_0(\mathbb{R})\otimes(C_0(\mathbb{Z}^n)\rtimes_{\hat\alpha}\mathbb{Z}_k) \to C_0(\mathbb{Z}^n)\rtimes_{\hat\alpha}\mathbb{Z} \to C_0(\mathbb{Z}^n)\rtimes_{\hat\alpha}\mathbb{Z}_k \to 0.$$

Then use Theorem 1.3 and (F3). □

We say that a connected amenable Lie group G is linearizable if it is isomorphic to a semi-direct product $N \rtimes K$ of a simply connected solvable Lie group N by a compact connected Lie group K. This is reasonable from the definition for connected solvable Lie groups to be linearlizable (cf. Theorem 1.2). See also [3] for the structure of amenable locally compact groups and their group C^*-algebras.

Theorem 1.5. *Let G be a linearizable connected amenable Lie group and $G \rtimes_\alpha \mathbb{Z}$ a semi-direct product of G by $\mathbb{Z}$. If G is noncompact, then*

$$\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) \geq 2.$$

Proof. The line of the proof is the same as that of Theorem 1.2. Note that

$$C^*(G \rtimes_\alpha \mathbb{Z}) \cong C^*(G) \rtimes_\alpha \mathbb{Z} \cong (C^*(N) \rtimes K) \rtimes_\alpha \mathbb{Z}.$$

Since K is compact, we can use the equivaliant K-theory for computing K-groups as given in the proof of Theorem 1.2 by replacing the torus $\mathbb{T}^k$ with K. Furthermore, since $C^*(K)$ is isomorphic to the direct sum $\oplus_{\lambda \in K^\wedge} M_{n_\lambda}(\mathbb{C})$ where $K^\wedge$ is the unitary dual of irreducible unitary representations of K up to unitary equivalence and n_λ is the dimension of λ. Therefore,

$$\begin{aligned} K_0(C^*(K)) &\cong K_0(\oplus_{\lambda \in K^\wedge} M_{n_\lambda}(\mathbb{C})) \cong \oplus_{K^\wedge} \mathbb{Z} \quad \text{and} \\ K_1(C^*(K)) &\cong K_1(\oplus_{\lambda \in K^\wedge} M_{n_\lambda}(\mathbb{C})) \cong \oplus_{K^\wedge} 0 \cong 0. \end{aligned}$$

□

Corollary 1.6. *Let G be a noncompact, linearizable connected amenable Lie group. If $\mathrm{sr}(C^*(G)) = 1$, then*

$$\mathrm{sr}(C^*(G \rtimes_\alpha \mathbb{Z})) = 2 = \mathrm{sr}(C^*(G)) + 1.$$

Proof. Use Theorem 1.5 and (F4). □

Remark. This consequence should be interesting. See [23], [24], [17] and [22] for G such that $\mathrm{sr}(C^*(G)) = 1$.

Moreover, we obtain

Theorem 1.7. *If K is a compact Lie group, then*

$$\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z})) = 1.$$

Moreover, we have $\mathrm{csr}(C^*(K \rtimes_\alpha \mathbb{Z})) = 2$. In addition, $\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1$, but

$$\mathrm{csr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1.$$

Proof. The line of the proof for $\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z})) = 1$ is the same as the general case in the proof of Theorem 1.3. Note that

$$\begin{aligned} C^*(K \rtimes_\alpha \mathbb{Z}) &\cong C^*(K) \rtimes_\alpha \mathbb{Z} \\ &\cong (\oplus_{\lambda \in K^\wedge} M_{n_\lambda}(\mathbb{C})) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong \Gamma_0(K^\wedge, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in K^\wedge}) \rtimes_{\hat{\alpha}} \mathbb{Z}, \end{aligned}$$

where the dual group $K^\wedge$ is discrete since K is compact, and the last crossed product by $\mathbb{Z}$ involves $\Gamma_0(K^\wedge, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in K^\wedge})$ the C^*-algebra of a continuous field on $K^\wedge$ with fibers $M_{n_\lambda}(\mathbb{C})$ (cf. [3] for C^*-algebras of continuous fields). Furthermore, this crossed product is decomposed into the following direct sum:

$$\begin{aligned} &\Gamma_0(F, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in F}) \rtimes_{\hat{\alpha}} \mathbb{Z} \\ &\oplus (\oplus_{n \in \mathbb{Z}_+} \Gamma_0(P_n, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in P_n}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \\ &\oplus (\oplus_{S/\mathbb{Z}} \Gamma_0(\mathbb{Z}, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in \mathbb{Z}}) \rtimes_{\hat{\alpha}} \mathbb{Z}), \end{aligned}$$

where F is the set of all fixed points in $K^\wedge$ under $\hat{\alpha}$, and P_n is the set of all points with period n under $\hat{\alpha}$, and S is the set of all points with no period under $\hat{\alpha}$ so that $\hat{\alpha}$ is free on S so that $S = \sqcup \mathbb{Z}$ a disjoint union of copies of $\mathbb{Z}$ by orbit decomposition, and those crossed products involve the C^*-algebras of continuous fields on F, P_n, $\mathbb{Z}$ with fibers $M_{n_\lambda}(\mathbb{C})$ respectively. Moreover, we have

$$\begin{aligned} \Gamma_0(F, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in F}) \rtimes_{\hat{\alpha}} \mathbb{Z} &\cong (\oplus_{\lambda \in F} M_{n_\lambda}(\mathbb{C})) \otimes C(\mathbb{T}), \\ \Gamma_0(P_n, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in P_n}) \rtimes_{\hat{\alpha}} \mathbb{Z} &\cong M_{n_\lambda}(\mathbb{C}) \otimes (\oplus_{P_n/\mathbb{Z}} C(X_n) \rtimes_{\hat{\alpha}} \mathbb{Z}), \\ \Gamma_0(\mathbb{Z}, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in \mathbb{Z}}) \rtimes_{\hat{\alpha}} \mathbb{Z} &\cong M_{n_\lambda}(\mathbb{C}) \otimes C_0(\mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong M_{n_\lambda}(\mathbb{C}) \otimes \mathbb{K}, \end{aligned}$$

where note that the dimension n_λ is the same for λ in an orbit under $\hat{\alpha}$ by definition of the action $\hat{\alpha}$.

Since $K_1(C^*(K) \rtimes \mathbb{Z}) \cong \oplus_{K^\wedge} \mathbb{Z}$ by using the P-V sequence as given in the proof of Theorem 1.3, we obtain $\mathrm{csr}(C^*(K) \rtimes \mathbb{Z}) \geq 2$ by [4, Corollary 1.6]. Then use (F1). See also the proof of Corollary 1.4 for $\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1$.

By [14, Theorem 2.10] it follows from $\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1$ that

$$GL_1(C^*(K \rtimes_\alpha \mathbb{Z}_k))/GL_1(C^*(K \rtimes_\alpha \mathbb{Z}_k))_0 \cong K_1(C^*(K \rtimes_\alpha \mathbb{Z}_k)).$$

Also, we have $\mathrm{csr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) \leq 2$ by (F1). By using the similar analysis above, $C^*(K \rtimes_\alpha \mathbb{Z}_k)$ is decomposed into a direct sum with the following direct summands:

$$\begin{aligned} \Gamma_0(F, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in F}) \rtimes_{\hat{\alpha}} \mathbb{Z}_k &\cong (\oplus_{\lambda \in F} M_{n_\lambda}(\mathbb{C})) \otimes C^*(\mathbb{Z}_k), \\ \Gamma_0(Q_l, \{M_{n_\lambda}(\mathbb{C})\}_{\lambda \in Q_l}) \rtimes_{\hat{\alpha}} \mathbb{Z}_k &\cong M_{n_\lambda}(\mathbb{C}) \otimes (\oplus_{Q_l/\mathbb{Z}_k} C(Y_l) \rtimes_{\hat{\alpha}} \mathbb{Z}_k), \end{aligned}$$

where F is the set of all fixed points in $K^\wedge$ under $\hat{\alpha}$, and for some $0 \leq l < k$, Q_l is the set of all points such that their orbits Y_l are homeomorphic to $\mathbb{Z}_k/\mathbb{Z}_l$. Furthermore, the imprimitivity theorem ([5]) implies

$$\begin{aligned} C(Y_l) \rtimes_{\hat{\alpha}} \mathbb{Z}_k &= C(\mathbb{Z}_k/\mathbb{Z}_l) \rtimes_{\hat{\alpha}} \mathbb{Z}_k \\ &\cong C^*(\mathbb{Z}_l) \otimes \mathbb{K}(l^2(\mathbb{Z}_k/\mathbb{Z}_l)) = C(\mathbb{Z}_l) \otimes M_{k/l}(\mathbb{C}), \end{aligned}$$

where $\mathbb{K}(l^2(\mathbb{Z}_k/\mathbb{Z}_l))$ means the C^*-algebra of compact operators on the Hilbert space $l^2(\mathbb{Z}_k/\mathbb{Z}_l) \cong \mathbb{C}^{k/l}$. Note that the dual group $\mathbb{Z}_s^\wedge \cong \mathbb{Z}_s$ for $s \in \mathbb{N}$. Therefore, the K_1-group of $C^*(K \rtimes_\alpha \mathbb{Z}_k)$ is decomposed into a direct sum with the following direct summands:

$$K_1((\oplus_{\lambda\in F} M_{n_\lambda}(\mathbb{C})) \otimes C^*(\mathbb{Z}_k)) \cong \oplus_{\lambda\in F} K_1(M_{n_\lambda}(\mathbb{C}) \otimes C(\mathbb{Z}_k)) \cong 0,$$
$$K_1(M_{n_\lambda}(\mathbb{C}) \otimes (\oplus_{Q_l/\mathbb{Z}_k} C(\mathbb{Z}_l) \otimes M_{k/l}(\mathbb{C}))) \cong \oplus_{Q_l/\mathbb{Z}_k} K_1(C(\mathbb{Z}_l) \otimes M_{k/l}(\mathbb{C})) \cong 0.$$

Thus, $K_1(C^*(K \rtimes_\alpha \mathbb{Z}_k))$ is trivial. Hence we conclude $\mathrm{csr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1$. $\square$

Corollary 1.8. *If K is a compact Lie group, then*

$$\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z})) = 1 = \mathrm{sr}(C^*(K)), \quad \mathrm{csr}(C^*(K \rtimes_\alpha \mathbb{Z})) = \mathrm{csr}(C^*(K)) + 1,$$
$$\mathrm{sr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1 = \mathrm{sr}(C^*(K)), \quad \mathrm{csr}(C^*(K \rtimes_\alpha \mathbb{Z}_k)) = 1 = \mathrm{csr}(C^*(K)).$$

Remark. This consequence should be interesting.

What's more, we first consider the case of semi-direct products of amenable locally compact groups as follows:

Proposition 1.9. *Let G, H be amenable locally compact groups. Then*

$$\mathrm{sr}(C^*(G \rtimes H)) \geq \mathrm{sr}(C^*(H)).$$

Thus, if $\mathrm{sr}(C^*(H)) \geq 2$*, then* $\mathrm{sr}(C^*(G \rtimes H)) \geq 2$*. In particular, if $H = \mathbb{Z}^k$, then*

$$\mathrm{sr}(C^*(G \rtimes \mathbb{Z}^k)) \geq [k/2] + 1.$$

Proof. Note that $C^*(G \rtimes H) \cong C^*(G) \rtimes H$, and the quotient: $C^*(G) \rtimes H \to C^*(H)$, which is deduced from that the trivial representation of $C^*(G)$ is closed in the spectrum of $C^*(G)$, and is stable under the action of H. Hence, by (F3) we obtain $\mathrm{sr}(C^*(G \rtimes H)) \geq \mathrm{sr}(C^*(H))$. On the other hand, $C^*(\mathbb{Z}^k) \cong C(\mathbb{T}^k)$ and $\mathrm{sr}(C(\mathbb{T}^k)) = [k/2] + 1$ by (F2). Hence, $\mathrm{sr}(C^*(G \rtimes \mathbb{Z}^k)) \geq [k/2] + 1$. $\square$

Remark. Note that $\mathrm{sr}(C^*(\mathbb{Z})) = \mathrm{sr}(C(\mathbb{T})) = 1$. Therefore, the proofs of Theorems 1.1, 1.2 and 1.5 are more complicated than that of this proposition.

We next consider the case of semi-direct products $G \rtimes H$ of amenable locally compact groups G by non-compact connected semi-simple Lie groups H. Note that if the quotient of a connected Lie group by the radical, that is, the maximal normal solvable Lie subgroup is compact, then the Lie group is amenable. (cf. [3, Section 18.3]). For H a non-compact connected semi-simple Lie group, let $\mathrm{r}(H)$ denote the real rank of H, which is defined to be the real dimension of A for the Iwasawa decomposition $H = KAN$. Let $C_r^*(G \rtimes H)$ be the reduced group C^*-algebra of $G \rtimes H$ (cf. [16]). Since H is non-amenable, $C_r^*(G \rtimes H) \neq C^*(G \rtimes H)$ (cf. [11]).

Proposition 1.10. *Let G be an amenable locally compact group and H a non-compact connected semi-simple Lie group. Then*

$$\mathrm{sr}(C_r^*(G \rtimes H)) \geq C_r^*(H).$$

If $\mathrm{r}(H) \geq 2$*, then* $\mathrm{sr}(C_r^*(G \rtimes H)) \geq 2$ *and, in addition,* $\mathrm{sr}(C^*(G \rtimes H)) \geq 2$.

Proof. Let $C^*(G \rtimes H)$ be the full group C^*-algebra of $G \rtimes H$. Then $C^*(G \rtimes H) \cong C^*(G) \rtimes H$ and $C_r^*(G \rtimes H) \cong C^*(G) \rtimes_r H$ the reduced crossed product of $C^*(G)$ by H. Moreover, since the trivial representation of G is fixed under the action of H, we have the following diagram:

$$\begin{array}{ccccc} C^*(G) \rtimes H & \longrightarrow & C^*(H) & \longrightarrow & 0, \\ \downarrow & & \downarrow & & \\ C^*(G) \rtimes_r H & \longrightarrow & C_r^*(H) & \longrightarrow & 0. \end{array}$$

Then use (F3). By [16], if $\mathrm{r}(H) \geq 2$, then $\mathrm{sr}(C_r^*(H)) = 2$. Therefore, we obtain the second claim by (F3). □

Remark. For example, we may take $SL_n(\mathbb{R})$ for $n \geq 3$ as H in the statement (cf. [16]). Note that $\mathrm{r}(SL_2(\mathbb{R})) = 1$. In this case, we know that

$$\mathrm{sr}(C_r^*(\mathbb{R}^2 \rtimes SL_2(\mathbb{R}))) = 1, \quad \mathrm{sr}(C_r^*((\mathbb{R}^2 \times \mathbb{R}) \rtimes SL_2(\mathbb{R}))) = 2,$$

where the actions of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ are the matrix multiplication, and the action on $\mathbb{R}$ is trivial (see [22]).

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