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Oscillatory Behavior of Linear Neutral Delay Dynamic Equations on Time Scales

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Abstract. By employing the Riccati transformation technique some new oscillation criteria for the second-order neutral delay dynamic equation

$$(y(t) + r(t)y(\tau(t)))^{\Delta\Delta} + p(t)y(\delta(t)) = 0,$$

on a time scale $\mathbb{T}$ are established. Our results as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ improve some well known oscillation criteria for second order neutral delay differential and difference equations, and when $\mathbb{T} = q^{\mathbb{N}}$, i.e., for second-order $q-$neutral difference equations our results are essentially new and can be applied on different types of time scales. Some examples are considered to illustrate the main results.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph. D. Thesis in 1988 in order to unify continuous and discrete analysis (see [17]). The theory of dynamic equations unifies the theories of differential equations and difference equations and it also extends these classical cases to cases "in between", e.g., to the so-called $q-$difference equations. Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors has expounded on various aspects of this new theory, see the paper by Agarwal et al. [1] and the references cited therein. The books on the subject of time scales, by Bohner and Peterson [4], [5], summarize and organize much of time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [4]). On any time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{1.1}$$

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A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left–dense if $\rho(t) = t$, right–dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-$continuous function provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of $rd-$continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is $rd-$continuous function is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \Re^+$) if it is $rd-$continuous function and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^{\Delta}(t) = \left(f(\sigma(t)) - f(t)\right) / (\sigma(t) - t), \tag{1.2}$$

if f is continuous at t and t is right–scattered. If t is not right–scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{t \to \infty} \frac{f(t) - f(s)}{t - s}, \tag{1.3}$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right–dense continuous if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. A useful formula is

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{1.4}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \tag{1.5}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \tag{1.6}$$

For $a, b \in \mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t) \Delta t = f(b) - f(a).$$

An integration by parts formulas read

$$\begin{cases} \int_a^b f(t) g^{\Delta}(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g^{\sigma} \Delta t, \\ \int_a^b f^{\sigma} g^{\Delta}(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g(t) \Delta t, \end{cases} \tag{1.7}$$

infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty}\int_a^b f(t)\Delta t,$$

and the integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t).$$

Note that if $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=\rho(t)=t$,

$$f^{\Delta}(t)=f^{'}(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

If $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1,\ \mu(t)\equiv 1$,

$$f^{\Delta}=\Delta f \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t),$$

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, we have $\sigma(t)=t+h$, $\mu(t)=h$,

$$f^{\Delta}=\Delta_h f = \frac{f(t+h)-f(t)}{h} \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{i=\frac{a}{h}}^{\frac{b}{h}-1} f(i),$$

If $\mathbb{T}=q^{\mathbb{N}}=\{t : t=q^n,\ n\in\mathbb{N},\ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$

$$x_q^{\Delta}(t)=\frac{x(qt)-x(t)}{(q-1)t}, \quad \int_a^b f(t)\Delta t = \sum_{t\in(a,b)} \mu(q^n)f(q^n).$$

If $\mathbb{T}=\{t_n : n\in\mathbb{N}_0\}$, where t_n be the so-called harmonic numbers defined by

$$t_0=0, \quad t_n=\sum_{k=1}^{n}\frac{1}{k}, \ n\in\mathbb{N},$$

we have $\sigma(t_n)=t_{n+1}$, $\mu(t_n)=\frac{1}{n+1}$ and

$$x^{\Delta}(t_n)=(n+1)\Delta x(t_n), \quad \int_a^b f(t)\Delta t = \sum_{t\in(a,b)} \mu(t)f(t).$$

If $\mathbb{T}=\mathbb{N}_0^2=\{n^2 : n\in\mathbb{N}_0\}$, we have $\sigma(t)=(\sqrt{t}+1)^2$ and $\mu(t)=1+2\sqrt{t}$ for $t\in\mathbb{T}$ and

$$x^{\Delta}(t)=\frac{x((\sqrt{t}+1)^2)-x(t)}{1+2\sqrt{t}} \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{n=0}^{b} \mu(t)f(t).$$

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of ordinary dynamic equations on time scales. We refer the reader to the papers [2], [3], [6]-[14], [18], [19], [20].

In this paper, we are concerned with oscillation of the second-order linear neutral delay dynamic equation

$$[y(t)+r(t)y(\tau(t))]^{\Delta\Delta}+p(t)y(\delta(t))=0, \tag{1.8}$$

on a time scale $\mathbb{T}$. Throughout this paper we assume that:

(h_1) The delay functions $\tau(t)\leq t$ and $\delta(t)\leq t$ satisfy $\tau(t):\mathbb{T}\to\mathbb{T}$ and $\delta(t):\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$ and $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty$,

(h_2) $r(t)$ and $p(t)$ are positive real-valued $rd-$ continuous functions defined on $\mathbb{T}$ and $0\leq r(t)<1$.

Recall that a solution of (1.8) is a nontrivial real function $y(t)$ such that $y(t)+r(t)y(\tau(t))\in C_{rd}^2[t_y,\infty)$ for $t_y\geq t_0$ and satisfying equation (1.8) for $t\geq t_y$. Our attention is restricted to those solutions of (1.8) which exist on some half line $[t_y,\infty)$ and satisfy $\sup\{|y(t)|:t>t_1\}>0$ for any $t_1\geq t_y$. A solution $y(t)$ of (1.8) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.8) is said to be oscillatory if all its solutions are oscillatory. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. Note that, if $\mathbb{T}=\mathbb{R}$, (1.8) becomes the second-order neutral delay differential equation

$$[y(t)+r(t)y(\tau(t))]''+p(t)y(\delta(t))=0. \tag{1.9}$$

If $\mathbb{T}=\mathbb{Z}$, (1.8) becomes the second-order neutral delay difference equation

$$\Delta^2[y(t)+r(t)y(\tau(t))]+p(t)y(\delta(t))=0. \tag{1.10}$$

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, (1.8) becomes the second-order neutral delay difference equation

$$\Delta_h^2[y(t)+r(t)y(\tau(t))]+p(t)y(\delta(t))=0. \tag{1.11}$$

If $\mathbb{T}=q^{\mathbb{N}}=\{t:t=q^n,\ n\in\mathbb{N},\ q>1\}$, (1.8) becomes the second order $q-$neutral delay difference equation

$$\Delta_q^2[y(t)+r(t)y(\tau(t))]+p(t)y(\delta(t))=0. \tag{1.12}$$

If $\mathbb{T}=\mathbb{N}_0^2=\{n^2:n\in\mathbb{N}_0\}$, (1.8) becomes the second-order neutral delay difference equation

$$\Delta_N^2(y(t)+r(t)y(\tau(t)))+p(t)y(\delta(t))=0. \tag{1.13}$$

If $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$ where t_n be the so-called harmonic numbers (1.8) becomes the second–order neutral difference equation

$$\Delta^2_{t_n}\left[y(t) + r(t)y(\tau(t))\right] + p(t)y(\delta(t)) = 0. \tag{1.14}$$

Grammatikopoulos et al. [15] considered (1.9) and proved that: If $p(t) > 0$, $0 \leq r(t) < 1$ and

$$\int_{t_0}^{\infty} p(s)[1 - r(\delta(s))]ds = \infty. \tag{1.15}$$

Then every solution of (1.9) oscillates. Note that the condition (1.15) can not be applied to the neutral equation

$$[y(t) + r(t)y(\tau(t))]'' + \frac{\beta}{t^2}y(\delta(t)) = 0, \tag{1.16}$$

where $\beta > 0$ and $0 \leq r(t) < 1$.

For oscillation of the second-order neutral delay difference equation (1.3) Zhang and Cheng [21] obtained the discrete analogy of (1.10) and proved that: If $p(t) > 0$, $0 \leq r(t) < 1$ are positive sequences and

$$\sum_{i=n_0}^{\infty} p(i)[1 - r(\delta(i))] = \infty, \tag{1.17}$$

then every solution of (1.10) oscillates. Note that the condition (1.17) can not be applied to the second-order neutral delay difference equation

$$\Delta^2[y(n) + r(n)y(\tau(n))] + \frac{\beta}{n^2}y(\delta(n)) = 0, \qquad n \geq n_0, \tag{1.18}$$

where $\beta > 0$ and $0 \leq r(n) < 1$.

Our aim in this paper, in Section 2 is to apply the Riccati transformation technique to establish some new sufficient condition for oscillation of neutral delay dynamic equation (1.8). Our result as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ improves the oscillation condition (1.15) established by Grammatikopoulos et al. [15] for second order delay differential equation (1.9) and the oscillation condition (1.17) established by Zhang and Cheng [21] for second-order difference equation (1.10). When $\mathbb{T} = h\mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^k,\ k \in \mathbb{N},\ q > 1\}$, $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$ and $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$, i.e., for equations (1.10)-(1.14) our oscillation results are essentially new. In Section 3, we will apply our results for equations (1.9)-(1.14). An example illustrating our main result is presented.

2. Main results

In this section, we use the Riccati technique to establish some new oscillation criteria of 1.8). In what follows, it will be assumed that the condition

$$\int_{t_0}^{\infty} \sigma(s)p(s)(1-r(\delta(s))\Delta s = \infty,$$

is fulfilled. We start with the following lemmas which the proof is similar to that of the proof of Lemma 2.1 in [2].

Lemma 2.1. *Let x be a positive solution of* $x^{\Delta\Delta}(t)+p(t)x(\delta(t)) \leq 0$, on $[t_0,\infty)$ and $T=\tau_{-1}(t_0)$. Then

(i) $x^{\Delta}(t) \geq 0$, $x(t) \geq tx^{\Delta}(t)$for $t \geq T$;

ii) x is nondecreasing, while $x(t)/t$ is nonincreasing on $[T,\infty)_{\mathbb{T}}$.

Theorem 2.1. *Assume that (h_1) and (h_2) hold. Furthermore assume that here exists a positive rd-continuous Δ-differentiable functions $\alpha(t)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\alpha(s)Q(s) - \frac{(((\alpha^{\Delta}(s))_+)^2}{4\alpha(s)}\right]\Delta s = \infty, \tag{2.1}$$

where

$$Q(s) = \frac{\delta(s)}{s}p(s)(1-r(\delta(s))),$$

$\left(\alpha^{\Delta}(t)\right)_+ = \max\{\alpha^{\Delta}(t),0\}$.*Then every solution of* (1.8) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.8) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that y is an eventually positive solution of (1.8) with $y(\delta(t))$, and $y(\tau(t)) > 0$ for all $t \geq t_1$ sufficiently large. Set

$$x(t) = y(t) + r(t)y(\tau(t)). \tag{2.2}$$

From (2.2), (1.8) and (h_2) we have

$$x^{\Delta\Delta}(t) + p(t)y(\delta(t)) \leq 0, \tag{2.3}$$

for all $t \geq t_1$, and so $x^{\Delta}(t)$ is an eventually decreasing function. We first show that $x^{\Delta}(t)$ is eventually nonnegative. Indeed, since $p(t)$ is a positive function, the decreasing function $x^{\Delta}(t)$ is either eventually positive or eventually negative. Suppose there exists an integer $t_2 \geq t_1$ such that $x^{\Delta}(t_2) = c < 0$, then from (2.3) we have $x^{\Delta}(t) < x^{\Delta}(t_2) = c$ for $t \geq t_2$, hence $x^{\Delta}(t) \leq c$,which implies that

$$x(t) \leq x(t_2) + c(t-t_2) \to -\infty \text{ as } t \to \infty, \tag{2.4}$$

which contradicts the fact that $x(t) > 0$ for all $t \geq t_2$. Hence $x^{\Delta}(t)$ is eventually nonnegative. Therefore, we see that there is some t_1 such that

$$x(t) > 0,\ x^{\Delta}(t) \geq 0,\ x^{\Delta\Delta}(t) < 0,\ t \geq t_1. \tag{2.5}$$

This implies that

$$\begin{aligned} y(t) &= x(t) - r(t)y(\tau(t)) = x(t) - r(t)[x(\tau(t)) - r(\tau(t))y(\tau(\tau(t))] \\ &\geq x(t) - r(t)x(\tau(t)) \geq (1 - r(t))x(t). \end{aligned}$$

Then, for $t \geq t_2$ sufficiently large, we see that

$$y(\delta(t)) \geq (1 - r(\delta(t)))x(\delta(t)). \tag{2.6}$$

From (2.3) and (2.6) we obtain for $t \geq t_2$

$$x^{\Delta\Delta}(t) + p(t)(1 - r(\delta(t)))x(\delta(t)) \leq 0. \tag{2.7}$$

Now, we define the function $w(t)$ by the Riccati substitution

$$w(t) = \alpha(t)\frac{x^{\Delta}(t)}{x(t)},\ t \geq t_2. \tag{2.8}$$

Then form (2.5), we have $w(t) > 0$ and using (1.5) and (1.6) yield that

$$\begin{aligned} w^{\Delta}(t) &= \left(x^{\Delta}\right)^{\sigma}\left[\frac{\alpha(t)}{x(t)}\right]^{\Delta} + \frac{\alpha(t)}{x(t)}x^{\Delta\Delta}(t) \\ &= \frac{\alpha(t)}{x(t)}x^{\Delta\Delta}(t) + \left(x^{\Delta}\right)^{\sigma}\left[\frac{x(t)\alpha^{\Delta}(t) - \alpha(t)(x(t))^{\Delta}}{x(t)x^{\sigma}}\right]. \end{aligned} \tag{2.9}$$

In view of (2.7), Lemma 2.1 and (2.9), we obtain

$$w^{\Delta}(t) \leq -\alpha(t)Q(t) + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}}w^{\sigma} - \frac{\alpha(t)\left(x^{\Delta}\right)^{\sigma}(x(t))^{\Delta}}{x(t)x^{\sigma}}. \tag{2.10}$$

From (2.6) since $x^{\Delta\Delta}(t) < 0$ we have for $t \geq t_2$

$$x^{\Delta}(t) \geq \left(x^{\Delta}\right)^{\sigma}. \tag{2.11}$$

It follows from (2.10), and (2.11) that

$$w^{\Delta}(t) \leq -\alpha(t)Q(t) + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}}w^{\sigma} - \frac{\alpha(t)\left((x^{\Delta})^{\sigma}\right)^2}{x(t)x^{\sigma}}.$$

Since $x(t)$ is nondecreasing we see that $x^{\sigma} \geq x(t)$, and this implies

$$w^{\Delta}(t) \leq -\alpha(t)Q(t) + \frac{\left(\alpha^{\Delta}(t)\right)_+}{\alpha^{\sigma}}w^{\sigma} - \frac{\alpha(t)\left((x^{\Delta})^{\sigma}\right)^2}{(x^2)^{\sigma}}. \tag{2.12}$$

From (2.8) and (2.12), we obtain

$$w^{\Delta}(t) \leq -\alpha(t)Q(t) + \frac{\left(\alpha^{\Delta}(t)\right)_{+}}{\alpha^{\sigma}} w^{\sigma} - \frac{\alpha(t)}{(\alpha^{\sigma})^{2}} \left(w^{\sigma}\right)^{2}. \tag{2.13}$$

Integrating from t_2 to t ($t \geq t_2$), we have

$$\begin{aligned} &\int_{t_2}^{t} \alpha(s)Q(s)\Delta s \\ \leq\ & -\int_{t_2}^{t} w^{\Delta}(s)\Delta s + \int_{t_1}^{t} \frac{\left(\alpha^{\Delta}(t)\right)_{+}}{\alpha^{\sigma}} w^{\sigma}\Delta s - \int_{t_2}^{t} \frac{\alpha(s)}{(\alpha^{\sigma})^{2}} \left(w^{\sigma}\right)^{2} \Delta s, \end{aligned} \tag{2.14}$$

hence

$$\int_{t_2}^{t} \alpha(s)Q(s)\Delta s \leq w(t_2) + \int_{t_2}^{t} \frac{\left(\alpha^{\Delta}(s)\right)_{+}}{\alpha^{\sigma}} w^{\sigma}(s)\, \Delta s - \int_{t_1}^{t} \frac{\alpha(s)}{(\alpha^{\sigma})^{2}} (w^{\sigma})^{2}. \tag{2.15}$$

Then, we have

$$\begin{aligned} &\int_{t_2}^{t} \alpha(s)Q(s)\Delta s \\ \leq\ & w(t_2) - \int_{t_2}^{t} \left[\frac{\sqrt{\alpha(s)}}{\alpha^{\sigma}} w^{\sigma} + \frac{\left(\alpha^{\Delta}(s)\right)_{+}}{2\sqrt{\alpha(s)}} \right]^{2} \Delta s + \int_{t_1}^{t} \frac{\left(\left(\alpha^{\Delta}(s)\right)_{+}\right)^{2}}{4\alpha(s)} \Delta s \end{aligned} \tag{2.16}$$

$$\int_{t_2}^{t} \alpha(s)Q(s)\Delta s < w(t_2) + \int_{t_1}^{t} \frac{\left(\left(\alpha^{\Delta}(s)\right)_{+}\right)^{2}}{4\alpha(s)} \Delta s. \tag{2.17}$$

Hence

$$\int_{t_2}^{t} \left[\alpha(s)Q(s) - \frac{\left(\left(\alpha^{\Delta}(s)\right)_{+}\right)^{2}}{4\alpha(s)} \right] < w(t_2), \tag{2.18}$$

which contradicts the condition (2.1). The proof is complete. □

In the following theorem, we present new oscillation criteria for (1.8) of Kamenev type.

Theorem 2.2. *Assume that* (h_1) *and* (h_2) *hold. Let* $\alpha(t)$ *be as defined in Theorem 2.1. If for* $m > 1$

$$\limsup_{t\to\infty} \frac{1}{t^{m}} \int_{t_0}^{t} \left[(t-s)^{m}\alpha(s)Q(s) - \frac{(\alpha^{\sigma})^{2}B^{2}(t,s)}{4\alpha(s)(t-s)^{m}} \right] \Delta s = \infty, \tag{2.19}$$

where

$$B(t,s) = (t-s)^{m} \frac{\left(\alpha^{\Delta}(s)\right)_{+}}{\alpha^{\sigma}} - m(t-\sigma(s))^{m-1},\ t \geq s \geq t_0.$$

Then every solution of (1.8) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.8) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that y is an eventually positive solution of (1.8) $y(\delta(t))$ and $y(\tau(t)) > 0$ for all $t \geq t_1$ sufficiently large. We proceed as in the proof of Theorem 2.1 to prove that there exists $t_2 \geq t_1$ such that (2.13) holds for all $t \geq t_2$. Multiplying (2.13) by $(t-s)^m$ and integrating from t_2 to t, we have

$$\int_{t_2}^{t}(t-s)^m\alpha(s)Q(s)\Delta s \leq -\int_{t_2}^{t}(t-s)^m w^{\Delta}(s)\Delta s + \int_{t_2}^{t}(t-s)^m\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}}w^{\sigma}\Delta s - \int_{t_2}^{t}\frac{(t-s)^m\alpha(s)}{(\alpha^{\sigma})^2}\left(w^{\sigma}\right)^2\Delta s. \tag{2.20}$$

Using the integration by parts formula (1.7), we have

$$-\int_{t_2}^{t}(t-s)^m w^{\Delta}(s)\Delta s = -\left.(t-s)^m w(s)\right|_{t_2}^{t} + \int_{t_2}^{t}\left((t-s)^m\right)^{\Delta_s} w^{\sigma}\Delta s. \tag{2.21}$$

Now, we prove that

$$\left((t-s)^m\right)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1}. \tag{2.22}$$

We consider the following to cases: (i) $\mu(t)=0$, (ii) $\mu(t)\neq 0$. If (i) holds, then

$$\left((t-s)^m\right)^{\Delta_s} = -m(t-s)^{m-1}. \tag{2.23}$$

If (ii) holds, then we have

$$\begin{aligned}\left((t-s)^m\right)^{\Delta} &= \frac{1}{\mu(s)}\left[\left((t-\sigma(s))^m\right)-\left((t-s)^m\right)\right]\\ &= -\frac{1}{\sigma(s)-s}\left[\left((t-s)^m\right)-\left((t-\sigma(s))^m\right)\right].\end{aligned}$$

Using Hardy, Littlewood and Polya inequality (cf. [16])

$$x^m - y^m \geq my^{m-1}(x-y) \quad for\ all\ x \geq y > 0 \text{ and } m \geq 1,$$

we have $[(t-s)^m-(t-\sigma(s))^m] \geq m((t-\sigma(s))^{m-1}(\sigma(s)-s)$, and then we obtain

$$\left((t-s)^m\right)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1}. \tag{2.24}$$

Then, from (2.23) and (2.24), since in general case $\sigma(s) \geq s$, we see that (2.22) holds. From (2.22)-(2.24), we can obtain

$$\begin{aligned}&\int_{t_2}^{t}(t-s)^m\alpha(s)Q(s)\Delta s \\ \leq\ & w(t_2)\left(t-t_2\right)^m + \int_{t_2}^{t}\left[(t-s)^m\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}} - m(t-\sigma(s))^{m-1}\right]w^{\sigma}(s)\,\Delta s\\ &-\int_{t_2}^{t}\frac{(t-s)^m\alpha(s)}{(\alpha^{\sigma})^2}(w^{\sigma})^2.\end{aligned} \tag{2.25}$$

Then, as in the proof of Theorem 2.1, we have

$$\int_{t_2}^{t}(t-s)^m\alpha(s)Q(s)\Delta s \le w(t_2)\,(t-t_2)^m + \int_{t_2}^{t}\frac{(\alpha^\sigma)^2B^2(t,s)}{4\alpha(s)(t-s)^m}\Delta s.$$

Hence

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_2}^{t}\left[(t-s)^m\alpha(s)Q(s)-\frac{(\alpha^\sigma)^2B^2(t,s)}{4\alpha(s)(t-s)^m}\right]\to w(t_2),$$

which contradicts the condition (2.19). The proof is complete. □

In following, we obtain new oscillation criteria which can be considered as generalization of Theorem 2.2. We define the function space $\Re$ as follows: $H\in\Re$ provided H is defined for $t_0\le s\le t$, t, $s\in[t_0,\infty)_{\mathbb{T}}$ $H(t,s)\ge 0$, $H(t,t)=0$, $H^{\Delta_s}(t,s)\le 0$ for $t\ge s\ge t_0$, and for each fixed t, $H^{\Delta_s}(t,s)$ is delta integrable with respect to s.

Theorem 2.3. *Assume that* (h_1), (h_2) *hold and let* $H\in\Re$ *and*

$$h(t,s)=-\frac{H^{\Delta_s}(t,s)}{\sqrt{H(t,s)}}. \tag{2.26}$$

If there exists a positive real-valued rd-continuous function $\alpha(t)$ *such that*

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\left[\alpha(s)Q(s)-\frac{(\alpha^\sigma)^2}{4\alpha(s)}R^2(t,s)\right]\Delta s=\infty, \tag{2.27}$$

where $\left(\alpha^\Delta(s)\right)_+=\max\{0,\left(\alpha^\Delta(s)\right)\}$ *and*

$$R(t,s)=\left[h(t,s)/\sqrt{H(t,s)}-\frac{\left(\alpha^\Delta(s)\right)_+}{\alpha^\sigma}\right].$$

Then every solution of (1.8) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. We proceed as in the proof of Theorem 2.1 to prove that there exists $t_2\ge t_1$ such that (2.13) holds for $t\ge t_2$. From (2.13), it follows that

$$\begin{aligned}\int_{t_2}^{t}H(t,s)\alpha(s)Q(s)\Delta s &\le -\int_{t_2}^{t}H(t,s)w^\Delta(s)\Delta s\\ &+\int_{t_2}^{t}H(t,s)\frac{\left(\alpha^\Delta(s)\right)_+}{\alpha^\sigma}w^\sigma\Delta s-\int_{t_2}^{t}H(t,s)\frac{\alpha(s)}{(\alpha^\sigma)^2}(w^\sigma)^2\Delta s.\end{aligned} \tag{2.28}$$

Using integration by parts formula (1.7), we have

$$\int_{t_2}^{t} H(t,s)w^{\Delta}(s)\Delta s = H(t,s)w(s)|_{t_2}^{t} - \int_{t_2}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s = -H(t,t_2)w(t_2) - \int_{t_2}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s, \tag{2.29}$$

where $H(t,t) = 0$. Substituting from (2.29) in (2.28) and use (2.26), we get

$$\int_{t_2}^{t} H(t,s)\alpha(s)Q(s)\Delta s \leq H(t,t_2)w(t_2) - \int_{t_2}^{t} h(t,s)\sqrt{H(t,s)}w^{\sigma}\Delta s + \int_{t_2}^{t} H(t,s)\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}}w^{\sigma}\Delta s - \int_{t_2}^{t} H(t,s)\frac{\alpha(s)}{(\alpha^{\sigma})^2}(w^{\sigma})^2\Delta s. \tag{2.30}$$

Hence,

$$\int_{t_2}^{t} H(t,s)\alpha(s)Q(s)\Delta s \leq H(t,t_2)w(t_2) - \int_{t_2}^{t}\left[h(t,s)\sqrt{H(t,s)} - H(t,s)\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}}\right]w^{\sigma}\Delta s - \int_{t_2}^{t} H(t,s)\frac{\alpha(s)}{(\alpha^{\sigma})^2}(w^{\sigma})^2\Delta s. \tag{2.31}$$

Therefore

$$\int_{t_2}^{t} H(t,s)\alpha(s)Q(s)\Delta s \leq H(t,t_2)w(t_2) - \int_{t_2}^{t}\left[\frac{\sqrt{H(t,s)\alpha(s)}}{\alpha^{\sigma}}w^{\sigma} + \frac{\alpha^{\sigma}\left[h(t,s)\sqrt{H(t,s)} - H(t,s)\frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}}\right]}{2\sqrt{H(t,s)\alpha(s)}}\right]^2\Delta s + \int_{t_2}^{t} H(t,s)\frac{(\alpha^{\sigma})^2}{4\alpha(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\left(\alpha^{\Delta}(s)\right)_+}{\alpha^{\sigma}}\right]^2\Delta s. \tag{2.32}$$

Then, for all $t \geq t_2$ we have

$$(2.33)\quad \int_{t_2}^{t} H(t,s)\left[\alpha(s)Q(s) - \frac{(\alpha^\sigma)^2}{4\alpha(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\left(\alpha^\Delta(s)\right)_+}{\alpha^\sigma}\right]^2\right]\Delta s < H(t,t_2)w(t_2),$$

and this implies that

$$(2.34)\quad \frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\left[\alpha(s)Q(s) - \frac{(\alpha^\sigma)^2}{4\alpha(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\left(\alpha^\Delta(s)\right)_+}{\alpha^\sigma}\right]^2\right]\Delta s < w(t_2),$$

for all large t, which contradicts (2.27). The proof is complete. □

As an immediate consequence of Theorem 2.3 we get the following.

Corollary 2.1. *Let the assumption* (2.27) *in Theorem 2.1 be replaced by*

$$(2.35)\quad \lim_{t\to\infty} \sup \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\alpha(s)Q(s)\Delta s = \infty,$$

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\frac{(\alpha^\sigma)^2}{4\alpha(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{\left(\alpha^\Delta(s)\right)_+}{\alpha^\sigma}\right]^2 \Delta s < \infty,$$

then every solution of (1.8) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Remark 2.2. With an appropriate choice of the functions H and h one can derive from the conditions on Remark 2.1 a number of oscillation criteria for (1.8) on different types of time scales. Consider, for example the function $H(t,s) = (t-s)^\lambda$, $(t,s) \in \mathbf{D}$ with $\lambda \geq 1$ is an odd integer. Evidently H belongs to the class $\Re$ and then (2.27) reduces to the oscillation criterion of Kamenev-type. Also, one can use the factorial function $H(t,s) = (t-s)^{(k)}$ where $t^{(k)} = t(t-1)\cdots(t-k+1)$, $t^{(0)} = 1$. In this case

$$\begin{aligned} H^{\Delta_2}(t-s)^{(\lambda)} &= \frac{(t-\sigma(s))^{(k)} - (t-s)^{(k)}}{\mu(s)} = -\frac{(t-s)^{(k)} - (t-\sigma(s))^{(k)}}{\mu(s)} \\ &\geq -(k)(t-s)^{(k-1)}. \end{aligned}$$

Example 2.1. Consider the following second-order neutral delay dynamic equation

$$(2.36)\quad \left[y(t) + \frac{1}{\delta^{-1}(t)}y(\tau(t))\right]^{\Delta\Delta} + \frac{\lambda}{t\delta(t)}y(\delta(t)) = 0, \qquad t \in \mathbb{T},$$

where $\mathbb{T} = [1, \infty)$ is a time scale and τ and δ are nonnegative constants such that $\tau(t)$ and $\delta(t) \in \mathbb{T}$ and $\lambda > 0$ is a constant. In (2.35) $p(t) = \frac{\lambda}{t^2}$, and $r(\delta(t)) = \frac{1}{t} < 1$. It is easy to see that the assumptions (h_1) and (h_2) hold. To apply Theorem 2.1, it remains to satisfy the condition (2.1). By choosing $\alpha(s) = s$, we have

$$\begin{aligned} & \lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\alpha(s)Q(s) - \frac{1}{4s} \right] \Delta s \\ = & \lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\frac{\lambda}{s}(1 - \frac{1}{s}) - \frac{1}{4s} \right] \Delta s \\ = & \lim_{t\to\infty} \sup \int_{t_0}^{t} \left(\frac{4\lambda - 1}{4s} - \frac{\lambda}{s^2} \right) \Delta s = \infty, \quad if \quad \lambda > \frac{1}{4} \end{aligned}$$

Hence, by Theorem 2.1 every solution of (2.36) oscillates if $\lambda > \frac{1}{4}$.

3. Applications

In this section, we apply the results in Section 2, to establish some oscillation criteria for equation (1.9)-(1.14).

Corollary 3.1. *Assume that $r(t)$ and $p(t)$ are positive functions defined on $[t_0, \infty) \subset \mathbb{R}$, and $0 \le r(t) < 1$ and $\tau(t)$ and $\delta(t)$ are delay function. Furthermore there exists a positive continuous differentiable functions $\alpha(t)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\alpha(s)Q(s) - \frac{(\left(\alpha^{'}(s)\right)_+)^2}{4\alpha(s)} \right) ds = \infty,$$

where

$$Q(s) = \frac{\delta(s)}{s} p(s)(1 - r(\delta(s))),$$

$\left(\alpha^{'}(t)\right)_+ = \max\{\alpha^{'}(t), 0\}$. *Then every solution of* (1.9) *is oscillatory on* $[t_0, \infty)$.

Corollary 3.2. *Assume that $r(t)$ and $p(t)$ are positive functions defined on $[t_0, \infty) \subset \mathbb{R}$, and $0 \le r(t) < 1$ and $\tau(t)$ and $\delta(t)$ are delay function. Furthermore there exists a positive continuous differentiable functions $\alpha(t)$ such that*

$$\lim_{t\to\infty} \sup \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s) \left[\alpha(s)Q(s) - \frac{\alpha(s)A^2(t, s)}{4} \right] ds = \infty,$$

where

$$A(t, s) = \left(h(t, s)/\sqrt{H(t, s)} - \frac{\left(\alpha^{'}(s)\right)_+}{\alpha(s)} \right).$$

Then every solution of (1.9) *oscillates.*

Corollary 3.3. *Assume that* $r(n)$ *and* $p(n)$ *are positive sequences defined on* $[t_0, \infty) \subset \mathbb{N}$, *and* $0 \leq r(n) < 1$ *and* $\tau(n) \in \mathbb{N}$, $\delta(n) \in \mathbb{N}$. *Furthermore, assume that there exists a positive sequence* $\alpha(n)$ *such that*

$$\limsup_{t\to\infty} \sum_{i=t_0}^{t-1} \left[\alpha(i)Q(i) - \frac{\left((\Delta\alpha(i))_+\right)^2}{4\alpha(i)} \right] = \infty,$$

where

$$Q(i) = \frac{\delta(i)}{i} p(i)(1 - r(\delta(i))),$$

$(\Delta\alpha(i))_+ = \max\{0,\, \Delta\alpha(i)\}$. *Then every solution of* (1.10) *oscillates.*

Corollary 3.4. *Assume that* $r(n)$ *and* $p(n)$ *are positive sequences defined on* $[t_0, \infty) \subset \mathbb{N}$, *and* $0 \leq r(n) < 1$ *and* $\tau(n) \in \mathbb{N}$, $\delta(n) \in \mathbb{N}$. *Furthermore, assume that there exists a positive sequence* $\alpha(n)$ *such that*

$$\lim_{m\to\infty} \sup \frac{1}{H(m,n_0)} \sum_{n=n_0}^{m-1} H(m,n) \left[\alpha(n)Q(n) - \frac{\alpha^2(n+1)}{4\alpha(n)} B^2(m,n) \right] = \infty,$$

where

$$B(m,n) = \left(h(m,n)/\sqrt{H(m,n)} - \frac{(\Delta\alpha(n))_+}{\alpha(n+1)} \right).$$

Then every solution of (1.10) *oscillates.*

Corollary 3.5. *Assume that* $r(n)$ *and* $p(n)$ *are positive sequences defined on* $[t_0, \infty) \subset h\mathbb{N}$, $h > 0$, *and* $0 \leq r(n) < 1$, $\tau(n)$ *and* $\delta(n) \in h\mathbb{N}$. *Furthermore, assume that there exists a positive sequence* $\alpha(n)$ *such that*

$$\limsup_{t\to\infty} \sum_{i=\frac{t_0}{h}}^{\frac{t}{h}-1} \left[\alpha(i)Q(i) - \frac{\left((\Delta_h\alpha(i))_+\right)^2}{4\alpha(i)} \right] = \infty,$$

where $(\Delta_h\alpha(i))_+ = \max\{0, \Delta_h\alpha(i)\}$. *Then every solution of* (1.11) *oscillates.*

Corollary 3.6. *Assume that* $r(n)$ *and* $p(n)$ *are positive sequences defined on* $[t_0, \infty) \subset h\mathbb{N}$, $h > 0$, *and* $0 \leq r(n) < 1$, $\tau(n)$ *and* $\delta(n) \in h\mathbb{N}$. *Furthermore, assume that there exists a positive sequence* $\alpha(n)$ *such that*

$$\lim_{t\to\infty} \sup \frac{1}{H(m,n_0)} \sum_{n=\frac{t_0}{h}}^{\frac{m}{h}-1} H(m,n) \left[\alpha(n)Q(n) - \frac{\alpha^2(n+h)C^2(m,n)}{4\alpha(n)} \right] = \infty,$$

where

$$C(m,n) = \left(h(m,n)/\sqrt{H(m,n)} - \frac{(\Delta_h\alpha(n))_+}{\alpha(n+h)} \right).$$

Then every solution of (1.11) *oscillates.*

Corollary 3.7. *Assume that $r(n)$ and $p(n)$ are positive sequences defined on $[t_0, \infty) \subset q^{\mathbb{N}}$ and $0 \le r(n) < 1$, τ and $\delta \in q^{\mathbb{N}}$. Furthermore, assume that there exists a positive sequence $\alpha(n)$ such that*

$$\sum_{i=0}^{\infty} \mu(q^i) \left[\alpha(q^i) Q(q^i) - \frac{\left(\left(\Delta_q \alpha(q^i) \right)_+ \right)^2}{4\alpha(q^i)} \right] = \infty,$$

where $(\Delta_q \alpha(i))_+ = \max\{0, \Delta_q \alpha(i)\}$. Then every solution of (1.12) *oscillates.*

Corollary 3.8. *Assume that $r(n)$ and $p(n)$ are positive sequences defined on $[t_0, \infty) \subset q^{\mathbb{N}}$ and $0 \le r(n) < 1$, τ and $\delta \in q^{\mathbb{N}}$. Furthermore, assume that there exists a positive sequence $\alpha(n)$ such that*

$$\lim_{t \to \infty} \sup \frac{1}{H(q^t, t_0)} \sum_{k=0}^{t} \mu(q^k) H(q^t, q^k) \left[\alpha(q^k) Q(q^k) - \frac{\alpha^2(qk) F^2(q^t, q^k)}{4\alpha(q^k)} \right] = \infty,$$

$$F(t,s) = \left(h(t,s)/\sqrt{H(t,s)} - \frac{(\Delta_q \alpha(s))_+}{\alpha(s)} \right), \quad (\Delta_q \alpha(s))_+ = \max\{0, \Delta_q \alpha(s)\}.$$

Then every solution of (1.12) *oscillates.*

The sufficient conditions for the oscillation of (1.13) and (1.14) are left to the interested reader. One uses

$$\int_{t_0}^{b} f(t) \Delta t = \sum_{t \in [t_0, b)} \mu(t) f(t).$$

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