

The Basis Number of the Cartesian Product of a Path with a Circular Ladder, a Möbius Ladder and a Net

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ABSTRACT. The basis number of a graph G is the least positive integer k such that G has a k -fold basis. In this paper, we prove that the basis number of the cartesian product of a path with a circular ladder, a Möbius ladder and path with a net is exactly 3. This improves the upper bound of the basis number of these graphs for a general theorem on the cartesian product of graphs obtained by Ali and Marougi, see [2]. Also, by this general result, the cartesian product of a theta graph with a Möbius ladder is at most 5. But in section 3 we prove that it is at most 4.

1. Introduction

Getting new graphs from known graphs through different kinds of graph products and operations on graphs originated as early as the beginning of graph theory as an independent subject. Actually graph products are the best natural way to enlarge the space of graphs. In the literature there are a lot of graph products. We mention out of these products; the cartesian product, the direct product, the strong product, the semi-strong product, the lexicographic product, the semi-composition product and the special product. Many researchers employed their efforts to study the properties of graphs obtained by the graph products and related some of these properties to those of the graphs incorporated in the products. The enthusiasm of studying graph products led Klavzar and Wilfried to write a whole book that focuses on materials regarding four of the above mentioned graph products, see [11].

In graph theory, there are many numbers that give rise to a better understanding and interpretation of the geometric properties of a given graph such as the crossing number, the thickness, the basis number, the genus, etc.. The basis number of a graph is of a particular importance because MacLane, in [17], made a connection between the basis number and the planarity of a graph; in fact, he proved that a

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graph is planar if and only if its basis number is at most 2.

In 1981, E. Schemichel utilized the ideas of MacLane and defined the basis number of a graph in its recent form, see [18]. Moreover, he investigated the basis number of certain important classes of non-planar graphs, specifically, complete graphs and complete bipartite graphs. Then, J. Banks and E. Schemichel [6] proved that for $n \geq 7$, the basis number of Q_n is 4, where Q_n is the n -cube. After that, the basis number attracted a lot of researchers to investigate the basis number of nonplanar graphs that come from different kinds of graph products, see [3], [4], [5], [9], [12], [13], [14], [15] and [16].

Ali and Marougi studied the basis number of the cartesian product of graphs in [2]. They proved that the basis number of the cartesian product of two given connected graphs, say H and G , is bounded above by the maximum of the two numbers $b(G) + \Delta(T_H)$ and $b(H) + \Delta(T_G)$, where T_H and T_G are spanning trees of H and G , respectively, such that the maximum degrees $\Delta(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G . In [3], Ali proved that the basis number of the cartesian product of a wheel with a path, a cycle, or another wheel is 3 under some restrictions on their orders, so he improved the upper bound of the basis number of these graphs from 4 to 3. Also, he proved that the basis number of the cartesian product of two complete graphs of orders $4m+2$ and $4n+2$ is at most 4. This improves the upper bound of the basis number of these graphs from 5 to 4.

In this paper, we investigate the basis number of the cartesian product of a path with a circular ladder, a path with a Möbius ladder, a theta graph with Möbius ladder and a path with a net. In fact, we improve the upper bound of the basis number obtained by Ali and Marougi for these graphs as we will see in section 3.

2. Definitions and preliminaries

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [7] or [10]. For a given graph G , we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. Given a graph G , let $e_1, e_2, \dots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0, 1)$ -vector $(b_1, b_2, \dots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$ -dimensional vector space, denoted by $(\mathbb{Z}_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(\mathbb{Z}_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the cycle space of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that for a connected graph G we have

$$(2.1) \quad \dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1.$$

A basis \mathcal{B} for $\mathcal{C}(G)$ is called a d -fold if each edge of G occurs in at most d of the cycles in the basis \mathcal{B} . The basis number, $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d -fold basis.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The cartesian product $G^* = G \times H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G) \text{ and } v_1 = v_2, \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(H)\}$.

The following results will be used frequently in our proofs:

Theorem 2.1. (MacLane) *If G is a graph, then G is planar if and only if $b(G) \leq 2$.*

Ali and Marougi [2] give an upper bound to the cartesian product of any two connected disjoint graphs in the following theorem.

Theorem 2.2. (Ali and Marougi) *If G and H are two connected disjoint graphs, then*

$$b(G \times H) \leq \max\{b(G) + \Delta(T_H), b(H) + \Delta(T_G)\},$$

where T_H and T_G are spanning trees of H and G , respectively, such that the maximum degrees $\Delta(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G .

Also, they cited a reference in [3] where they have proved the following result:

Theorem 2.3. *$G \times H$ is nonplanar if G and H are any graphs with $\Delta(G) \geq 2$ and $\Delta(H) \geq 3$.*

3. Main results

In this section we investigate the basis number of the cartesian product of a path with a circular ladder, a path with a Möbius ladder, a theta graph with Möbius ladder and a path with a net. Throughout this section we denote by P_n the path $123 \cdots n$, and C_n the cycle $123 \cdots n1$. The circular ladder, CL_m , is a graph with vertex set $\{u_1, v_1, \dots, u_m, v_m\}$ and edge set $\{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq m-1\} \cup \{u_mu_1, v_mv_1\} \cup \{u_iv_i : 1 \leq i \leq m\}$. We consider the graphs of the ladder and the Möbius ladder as graphs obtained from the graph of the circular ladder; the first is obtained by deleting the set of edges $\{u_mu_1, v_mv_1\}$, and the second is obtained by replacing the set $\{u_mu_1, v_mv_1\}$ by the set $\{u_mv_1, v_mu_1\}$. The theta graph, θ_n , is considered as a graph obtained from the graph of C_n , by adding a new edge that joins two nonadjacent vertices of C_n .

It is clear that $|V(P_n \times CL_m)| = 2mn$ and $|E(P_n \times CL_m)| = 5mn - 2m$, and so $\dim \mathcal{C}(P_n \times CL_m) = 3mn - 2m + 1$.

If we apply Ali and Marougi Theorems that were stated in Section 2 we notice that $P_n \times CL_m$ is nonplanar and $b(P_n \times CL_m) \leq 4$. Our goal in the following theorem is to prove that the basis number of $P_n \times CL_m$ is exactly 3. The following lemma is useful in our work and is easy to show.

Lemma 3.1. *Let $\mathcal{B}_{(1 \times CL_m)} = \{(1, u_i)(1, v_i)(1, v_{i+1})(1, u_{i+1})(1, u_i) | i = 1, 2, \dots, m-1\} \cup \{a_1 = (1, u_1)(1, u_2)(1, v_3) \cdots (1, u_m)(1, u_1), a_2 = (1, v_1)(1, v_2)(1, v_3) \cdots (1, v_m)(1, v_1)\}$. Then $\mathcal{B}_{(1 \times CL_m)}$ is a 2-fold basis for $\mathcal{C}(1 \times CL_m)$.*

Theorem 3.1. *For each $n \geq 3$ and $m \geq 3$, we have $b(P_n \times CL_m) = 3$.*

Proof. Since $P_n \times CL_m$ is nonplanar we have $b(P_n \times CL_m) \geq 3$ by MacLane's Theorem. To prove that $b(P_n \times CL_m) \leq 3$, we have to prove that the cycle space $\mathcal{C}(P_n \times CL_m)$ has a 3-fold basis. To do so, we define the following set of cycles:

$$\mathcal{B}(P_n \times CL_m) = \left(\bigcup_{i=1}^3 \mathcal{B}_i \right) \cup \left(\bigcup_{i=1}^2 \mathcal{B}_{mi} \right) \cup \mathcal{B}_{(1 \times CL_m)},$$

where $\mathcal{B}_1 = \bigcup_{i=1}^{n-1} \mathcal{B}_{1i}$, $\mathcal{B}_2 = \bigcup_{i=1}^{n-1} \mathcal{B}_{2i}$, $\mathcal{B}_3 = \bigcup_{i=1}^{n-1} \mathcal{B}_{3i}$ and $\mathcal{B}_{(1 \times CL_m)}$ is the required basis of the cycle space of $1 \times CL_m$ as in Lemma 3.1. For each $1 \leq i \leq n-1$, we define the sets of cycles

$$\begin{aligned} \mathcal{B}_{1i} &= \{(i, u_j)(i+1, u_j)(i+1, u_{j+1})(i, u_{j+1})(i, u_j) : 1 \leq j \leq m-1\}, \\ \mathcal{B}_{2i} &= \{(i, v_j)(i+1, v_j)(i+1, v_{j+1})(i, v_{j+1})(i, v_j) : 1 \leq j \leq m-1\}, \\ \mathcal{B}_{3i} &= \{(i, u_j)(i, v_j)(i+1, v_j)(i+1, u_{j+1})(i, u_j) : 1 \leq j \leq m\}. \end{aligned}$$

The sets \mathcal{B}_{m1} , and \mathcal{B}_{m2} are given by the following sets:

$$\begin{aligned} \mathcal{B}_{m1} &= \{(i, u_m)(i, u_1)(i+1, u_1)(i+1, u_m)(i, u_m) : 1 \leq i \leq n-1\}, \\ \mathcal{B}_{m2} &= \{(i, v_m)(i, v_1)(i+1, v_1)(i+1, v_m)(i, v_m) : 1 \leq i \leq n-1\}. \end{aligned}$$

Now, let $P_m^u = u_1 u_2 \cdots u_m$, and $P_m^v = v_1 v_2 \cdots v_m$. Then, it is easy to notice that the set $\bigcup_{i=1}^{n-1} \mathcal{B}_{1i}$ is the set of all cycles obtained by taking the boundaries of all the finite faces in the planar subgraph $P_n \times P_m^u$, this implies that $\bigcup_{i=1}^{n-1} \mathcal{B}_{1i}$ is a basis of $\mathcal{C}(P_n \times P_m^u)$. Thus, $\mathcal{B}_1 = \bigcup_{i=1}^{n-1} \mathcal{B}_{1i}$ is linearly independent. To prove that $\mathcal{B}_2 = \bigcup_{i=1}^{n-1} \mathcal{B}_{2i}$ is linearly independent, we repeat the same arguments that we used above for \mathcal{B}_1 with replacing P_m^u by P_m^v and \mathcal{B}_{1i} by \mathcal{B}_{2i} . Since $P_n \times P_m^u$ has no common edges with $P_n \times P_m^v$, $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent. Since $(E(P_n \times P_m^u) \cup E(P_n \times P_m^v)) \cap E(1 \times CL_m) = E(1 \times P_m^u) \cup E(1 \times P_m^v)$ which is a forest subgraph of $1 \times CL_m$ and since any linear combination of cycles of a linearly independent set of cycles is a cycle or an edge disjoint union of cycles, any linear combination of $1 \times \mathcal{B}_{CL_m}$ must contain an edge of the complement of $1 \times P_m^u \cup 1 \times P_m^v$ in $(1 \times CL_m)$ which is not in $P_n \times P_m^u \cup P_n \times P_m^v$. Thus, $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{(1 \times CL_m)}$ is linearly independent. We now show that the set $\mathcal{B}_3 = \bigcup_{i=1}^{n-1} \mathcal{B}_{3i}$ is linearly independent by using mathematical induction on n . \mathcal{B}_{3i} is linearly independent because \mathcal{B}_{3i} is a vertex pairwise disjoint union of cycles for each $i = 1, 2, \dots, n-1$. If $n = 2$, then $\mathcal{B}_3 = \mathcal{B}_{31}$ which is independent by the above argument. Assume that $n \geq 3$ and the result is true for less than n . Then by induction step and above argument both of $\bigcup_{i=1}^{n-2} \mathcal{B}_{3i}$ and $\mathcal{B}_{3(n-1)}$ are linearly independent. Note that $E(\mathcal{B}_{3(n-1)}) \cap E(\bigcup_{i=1}^{n-2} \mathcal{B}_{3i}) = \{(n-1, u_j)(n-1, v_j) \mid i = 1, 2, \dots, n\}$ is a perfect matching. Thus, if

$$O_{(n-1)} = \sum_{i=1}^{n-2} O_i \pmod{2},$$

where O_i is a linear combination of cycles of \mathcal{B}_{3i} , then

$$O_{(n-1)} = O_1 \oplus O_2 \oplus \cdots \oplus O_{(n-2)},$$

where \oplus is the ring sum. And so

$$E(O_{(n-1)}) = E(O_1 \oplus O_2 \oplus \cdots \oplus O_{(n-2)}) \subseteq \{(n-1, u_j)(n-1, v_j) \mid i = 1, 2, \dots, n\}$$

which is a contradiction. Since any linear combination of cycles of \mathcal{B}_3 contains at least two edges of $\{(i, u_j)(i, v_j) \mid i = 1, 2, \dots, n; j = 1, 2, \dots, m\}$, but $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{(1 \times CL_m)}$ contains at most one edge of the above set. In fact, of the set $\{(1, u_j)(1, v_j) \mid j = 1, 2, \dots, m\}$, as a result $\bigcup_{i=1}^3 \mathcal{B}_i$ is linearly independent. Clearly that \mathcal{B}_{m1} and \mathcal{B}_{m2} are bases of the cycle spaces $P_n \times u_1 u_m$ and $P_n \times v_1 v_m$, respectively, and $P_n \times u_1 u_m$ has no common edge with $P_n \times v_1 v_m$. Hence, $\mathcal{B}_{m1} \cup \mathcal{B}_{m2}$ is linearly independent. Similar any linear combination of cycles of $\mathcal{B}_{m1} \cup \mathcal{B}_{m2}$ contains at least two edges of $\{(i, u_1)(i, u_m) \mid i = 1, 2, \dots, n\}$ or at least two edges of $\{(i, v_1)(i, v_m) \mid i = 1, 2, \dots, n\}$. On the other hand, any linear combination of cycles of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{1 \times CL_m}$ contains at most one edge of $\{(i, u_1)(i, u_m) \mid i = 1, 2, \dots, n\}$, in fact $(1, u_1)(1, u_m)$ and at most one edge of $\{(i, v_1)(i, v_m) \mid i = 1, 2, \dots, n\}$, in fact $(1, v_1)(1, v_m)$. Thus, any linear combination of cycles of $\mathcal{B}_{m1} \cup \mathcal{B}_{m2}$ can not be written as a linear combination of cycles of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_{(1 \times CL_m)}$. Therefore, $\mathcal{B}(P_n \times CL_m)$ is linearly independent. To this end,

$$\begin{aligned} |\mathcal{B}(P_n \times CL_m)| &= 2(m-1)(n-1) + m(n-1) + 2(n-1) + m + 1 \\ &= \dim \mathcal{C}(P_n \times CL_m). \end{aligned}$$

Hence, $\mathcal{B}(P_n \times CL_m)$ is a basis of $\mathcal{C}(P_n \times CL_m)$. To complete the proof, we show that $\mathcal{B}(P_n \times CL_m)$ is a 3-fold basis. Let $e \in E(P_n \times CL_m)$. Then (1) if $e = (i, u_j)(i, u_{j+1})$ or $(i, v_j)(i, v_{j+1})$ such that $i \neq 1$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) \leq 2$, and $f_{\mathcal{B}_3}(e) = f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) = f_{\mathcal{B}_{(1 \times CL_m)}}(e) = 0$. (2) If $e = (1, u_j)(1, u_{j+1})$ or $(1, v_j)(1, v_{j+1})$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) \leq 1$, $f_{\mathcal{B}_3}(e) = f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) = 0$, and $f_{\mathcal{B}_{(1 \times CL_m)}}(e) \leq 2$. (3) If $e = (i, u_j)(i, v_j)$ such that $i \neq 1$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) = f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) = f_{\mathcal{B}_{(1 \times CL_m)}}(e) = 0$, and $f_{\mathcal{B}_3}(e) \leq 2$. (4) If $e = (1, u_j)(1, v_j)$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) = f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) = 0$, $f_{\mathcal{B}_3}(e) \leq 1$, and $f_{\mathcal{B}_{(1 \times CL_m)}}(e) \leq 2$. (5) If $e = (i, u_1)(i, u_m)$ or $(i, v_1)(i, v_m)$ such that $i \neq 1$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) = f_{\mathcal{B}_3}(e) = f_{\mathcal{B}_{(1 \times CL_m)}}(e) = 0$ and $f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) \leq 2$. (6) If $e = (1, u_1)(1, u_m)$ or $(1, v_1)(1, v_m)$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) = f_{\mathcal{B}_3}(e) = 0$, $f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) = 1$ and $f_{\mathcal{B}_{(1 \times CL_m)}}(e) \leq 2$. (7) If $e = (i, u_j)(i+1, u_j)$ or $(i, v_j)(i+1, v_j)$ or $(1, u_j)(n, u_j)$ or $(1, v_j)(n, v_j)$ such that $j \neq 1, m$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) \leq 2$, $f_{\mathcal{B}_3}(e) \leq 1$, $f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) = f_{\mathcal{B}_{(1 \times CL_m)}}(e) = 0$. (8) If $e = (i, u_j)(i+1, u_j)$ or $(i, v_j)(i+1, v_j)$ or $(1, u_j)(n, u_j)$ or $(1, v_j)(n, v_j)$ such that $j = 1, m$, then $f_{\mathcal{B}_1 \cup \mathcal{B}_2}(e) \leq 1$, $f_{\mathcal{B}_3}(e) \leq 1$, $f_{\mathcal{B}_{m1} \cup \mathcal{B}_{m2}}(e) \leq 1$, and $f_{\mathcal{B}_{(1 \times CL_m)}}(e) = 0$. The proof is completed. \square

We have assumed that the Möbius ladder graph is obtained from the circular ladder by deleting the edges $u_m u_1, v_m v_1$ and replacing them by $u_m v_1, v_m u_1$ that cross each other. The following lemma will be used in the next theorem.

Lemma 3.2. *Let $\mathcal{B}_{(1 \times ML_m)} = \{(1, u_i)(1, v_i)(1, v_{i+1})(1, u_{i+1})(1, u_i) \mid i = 1, 2, \dots, m-1\} \cup \{b_1 = (1, u_1)(1, u_2)(1, u_3) \cdots (1, u_m)(1, v_m)(1, u_1), b_2 = (1, v_1)(1, v_2)(1, v_3) \cdots (1, v_m)(1, u_m)(1, v_1)\}$. Then $\mathcal{B}_{(1 \times ML_m)}$ is a 3-fold basis for $\mathcal{C}(1 \times ML_m)$.*

Theorem 3.2. *For each $n \geq 3$ and $m \geq 3$, we have $b(P_n \times ML_m) = 3$.*

Proof. Define $\mathcal{B}(P_n \times ML_m) = (\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup (\mathcal{B}_{(1 \times ML_m)} - b_1) \cup \{C\}$, where $\bigcup_{i=1}^3 \mathcal{B}_i$ is as defined in Theorem 3.1, $(\mathcal{B}_{(1 \times ML_m)} - b_1)$ is as defined in Lemma 3.2,

$$\begin{aligned} \mathcal{B}_{m1}^* &= \{(i, v_m)(i, u_1)(i+1, u_1)(i+1, v_m)(i, v_m) : 1 \leq i \leq n-1\}, \\ \mathcal{B}_{m2}^* &= \{(i, u_m)(i, v_1)(i+1, v_1)(i+1, u_m)(i, u_m) : 1 \leq i \leq n-1\}, \end{aligned}$$

and

$$C = (2, u_1)(2, u_2)(2, u_3) \cdots (2, u_m)(2, v_m)(2, u_1).$$

By arguments as in Theorem 3.1, we prove that $(\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup \mathcal{B}_{1 \times ML_m}$ is linearly independent. And so $(\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup (\mathcal{B}_{1 \times ML_m} - b_1)$ is linearly independent. We now show that C is linearly independent with the cycles of $(\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup (\mathcal{B}_{1 \times ML_m} - b_1)$. Assume that C is a linear combination of cycles of $(\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup (\mathcal{B}_{1 \times ML_m} - b_1)$, say $R = \{R_1, R_2, \dots, R_k\}$. Since $(1, v_m)(1, u_1)(2, u_1)(2, v_m)(1, v_m)$ contains $(1, v_m)(1, u_1)$ which is not in $E(C)$ and belongs to no other cycles of $(\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup (\mathcal{B}_{1 \times ML_m} - b_1)$, as a result $(1, v_m)(1, u_1)(2, u_1)(2, v_m)(1, v_m)$ can not belong to R . Now, since, $(2, v_m)(2, u_1) \in E(C)$ and $(2, v_m)(2, u_1)$ belongs only to $(1, v_m)(1, u_1)(2, u_1)(2, v_m)(1, v_m)$ and $(2, v_m)(2, u_1)(3, u_1)(3, v_m)(2, v_m)$, then $(2, v_m)(2, u_1)(3, u_1)(3, v_m)(2, v_m)$ must belong to R , say $R_1 = (2, v_m)(2, u_1)(3, u_1)(3, v_m)(2, v_m)$. Since $(3, u_1)(3, v_m) \in E(R_1)$ and $(3, u_1)(3, v_m) \notin E(C)$ and since $(3, u_1)(3, v_m)$ belongs only to R_1 and $(3, v_m)(3, u_1)(4, u_1)(4, v_m)(3, v_m)$, as a result $(3, v_m)(3, u_1)(4, u_1)(4, v_m)(3, v_m) \in R$, say $R_2 = (3, v_m)(3, u_1)(4, u_1)(4, v_m)(3, v_m)$. Since $(4, u_1)(4, v_m) \in E(R_1 \oplus R_2)$ and $(4, u_1)(4, v_m) \notin E(C)$ and since $(4, u_1)(4, v_m)$ belongs only to R_2 and $(4, v_m)(4, u_1)(5, u_1)(5, v_m)(4, v_m)$, as a result $(4, v_m)(4, u_1)(5, u_1)(5, v_m)(4, v_m) \in R$. By continuing in this process, it implies that $(n-1, v_m)(n-1, u_1)(n, u_1)(n, v_m)(n-1, v_m) \in R$, say $R_{n-2} = (n-1, v_m)(n-1, u_1)(n, u_1)(n, v_m)(n-1, v_m)$. Since $(n, u_1)(n, v_m) \in E(R_1 \oplus R_2 \oplus \dots \oplus R_{n-2})$ and since $(n, u_1)(n, v_m)$ does not belong to any cycles of $(\bigcup_{i=1}^3 \mathcal{B}_i) \cup (\bigcup_{i=1}^2 \mathcal{B}_{mi}^*) \cup (\mathcal{B}_{1 \times ML_m} - b_1) - R_{n-2}$, as a result $(n, u_1)(n, v_m) \in E(R_1 \oplus R_2 \oplus \dots \oplus R_k)$. And so $(n, u_1)(n, v_m) \in E(C)$. This is a contradiction. It is an easy task to show that $\mathcal{B}(P_n \times ML_m)$ is a 3-fold basis. The proof is completed. \square

We consider the graph θ_n as a graph obtained from the cycle C_n by adding a new edge that joins the nonadjacent vertices r and t , with $1 \leq r < t < n$. Then, the graph $\theta_n \times CL_m$ is obtained from $P_n \times CL_m$ by adding the following two sets of edges:

$$\begin{aligned} \mathcal{A}_1 &= \{(r, u_j)(t, u_j), (r, v_j)(t, v_j) : 1 \leq j \leq m\}; \quad |\mathcal{A}_1| = 2m. \\ \mathcal{A}_2 &= \{(1, u_j)(n, u_j), (1, v_j)(n, v_j) : 1 \leq j \leq m\}; \quad |\mathcal{A}_2| = 2m. \end{aligned}$$

It is clear that $|V(\theta_n \times ML_m)| = 2mn$ and $|E(\theta_n \times ML_m)| = 3mn + 2mn + 2m$, and so

$$\begin{aligned} \dim \mathcal{C}(\theta_n \times ML_m) &= (3mn - 2m + 1) + 4m \\ &= \dim \mathcal{C}(P_n \times ML_m) + 4m. \end{aligned}$$

From Ali and Marougi Theorems, those were stated in the introduction, we have $\theta_n \times ML_m$ is nonplanar and $b(\theta_n \times ML_m) \leq 5$, but in the following theorem we will prove that $b(\theta_n \times ML_m) \leq 4$.

Theorem 3.3. *For each $n \geq 4$ and $m \geq 3$, we have $3 \leq b(\theta_n \times ML_m) \leq 4$.*

Proof. The graph $\theta_n \times ML_m$ is nonplanar, so by MacLane's Theorem we have $b(\theta_n \times ML_m) \geq 3$. On the other hand we define the set

$$\mathcal{B}(\theta_n \times ML_m) = \mathcal{B}(P_n \times ML_m) \cup \mathcal{B}_u \cup \mathcal{B}_v \cup \mathcal{B}^u \cup \mathcal{B}^v,$$

where $\mathcal{B}(P_n \times ML_m)$ is the basis of $\mathcal{C}(P_n \times ML_m)$ that constructed in Theorem 3.2 and the other sets are defined as follows:

$$\begin{aligned} \mathcal{B}_u &= \{(r, u_j)(r+1, u_j) \cdots (t, u_j)(r, u_j) : 1 \leq j \leq m-1\} \\ \mathcal{B}_v &= \{(r, v_j)(r+1, v_j) \cdots (t, v_j)(r, v_j) : 1 \leq j \leq m\} \\ \mathcal{B}^u &= \{(1, u_j)(2, u_j) \cdots (r, u_j)(t, u_j) \cdots (n, u_j)(1, u_j) : 1 \leq j \leq m-1\} \\ \mathcal{B}^v &= \{(1, v_j)(2, v_j) \cdots (r, v_j)(t, v_j) \cdots (n, v_j)(1, v_j) : 1 \leq j \leq m-1\}. \end{aligned}$$

All the cycles of $\mathcal{B}_u \cup \mathcal{B}_v$ are edge-pairwise disjoint, so they are linearly independent. Moreover, every cycle in $\mathcal{B}_u \cup \mathcal{B}_v$ contains an edge from the set \mathcal{A}_1 that does not occur in any other cycle of $\mathcal{B}(P_n \times ML_m)$. Thus, $\mathcal{B}(P_n \times ML_m) \cup \mathcal{B}_u \cup \mathcal{B}_v$ is linearly independent. Similarly, all the cycles of $\mathcal{B}^u \cup \mathcal{B}^v$ are edge-pairwise disjoint, so they are linearly independent. Moreover, every cycle in $\mathcal{B}^u \cup \mathcal{B}^v$ contains an edge from the set \mathcal{A}_2 that does not occur in any other cycle of $\mathcal{B}(P_n \times ML_m) \cup \mathcal{B}_u \cup \mathcal{B}_v$. Therefore, $\mathcal{B}(\theta_n \times ML_m)$ is linearly independent. And since $|\mathcal{B}(\theta_n \times ML_m)| = \dim \mathcal{C}(\theta_n \times ML_m)$, the set $\mathcal{B}(\theta_n \times ML_m)$ is a basis of $\mathcal{C}(\theta_n \times ML_m)$. It is easy to verify that $\mathcal{B}(\theta_n \times ML_m)$ is a 4-fold basis of $\mathcal{B}(\theta_n \times ML_m)$. Hence, $b(\theta_n \times ML_m) \leq 4$. The proof is completed. \square

We turn our attention to investigate the basis number of the cartesian product of a path and a net. We denote by $N_{m,s}$ a net that has ms vertices and $2ms - m - s$ edges. In fact $N_{m,s}$ is the cartesian product of the paths P_m and P_s . And so we consider the vertex and the edge sets of $N_{m,s}$ given, respectively, as follows:

$$V(N_{m,s}) = \{(j, k) : 1 \leq j \leq m, 1 \leq k \leq s\}$$

$$\begin{aligned} E(N_{m,s}) &= \{(j, k)(j, k+1) : 1 \leq j \leq m, 1 \leq k \leq s-1\} \cup \\ &\quad \{(j, k)(j+1, k) : 1 \leq j \leq m-1, 1 \leq k \leq s\} \end{aligned}$$

These notations allow us to consider the cartesian product of the cycle $P_n = 123 \cdots n1$ and the net $N_{m,s}$ as a graph with vertex set

$$V(P_n \times N_{m,s}) = \{(i, j, k) : 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq s\}.$$

It is clear that $|V(P_n \times N_{m,s})| = nms$, $|E(P_n \times N_{m,s})| = 3nms - nm - ns - ms$ and $\dim \mathcal{C}(P_n \times N_{m,s}) = 2nms - nm - ns - ms + 1$. Following Ali and Marougi Theorem we have $b(P_n \times N_{m,s}) \leq 4$. In the following result we prove that the basis number of $P_n \times N_{m,s}$ is exactly 3.

Theorem 3.4. *For each $n, m, s \geq 3$, we have $b(P_n \times N_{m,s}) = 3$.*

Proof. Since $P_n \times N_{m,s}$ is nonplanar by Theorem 2.3, MacLane's Theorem implies that $b(P_n \times N_{m,s}) \geq 3$. To prove that $b(P_n \times N_{m,s}) = 3$, we have to exhibit a 3-fold basis for $\mathcal{C}(P_n \times N_{m,s})$. We consider $P_n \times N_{m,s}$ as a graph obtained from the disjoint union of the subgraphs $P_n \times P_{ms}$ and $P_n \times \overline{P_{ms}}$, where $\overline{P_{ms}}$ is the complement of P_{ms} in $N_{m,s}$ and P_{ms} is a path passing through all the vertices of $N_{m,s}$ in a zig-zag way as follows:

$$\begin{aligned} P_{ms} = & (1, 1)(1, 2) \cdots (1, m)(2, m)(2, m-1) \cdots (2, 1)(3, 1)(3, 2) \\ & \cdots (3, m)(4, m)(4, m-1) \cdots (m, s). \end{aligned}$$

We define $\mathcal{B}(P_n \times N_{m,s}) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, where $\mathcal{B}_1 = \mathcal{B}(P_n \times P_{ms})$ which consists of all the 4-cycles in $P_n \times P_{ms}$, \mathcal{B}_2 is defined in the following way: For each odd number j with $1 \leq j < m$, we define

$$\mathcal{B}_{oj} = \left\{ (i, j, k)(i, j+1, k)(i+1, j+1, k)(i+1, j, k)(i, j, k) : \begin{array}{l} 1 \leq i \leq n-1, \\ 1 \leq k \leq s-1 \end{array} \right\}$$

and for each even number j , with $1 \leq j < m$, we define

$$\mathcal{B}_{ej} = \left\{ (i, j, k)(i, j, k+1)(i, j+1, k+1)(i, j+1, k)(i, j, k) : \begin{array}{l} 2 \leq i \leq n, \\ 1 \leq k \leq s-1 \end{array} \right\},$$

thus, we define \mathcal{B}_2 by the set

$$\mathcal{B}_2 = \left(\bigcup_{j=1}^{n-1} \bigcup_{\text{and } j \text{ is odd}} \mathcal{B}_{oj} \right) \cup \left(\bigcup_{j=2}^n \bigcup_{j \text{ is even}} \mathcal{B}_{ej} \right)$$

and

$$\mathcal{B}_3 = \{(1, j, k)(1, j, k+1)(1, j+1, k+1)(1, j+1, k)(1, j, k) \mid 1 \leq j < m, 1 \leq k < s\}.$$

Clearly that the cycles of \mathcal{B}_3 are obtained from taking the bounds of all finite faces in the planar graph $1 \times N_{m,s}$. Thus, \mathcal{B}_3 is linearly independent. Now, any linear combination of cycles of \mathcal{B}_3 contains an edge of $1 \times \overline{P_{ms}}$ which is not in any cycle of \mathcal{B}_1 . Hence, $\mathcal{B}_1 \cup \mathcal{B}_3$ is linearly independent. For each odd j , $1 \leq j < m$, and k with

$1 \leq k < s$, the union of all the cycles obtained by varying i from 1 to n gives a ladder of the form $P_n \times \{(j, k)(j + 1, k)\} = L_{j,k}$. It is clear that all the cycles forming the ladder $L_{j,k}$ are linearly independent. If we vary j and k we get an edge-disjoint union set of ladders, so $\left(\bigcup_{j=1}^{n-1} \text{and } i \text{ is odd } \mathcal{B}_{oj}\right)$ is a linearly independent set. Also, using similar arguments we can verify that $\left(\bigcup_{j=2}^n \text{and } j \text{ is even } \mathcal{B}_{ej}\right)$ is linearly independent. Every linear combination of cycles of $\left(\bigcup_{j=1}^{n-1} \text{and } j \text{ is odd } \mathcal{B}_{oj}\right)$ contains an edge of the form $(i, j, k)(i + 1, j, k)$ for some $i \geq 2$, $k \leq s - 1$ and j is odd which does not occur in any cycle of $\mathcal{B}_1 \cup \mathcal{B}_3$. Thus, $\mathcal{B}_1 \cup \mathcal{B}_3 \cup \left(\bigcup_{j=1}^{n-1} \text{and } j \text{ is odd } \mathcal{B}_{oj}\right)$ is linearly independent. Similarly each linear combination of cycles of $\left(\bigcup_{j=2}^n \text{and } j \text{ is even } \mathcal{B}_{ej}\right)$ contains an edge of the form $(i, j, k)(i + 1, j, k)$ for some $i \geq 2$, $k \geq 2$ and j is even which is not in any cycle of $\mathcal{B}_1 \cup \mathcal{B}_3 \cup \left(\bigcup_{j=1}^{n-1} \text{and } j \text{ is odd } \mathcal{B}_{oj}\right)$. Thus, $\mathcal{B}(P_n \times N_{m,s})$ is linearly independent. Moreover,

$$\begin{aligned} |\mathcal{B}(P_n \times N_{m,s})| &= |\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{B}_3| = (nms - n + 1) + (n - 1)(m - 1)(s - 1) \\ &\quad + (s - 1)(m - 1) \\ &= 2nms - nm - ns - ms + 1 = \dim \mathcal{C}(P_n \times N_{m,s}). \end{aligned}$$

Hence, $\mathcal{B}(P_n \times N_{m,s})$ is a basis of $\mathcal{C}(P_n \times N_{m,s})$. One can easily verify that $\mathcal{B}(P_n \times N_{m,s})$ is a 3-fold basis. The proof is complete. \square

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