Applications of Block Pulse Response Circulant Matrix and its Singular Value Decomposition to MIMO Control and Identification

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Abstract: Properties and potential applications of the block pulse response circulant matrix (PRCM) and its singular value decomposition (SVD) are investigated in relation to MIMO control and identification. The SVD of the PRCM is found to provide complete directional as well as frequency decomposition of a MIMO system in a real matrix form. Three examples were considered: design of MIMO FIR controller, design of robust reduced-order model predictive controller, and input design for MIMO identification. The examples manifested the effectiveness and usefulness of the PRCM in the design of MIMO control and identification. irculant matrix, SVD, MIMO control, identification.

Keywords: Circulant matrix, identification, MIMO control, SVD.

1. INTRODUCTION

In recent years, the state space model-based approaches have dominated the controller design techniques. The transfer function-based methods are still important, however, in that the transfer function represents the complete frequency characteristics of a linear system, enabling us to view and analyze the system from a different angle. Despite its advantage, usefulness of the frequency domain approach is confined to SISO problems in many cases and is not easily extended to MIMO problems. It is primarily because a MIMO transfer function is represented by a matrix of complex functions and the techniques to analyze and interpret complex matrices are not as convenient as those for real matrices. Additionally, the transfer function itself does not provide the directional information although both the frequency and directional characteristics are inherent in MIMO systems.

As an alternative representation of the MIMO transfer function that can overcome the problems mentioned above, the block pulse response circulant matrix (PRCM) [1,2] and its properties are introduced and investigated. The PRCM is a matrix that maps an input sequence to an output sequence, both of which are periodic over an interval. It is composed of the finite impulse response (FIR) coefficients and has one-to-one correspondence to an FIR model. The PRCM has been used in the filter design in signal processing [3,4]. The related concepts have also been introduced to controller design [5,6]. The applications have been restricted to SISO problems and MIMO extensions have not yet been made as far as the authors have surveyed. This paper shows that the singular value decomposition (SVD) of the block PRCM gives the directional and frequency decomposition of a MIMO system in a real matrix form, and this property can be utilized in control and identification of MIMO systems. As potential applications of block PRCM and its SVD, three design problems are considered: MIMO feedback control, reduced-order robust model predictive control (MPC), and input excitation signal for MIMO identification. Effectiveness of the PRCM is demonstrated through numerical examples for the respective applications.

In the subsequent sections, R,C, and the superscript * will be used to represent the spaces of real and complex numbers and the conjugate transpose of a complex matrix, respectively.

2. PULSE RESPONSE CIRCULANT MATRIX

2.1. Definition and basic properties

Consider an n_u-input/n_y-output asymptotically stable MIMO discrete-time system described by an impulse response model

$$y(z^{-1}) = \hat{H}(z^{-1})u(z^{-1}), \tag{1}$$

where

$$\hat{H}(z^{-1}) = \sum_{k=0}^{\infty} h_k z^{-k},\ h_k \in R^{n_y \times n_u}. \tag{2}$$

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Let the SVD of $\hat{H}(e^{-j\omega})$ be

$$\hat{H}(e^{-j\omega}) = \hat{W}(\omega)\hat{D}(\omega)\hat{V}^*(\omega) \tag{3}$$

with $\hat{W}(\omega) \in C^{n_y \times n_y}, \hat{V}(\omega) \in C^{n_u \times n_u}, \hat{D}(\omega) \in R^{n_y \times n_u}$. The singular value matrix $\hat{D}(\omega)$ has nonnegative real elements in the principal diagonal, which are the directional gains of the system at frequency ω. The input and output singular matrices, $\hat{V}(\omega)$ and $\hat{W}(\omega)$, are unitary. The above SVD provides the directional decomposition of a frequency response model, but the analytic SVD of $\hat{H}(e^{-j\omega})$ is hard to obtain. What we can do best is the numerical decomposition at each ω.

For asymptotically stable $\hat{H}$, (2) can be approximated by the following finite impulse response (FIR) model for some N:

$$H(z^{-1}) = \sum_{k=0}^{N-1} h_k z^{-k}. \tag{4}$$

Associated with (4), we consider the following block PRCM $\mathbf{H}$ that represents the map between a sequence of N-periodic input and a sequence of N-periodic output:

$$\mathbf{Y} = \mathbf{HU}, \tag{5}$$

where

$$\mathbf{Y} \triangleq \left[y(k)^T \quad \cdots \quad y(k+N-1)^T \right]^T,$$
$$\mathbf{U} \triangleq \left[u(k)^T \quad \cdots \quad u(k+N-1)^T \right]^T, \tag{6}$$
$$\mathbf{H} \triangleq \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_1 \\ h_1 & h_0 & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & \cdots & h_0 \end{bmatrix} \in R^{n_y N \times n_u N}.$$

In the followings, MIMO block PRCM will be simply called PRCM wherever there is no confusion.

Consider the discrete Fourier transform (DFT) matrix:

$$F \triangleq \frac{1}{\sqrt{N}} \begin{bmatrix} d_0^0 & d_0^1 & \cdots & d_0^{N-1} \\ d_1^0 & d_1^1 & \cdots & d_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N-1}^0 & d_{N-1}^1 & \cdots & d_{N-1}^{N-1} \end{bmatrix} \in C^{mN \times mN}, \tag{7}$$

where $d_k \triangleq e^{-j\omega_k} I_m$; $\omega_k \triangleq \frac{2\pi k}{N}$; I_m denotes the $m \times m$ identity matrix. The DFT matrix is symmetric and unitary, i.e., $F^T = F$ and $F^*F = FF^* = I$.

Then the following holds for the PRCM:

Lemma 1: For a SISO PRCM, the eigenvalues and the normalized eigenvectors are $H(e^{-j\omega_k})$ and the column vectors of F^* for $k = 0, 1, \cdots, N-1$, respectively. For a block PRCM, the same holds block-wise.

The lemma is well known for the SISO case [1,2]. Proof for the MIMO case is given in the appendix.

According to the lemma, $\mathbf{H}$ can be block-diagonalized by the similarity transform with F such that

$$Y = HU, \tag{8}$$

where

$$Y \triangleq F\mathbf{Y}, U \triangleq F\mathbf{U}, H \triangleq F\mathbf{H}F^* = \mathrm{bd}\left[H(e^{-j\omega_k}) \right], \tag{9}$$

and 'bd' denotes the block-diagonal matrix. It can be seen that $\mathbf{H}$ corresponds to the FIR model H at the frequencies ω_k, $k = 0, \cdots, N-1$.

The following is essential in the analysis and synthesis of a MIMO controller using the PRCM:

Lemma 2: Any series, parallel, and/or feedback interconnection of PRCM's is a PRCM. Conversely, if a series, parallel, and/or feedback interconnection of some matrix A and PRCM's is a PRCM, A is also a PRCM.

Proof is given in the appendix.

2.2. SVD of PRCM

Let the SVD of PRCM $\mathbf{H}$ be represented as

$$\mathbf{H} = \mathbf{WDV}^T, \tag{10}$$

where $\mathbf{W} \in R^{n_y N \times n_y N}$, $\mathbf{V} \in R^{n_u N \times n_u N}$, and $\mathbf{D} \in R^{n_y N \times n_u N}$ denote orthogonal output and input singular matrices, and the singular value matrix, respectively.

SISO Case: Let $n_u = n_y = 1$. To investigate the properties of the matrices in (10), we take the DFT on both sides of (10).

$$H = (F\mathbf{W})\mathbf{D}(F\mathbf{V})^* \tag{11}$$

Both $F\mathbf{W}$ and $F\mathbf{V}$ are unitary and $\mathbf{D}$ is real and diagonal, and (17) represents the SVD of H. From the fact that H is diagonal with complex elements, $\mathbf{D} = \mathrm{diag}\left[|H(e^{-j\omega_k})| \right]$, i.e., the diagonal elements of $\mathbf{D}$ are the amplitude ratios of the FIR model at each ω_k rearranged in order of descending magnitude according to the definition of the SVD. More details on the SVD are summarized in the following lemma:

Lemma 3: Consider the SVD of a SISO PRCM in (10). It is assumed that the SVD is rearranged in order of increasing ω_k for notational simplicity. Let $\mathbf{w}_k$

and $\mathbf{v}_k$ be the column vectors of $\mathbf{W}$ and $\mathbf{V}$, respectively, corresponding to $|H(e^{-j\omega_k})|$. Then

$$\mathbf{D} = \text{diag}\left[|H(e^{-j\omega_k})|\right],\ k = 0,1,\cdots N-1,$$

$$\mathbf{w}_k = \sqrt{\frac{2}{N}}\begin{bmatrix} \cos(\omega_k \cdot 0 + \phi_{1,k}) \\ \vdots \\ \cos(\omega_k (N-1) + \phi_{1,k}) \end{bmatrix},$$

$$\mathbf{w}_{N-k} = \sqrt{\frac{2}{N}}\begin{bmatrix} \sin(\omega_k \cdot 0 + \phi_{1,k}) \\ \vdots \\ \sin(\omega_k (N-1) + \phi_{1,k}) \end{bmatrix}, \quad (12)$$

$$\mathbf{v}_k = \sqrt{\frac{2}{N}}\begin{bmatrix} \cos(\omega_k \cdot 0 + \phi_{2,k}) \\ \vdots \\ \cos(\omega_k (N-1) + \phi_{2,k}) \end{bmatrix},$$

$$\mathbf{v}_{N-k} = \sqrt{\frac{2}{N}}\begin{bmatrix} \sin(\omega_k \cdot 0 + \phi_{2,k}) \\ \vdots \\ \sin(\omega_k (N-1) + \phi_{2,k}) \end{bmatrix},$$

where $\phi_k \triangleq \arg H(e^{-j\omega_k}) = \phi_{1,k} - \phi_{2,k}$ represents the phase angle. The SVD is not unique since $\phi_{1,k}$ and $\phi_{2,k}$ can be arbitrary within $\phi_k = \phi_{1,k} - \phi_{2,k}$.

Proof is given in the appendix. The above tells that the SVD of a SISO PRCM provides the complete information on the amplitude ratio and phase angle in a real matrix form.

MIMO Case: The result of the SISO case can be extended to the MIMO case with additional complexity. $H(e^{-j\omega_k})$ for a MIMO system is an $n_y \times n_u$ complex matrix and H is a block-diagonal matrix. Let the SVD of $H(e^{-j\omega_k})$ be

$$H(e^{-j\omega_k}) = W(\omega_k)D(\omega_k)V(\omega_k)^*, \quad (13)$$

where $W(\omega_k) \in C^{n_y \times n_y}, V(\omega_k) \in C^{n_u \times n_u}$, and $D(\omega_k) \in R^{n_y \times n_u}$. The above represents the directional decomposition of $H(e^{-j\omega_k})$. From (9), (10), and (13), we have

$$\begin{aligned}\mathbf{H} &= \mathbf{W}\mathbf{D}\mathbf{V}^T = F^* H F \\ &= \underbrace{F^*\text{bd}[W(\omega_k)]}_{=\mathbf{W}}\underbrace{\text{bd}[D(\omega_k)]}_{=\mathbf{D}}\underbrace{\text{bd}[V(\omega_k)^*]F}_{=\mathbf{V}^\mathbf{T}}. \end{aligned} \quad (14)$$

In the above, *mbD* consists of the directionally decomposed frequency gains. Expansion of (14) and rearrangement leads to the following lemma:

Lemma 4: Consider the SVD of a MIMO PRCM as in (10). The SVD is assumed to be rearranged in order of increasing ω_k. Let $\mathbf{w}_k$ and $\mathbf{v}_k$ be the block column vectors of $\mathbf{W}$ and $\mathbf{V}$, respectively, and $\mathbf{d}_k$ be the corresponding k^{th} diagonal block of $\mathbf{D}$. Then using the SVD in (13), we have

$$\begin{aligned}\mathbf{D} &= \text{bd}[D(\omega_k)]\mathbf{w}_k\mathbf{d}_k\mathbf{v}_k^T + \mathbf{w}_{N-k}\mathbf{d}_{N-k}\mathbf{v}_{N-k}^T \\ &= \frac{2}{N}\text{Re}\begin{bmatrix} e^{j\omega_k \cdot 0}W(\omega_k) \\ \vdots \\ e^{j\omega_k (N-1)}W(\omega_k) \end{bmatrix} \times \\ &\quad D(\omega_k)\left[e^{-j\omega_k \cdot 0}V(\omega_k)^* \cdots e^{-j\omega_k (N-1)}V(\omega_k)^*\right].\end{aligned} \quad (15)$$

Proof is given in the appendix. In both the SISO and MIMO cases, column vectors of the input and output singular matrices are sampled sinusoids of different frequencies. Therefore, the frequency associated with each singular value can be identified by inspection.

3. APPLICATIONS

3.1. Design of FIR controller

Consider a MIMO discrete-time stable process

$$y(z) = G(z)u(z). \quad (16)$$

We want to design a feedback regulator

$$u(z) = K(z)(r(z) - y(z)), \quad (17)$$

such that the closed-loop transfer function is $G_{cl}(z)$, i.e.,

$$\begin{aligned} y(z) &= (I + G(z)K(z))^{-1}G(z)K(z)r(z) \\ &= G_{cl}(z)r(z). \end{aligned} \quad (18)$$

It is assumed that $G_{cl}(z)$ is given so that $K(z)$ is asymptotically stable.

Let $\mathbf{G}$ and $\mathbf{G}_{cl}$ be the PRCM representations of $G(z)$ and $G_{cl}(z)$, respectively, and $\mathbf{K}$ be the matrix representation of yet undetermined $K(z)$. The problem in (18) can be converted to

$$(\mathbf{I} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}\mathbf{K} = \mathbf{G}_{cl}. \quad (19)$$

Under the assumption that $\mathbf{I} - \mathbf{G}_{cl}$ is invertible, $\mathbf{K}$ is obtained as

$$\mathbf{K} = \mathbf{G}^+\mathbf{G}_{cl}(\mathbf{I} - \mathbf{G}_{cl})^{-1}, \quad (20)$$

where $^+$ denotes a pseudo-inverse. From Lemma 2, $\mathbf{K}$ is a PRCM, and FIR controller can be constructed with the first column block of $\mathbf{K}$.

3.2. Design of robust MPC

When a system is identified experimentally, the high gain modes are easily excited and precisely identified whereas the low gain modes are not. Consequently, large error is usually incurred in the estimates of the low gain modes, and a model-based controller may result in poor performance or instability of the closed loop. One way to avoid this problem is to block the signals associated with the low gain modes. For this, it is necessary to know the directions and frequency range over which the process gains are significantly small.

We apply the above idea to the design of robust MPC. Let $\Delta\mathbf{U}(t)$ denote the future input moves determined at t. In the standard MPC method, the dimension of $\Delta\mathbf{U}(t)$ is high and the required computation is heavy [7,8]. We address the problem of computation reduction together with robustness enhancement against model uncertainty, and the following input blocking is proposed on the basis of the SVD of the PRCM.

Let $\mathbf{H}$ be the PRCM of the process model. Assume that the SVD of $\mathbf{H}$ is given as

$$\mathbf{H}=\begin{bmatrix}\mathbf{W}_1 & \mathbf{W}_2\end{bmatrix}\begin{bmatrix}\mathbf{D}_1 & 0\\ 0 & \mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\end{bmatrix}^T, \mathbf{D}_1 \gg \mathbf{D}_2. \quad (21)$$

In the above, the partition associated with $\mathbf{D}_2$ may include large model error. If the control input is restricted to the span of $\mathbf{V}_1$, the low gain part will not be excited and the potential instability of the closed-loop can be avoided. The input movement is defined as

$$\Delta\mathbf{U}(t)=\mathbf{V}_1\mathbf{c}(t). \quad (22)$$

MPC determines $\mathbf{c}(t)$, whose dimension is smaller than that of $\Delta\mathbf{U}(t)$.

Example 1: The process and the nominal model are zero-order hold equivalents of the following transfer function matrices with sampling interval of 2:

$$\mathbf{G}^{pro}(s)=\begin{bmatrix}\dfrac{1.7}{(10s+1)(10s^2+s+1)} & \dfrac{2.3}{30s+1}\\ \dfrac{1.3}{20s+1} & \dfrac{2.8}{(10s+1)(5s^2+s+1)}\end{bmatrix},$$

$$\mathbf{G}^{nom}(s)=\begin{bmatrix}\dfrac{1.5}{10s+1} & \dfrac{2.2}{30s+1}\\ \dfrac{1.2}{20s+1} & \dfrac{2.6}{10s+1}\end{bmatrix}. \quad (23)$$

Both the prediction and control horizons were chosen to be 90, and both input and output weighting matrices were given as $\mathbf{I}$. From Fig. 1, it can be seen that significant error exists in the nominal model beyond $\omega = 0.1$ rad/sec. Because of this, regular MPC

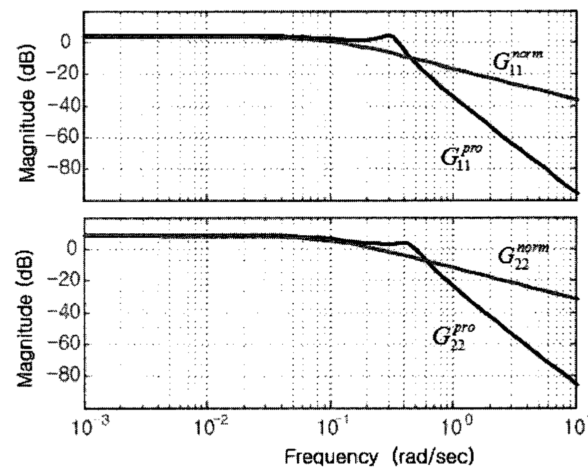
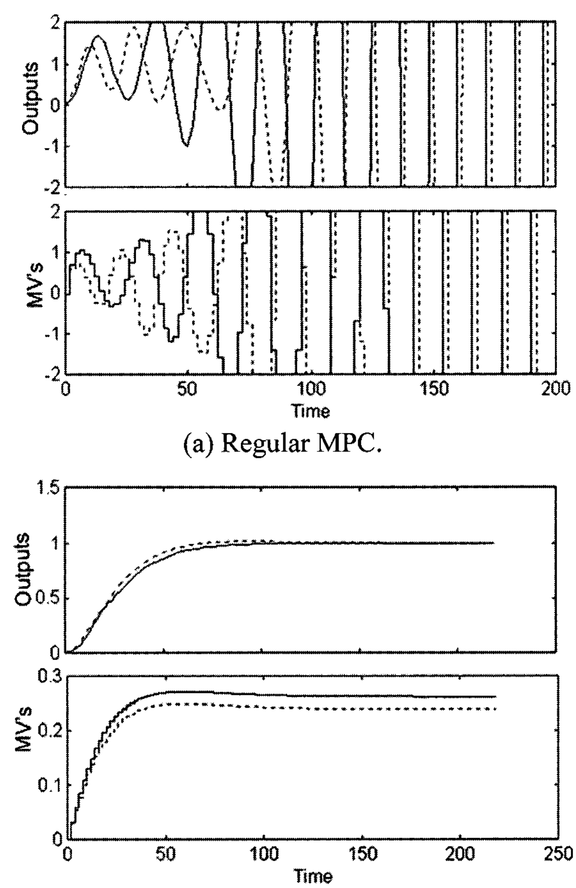


Fig. 1. Magnitude ratios of diagonal elements of process and nominal model transfer function matrices.



(a) Regular MPC.

(b) Reduced-order MPC.

Fig. 2. Responses of regular and reduced-order MPC (solid line: y_1, dashed line: y_2).

renders the closed-loop unstable as shown in Fig. 2(a). To improve the robustness, the SVD was conducted on the 180×180 PRCM formed with ninety 2×2 pulse response coefficient matrices of the nominal model, and the first 12 column vectors of $\mathbf{V}$ was selected as $\mathbf{V}_1$, which roughly covers the frequency up to 0.2 rad/sec. The performance of the robust reduced-order MPC by input blocking is shown in Fig. 2(b). It can be seen that the low gain modes are effectively suppressed and the instability is removed leading to smoothly converging responses to the set point changes.

3.3. Input design for MIMO identification

The input design to excite all latent system modes with prescribed S/N ratios has been a constant research subject in system identification [9,10]. We propose a novel approach to this, where the SVD of the PRCM is repeatedly applied to the estimated model.

Consider the following system with a PRCM representation:

$$\mathbf{Y} = \mathbf{HU} + \mathbf{E} = \mathbf{WDV}^T\mathbf{U} + \mathbf{E}, \tag{24}$$

where $\mathbf{E}$ is assumed to be a zero-mean output noise with uniform variance over all modes.

In order to estimate all the principal gains with uniform accuracy, it is necessary to design the input so that the S/N ratio of the output for each principal direction is uniformly large. This can be achieved by the following input signal:

$$\mathbf{U} = \sum_{k=1}^{n_u N} \alpha \frac{\mathbf{v}_k}{d_k} \gg \mathbf{W}^T\mathbf{E} \tag{25}$$

for some large $\alpha > 0$ where d_k's and $\mathbf{v}_k$'s denote the singular values and the input singular vectors, respectively. In reality, accurate $\mathbf{H}$ is not available in advance, hence $\mathbf{U}$ may not be designed as intended. Instead, we can take the following iterative steps for sequential improvement of the input signal:

Step 1: Apply a uniformly distributed random signal to all inputs and identify the system. Let $n = 1$.

Step 2: Let the PRCM of the estimated model be $\hat{\mathbf{H}}_n$. Take the SVD of $\hat{\mathbf{H}}_n$.

Step 3: Construct the input signal as in (25) using the SVD of $\hat{\mathbf{H}}_n$ for the next identification experiment.

Step 4: Repeat steps 2 and 3 with increasing n until the estimates of d_k's converge.

In the first run, only the high gain modes will be identified accurately. The input directions for the low gains will be estimated inaccurately as a result, but the span of the directions will be correct because it is given to be orthogonal to the high gain input directions. The low gain modes can be appropriately excited and be identified in the next run. Repeated application of (25) with $\mathbf{v}_k$ and d_k, which are replaced by the estimates from the previous run, will progressively achieve balanced excitation of all modes.

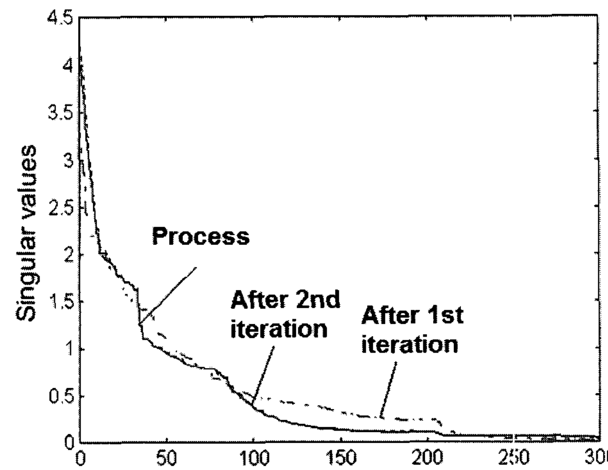


Fig. 3. Singular values of PRCM's.

α can be given differently with k if it is desired to concentrate the input energy over a certain frequency range and/or input directions. Koung and MacGregor [11] have proposed a similar idea to the above, but the method was limited to the identification of the steady state gain.

Example 2: The process to identify is the zero-order hold equivalent of $\mathbf{G}^{pro}(s)$ in (23) with sampling period of 2. It is assumed that independent zero-mean Gaussian noises with variance of 0.5 are imposed on each output. In the first experiment, independent PRBS's with amplitude 1.5 was applied to each input for 1,500 sampling times and a model was identified using a subspace technique called N4SID [12]. In the subsequent experiments, the proposed method was applied with $\alpha = 2$ using PRCM's with $N = 150$.

In Fig. 3, the singular values of the PRCMs for the true process and the estimated model are compared. One can see the singular value estimates converge to the true values in just two runs.

4. CONCLUSIONS

It has been shown that the block PRCM is a useful alternative representation of a MIMO dynamic system and that its SVD can play as a powerful instrument in the design of MIMO control and identification. The PRCM's can replace the asymptotically stable transfer function matrices with one-to-one correspondence. This property enables us to construct a MIMO FIR controller through a simple matrix algebra. It was also proved that the SVD of a PRCM provides the complete directional and frequency decomposition of the PRCM in a real matrix form. Methods to design a robust controller and input signals for MIMO identification were considered as exemplary applications of this property.

APPENDIX A

In the subsequent proofs, the ordinal numbers are counted from zero.

Proof of Lemma 1: Let f_k be the k^{th} column block of F.

$$f_k = \frac{1}{\sqrt{N}}\begin{bmatrix} I & e^{-j\omega_k}I & \cdots & e^{-j\omega_k(N-1)}I \end{bmatrix}^T \quad \text{(A.1)}$$

What we have to show is that

$$\mathbf{H}\bar{f}_k = \bar{f}_k H(e^{-j\omega_k}), \quad \text{(A.2)}$$

where $^-$ denotes the complex conjugate. Since $e^{j\omega_k m} = e^{-j\omega_k(N-m)}$, the p^{th} block of $\mathbf{H}\bar{f}_k$ is obtained as

$$\begin{aligned} \mathbf{e}_p^T\mathbf{H}\bar{f}_k &= \frac{1}{\sqrt{N}}\Big(h_p + \cdots h_0 e^{j\omega_k p} \\ &\quad + h_{N-1}e^{j\omega_k(p+1)} + \cdots h_{p+1}e^{j\omega_k(N-1)}\Big) \\ &= e^{j\omega_k p}\frac{1}{\sqrt{N}}\Big(h_0 + h_1 e^{-j\omega_k}\cdots \\ &\quad + h_{N-1}e^{-j\omega_k(N-1)}\Big) \\ &= \left(\frac{1}{\sqrt{N}}e^{-j\omega_k p}I\right)\left(\sum_{m=0}^{N-1} h_i e^{-j\omega_k m}\right) \\ &= e^{-j\omega_k p}H(e^{-j\omega_k}), \end{aligned} \quad \text{(A.3)}$$

where $\mathbf{e}_p^T$ denotes a matrix that picks the p^{th} block. The above tells that $\bar{f}_k$ is the k^{th} block eigenvector of $\mathbf{H}$ and the corresponding block eigenvalue is $H(e^{-j\omega_k})$. □

Proof of Lemma 2: We prove the lemma using an exemplary matrix equation $\mathbf{Q} = (\mathbf{I}+\mathbf{HK})^{-1}\mathbf{HK}$. First, let $\mathbf{H}$ and $\mathbf{K}$ be PRCM's. DFT of the matrix equation yields

$$\begin{aligned} F\mathbf{Q}F^* &= F(\mathbf{I}+\mathbf{HK})^{-1}\mathbf{HK}F^* \\ &= (\mathbf{I}+F\mathbf{H}F^*F\mathbf{K}F^*)^{-1}F\mathbf{H}F^*F\mathbf{K}F^* \\ &= (\mathbf{I}+HK)^{-1}HK. \end{aligned} \quad \text{(A.4)}$$

Since H and K are block-diagonal, $Q = F\mathbf{Q}F^*$ is also block-diagonal, which implies that $\mathbf{Q}$ is a PRCM. Conversely, if $\mathbf{Q}$ and $\mathbf{H}$ are PRCM's, DFT shows that $F\mathbf{K}F^*$ is block-diagonal, which means $\mathbf{K}$ is a PRCM. □

Proof of Lemmas 3 and 4: Let f_k be the k^{th} column block of F. It holds that $f_k = f_{N-k}$ and $H(e^{-j\omega_k}) = H(e^{j\omega_{N-k}})$ for $k = 0, \cdots N-1$. Without loss of generality, N is assumed to be an odd number. From (9)-(11) and (13), we have

$$\begin{aligned} \mathbf{H} = \mathbf{WDV}^T &= F^*HF = \sum_{k=0}^{N-1}\bar{f}_k H(e^{-j\omega_k})f_k^T \\ &= H(1) + \sum_{k=1}^{(N-1)/2}\Big(\bar{f}_k H(e^{-j\omega_k})f_k^T \\ &\quad + \bar{f}_{N-k}H(e^{-j\omega_{N-k}})f_{N-k}^T\Big) \\ &= H(1) + \sum_{k=1}^{(N-1)/2}\Big(\bar{f}_k H(e^{-j\omega_k})f_k^T + f_k H(e^{j\omega_k})\bar{f}_k^T\Big) \\ &= H(1) + 2\sum_{k=1}^{(N-1)/2}\mathrm{Re}\Big(\bar{f}_k H(e^{-j\omega_k})f_k^T\Big) \\ &= H(1) + 2\sum_{k=1}^{(N-1)/2}\mathrm{Re}\Big(\bar{f}_k W(\omega_k)D(\omega_k)V(\omega_k)^* f_k^T\Big). \end{aligned} \quad \text{(A.5)}$$

For the SISO case, $H(e^{-j\omega_k}) = r_k e^{j\phi_k}$ and $D(\omega_k) = r_k$. $e^{j\phi_k}$ can be split between $W(\omega_k)$ and $V(\omega_k)^*$ with any ratio. Let

$$W(\omega_k) = e^{j\phi_{1,k}}, V(\omega_k)^* = e^{-j\phi_{2,k}} \quad \text{(A.6)}$$

with $\phi_k = \phi_{1,k} - \phi_{2,k}$. Then

$$\begin{aligned} &\mathrm{Re}\bar{f}_k W(\omega_k)D(\omega_k)V(\omega_k)^* f_k^T \\ &= \tfrac{r_k}{N}\mathrm{Re}\Big(\mathrm{col}\Big[e^{j(\omega_k n_r + \phi_{1,k})}\Big]\mathrm{row}\Big[e^{-j(\omega_k n_c + \phi_{2,k})}\Big]\Big) \\ &= \tfrac{r_k}{N}\Big[\cos((\omega_k n_r + \phi_{1,k}) - (\omega_k n_c + \phi_{2,k}))\Big] \\ &= \tfrac{r_k}{N}\Big[\cos(\omega_k n_r + \phi_{1,k})\cos(\omega_k n_c + \phi_{2,k}) \\ &\quad + \sin(\omega_k n_r + \phi_{1,k})\sin(\omega_k n_c + \phi_{2,k})\Big] \\ &= \tfrac{r_k}{N}\begin{bmatrix} \cos(\omega_k\cdot 0 + \phi_{1,k}) \\ \vdots \\ \cos(\omega_k(N-1) + \phi_{1,k}) \end{bmatrix}\times \\ &\quad \Big[\cos(\omega_k\cdot 0 + \phi_{2,k}) \ \cdots \ \cos(\omega_k(N-1) + \phi_{2,k})\Big] \\ &\quad + \tfrac{r_k}{N}\begin{bmatrix} \sin(\omega_k\cdot 0 + \phi_{1,k}) \\ \vdots \\ \sin(\omega_k(N-1) + \phi_{1,k}) \end{bmatrix}\times \\ &\quad \Big[\sin(\omega_k\cdot 0 + \phi_{2,k}) \ \cdots \ \sin(\omega_k(N-1) + \phi_{2,k})\Big], \end{aligned} \quad \text{(A.7)}$$

where $n_r, n_c = 0, 1, \cdots, N-1$ and $[\cdot]$ represents a matrix. If we define

$$\mathbf{W} = [\mathbf{w}_0 \cdots \mathbf{w}_{N-1}], \mathbf{V} = [\mathbf{v}_0 \cdots \mathbf{v}_{N-1}], \mathbf{D} = \mathrm{diag}[d_k] \quad \text{(A.8)}$$

then we have

$$\mathbf{H} = \sum_{k=0}^{N-1} d_k \mathbf{w}_k \mathbf{v}_k^T. \tag{A.9}$$

Comparing (A.9) with (A.5) after inserting (A.7) gives (12). For the MIMO case, we rearrange (A.5) using the following expression

$$\begin{aligned}&\mathrm{Re}\,\bar{f}_k W(\omega_k) D(\omega_k) V(\omega_k)^* f_k^T \\ &= \frac{1}{N}\mathrm{Re}\begin{bmatrix} e^{j\omega_k \cdot 0} W(\omega_k) \\ \vdots \\ e^{j\omega_k (N-1)} W(\omega_k) \end{bmatrix} D(\omega_k) \times \\ &\quad \left[e^{-j\omega_k \cdot 0} V(\omega_k)^* \cdots e^{-j\omega_k (N-1)} V(\omega_k)^* \right]\end{aligned} \tag{A.10}$$

and the fact that the k^{th} and the $N-k^{\text{th}}$ column blocks of $\mathbf{W}$ and $\mathbf{V}$ are associated with $\bar{f}_k W(\omega_k)$ and $\bar{f}_k V(\omega_k)$, respectively. This results in (15) and proves Lemma 4. □

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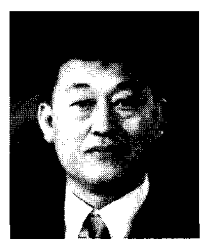

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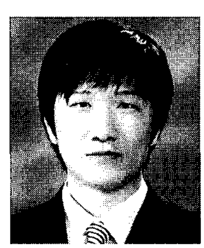

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