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조립 생산 시스템에서 최적 Base-Stock 수준

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The Optimal Base-Stock Level in Assembly lines

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In this study, we consider an assembly line operated under a base-stock policy. A product consists of two parts, and a finished product transfers to a warehouse in which demands are satisfied. Assume that demands arrive according to a Poisson process and processing times at each production line are exponentially distributed. Whenever a demand arrives, it is satisfied immediately from an inventory in the warehouse if available; otherwise, it is backlogged and satisfied later by the next product exiting from production lines. In either case, an arriving demand automatically triggers the production of a part at both production lines. These two parts will be assembled into a product that eventually transfers to the warehouse. We obtain a closed form formula of approximation for delay time or lead time distribution of a demand when a base- stock level is s. Moreover, it can be applied to the optimal base-stock level which minimizes the total inventory cost. Numerical examples are presented to show our optimal base-stock level's quality.

Keywords : Assembly-Production System, M/M/1 Queue, Base-Stock Policy, Optimization

1. Introduction

We consider an assembly line operated under a base-stock inventory policy shown in <Figure 1>, which is an simple example of a fork-join (see for reference [6]) production system with a warehouse at the end. A product consists of two parts. These parts are produced in each production line, and assembled into a product, and this finished product transfers to a warehouse. The incoming demands are satisfied at the warehouse. We assume that the inter-arrival times of demands are exponentially distributed, and i.i.d. (independent and identically distributed), and processing times at each production line are exponentially distributed.

Whenever a demand arrives, it is satisfied instantaneously from inventories at the warehouse if there is inventory; otherwise, it is backlogged and satisfied later by the next product that exits from the production lines. In either case, the arriving demand automatically triggers the production of a part at both production lines. These two parts are assembled into a product and this assembled product will move to the ware-

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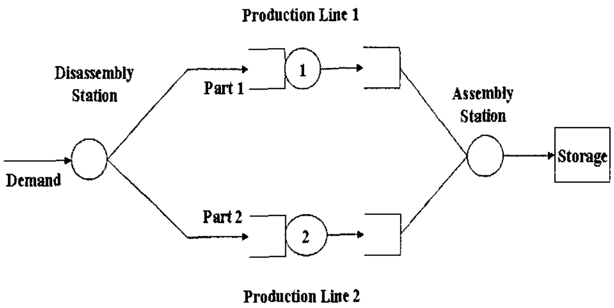
house. We also assume that the system is controlled by a base-stock policy with base-stock level s. This means that the inventory level at time 0 is s and the system make products until to reach the level s whenever demands occur. In addition, because of the base-stock policy assumption the inventory level never exceed s during the system's operation time.



<Figure 1> Assembly Production System

Although there has been extensive research on inventory management, only a few papers are related to our study. Initial work on stochastic multi-echelon inventory systems of discrete-time appeared in Clark and Scarf [1], and a base-stock policy was shown to minimize holding and back-order costs (see also Federgruen and Zipkin [2, 3]). Schmidt and Nahmias [8] extended their study to a two-stage assembly system. For uncapacitated multistage assembly systems, Rosling [7] identified an optimal policy, which is a base-stock policy in the absence of fixed order costs. For continuous review systems, Glasserman and Wang [4] studied the trade-offs between inventory levels and the delivery lead-time in a limiting sense.

The rest of this paper is organized as follows. In Section 2, we introduce performance measures of interest such as fill rate, back-order quantities and inventory level, and so on. A closed form formula for the optimal base-stock level and numerical examples for a Markovian system are given in Section 3 and 4. Section 5 includes concluding remarks.

2. Notations and Preliminaries

Throughout this paper the following notations are used.

- $IO(t)$: the amount of on-hand inventory at time t
- $B(t)$: the amount of backorder at time t
- $I(t)$: the inventory position at time t, which is $I(t) = IO(t) - B(t)$
- $Q^i(t)$: the number of parts waiting or in-process in production line i at time t
- $Q(t) = max_i\ Q^i(t)$
- W_n^i : the nth part sojourn times in production line i.
- W_n : the response time of the nth product in the system, which is $W_n = max\ _i\ W_n^i$
- A_n : the arriving time of the nth demand
- V_n^i : the nth parts processing time at production line i

We assume that the inter-arrival times of arriving demands, $U_n = A_n - A_{n-1}$, are independent and identically distributed, and independent of the processing time V_n^i. And, V_n^i for $n \geq 1$ and $i = 1, 2$ are independent of each other. In addition, we assume $EV_1^i < EU_1$ for each i for the system's stability.

Now, for the system shown in <Figure 1> we describe various interesting performance measures such as delay-time, back-order quantity, inventory level, and so forth as well as some related results.

When base-stock level s is zero, one can see that this assembly system becomes the same one as the two-node $M/M/1$ fork-join system. Through an evident connection with a two- node fork-join system, the following results can be easily obtained (see Ko and Serfozo [6]).

The part sojourn times at the nodes satisfy

$$(W_n^1, W_n^2) \xrightarrow{d} (W^1, W^2)$$

where

$$W^i =^d V_1^i + max\ _{l \geq 0} \sum_{k=1}^{l} (V_k^i - U_k).$$

In here, $\xrightarrow{d}$ means "converge in distribution" and $=^d$ means "equal in distribution."

Consequently,

$$W_n = max\ \{W_n^1, W_n^2\} \xrightarrow{d} W = \max\{W^1, W^2\},$$

where W is the equilibrium response time of a demand.

In the sense of service of quality(QoS), a delay is one of the most important measures in the analysis of system performance. A QoS can be measured by the fill rate, which is the fraction of demands that are met immediately. Let D_n be the delay experienced by the nth demand and D be a steady-state delay under the base-stock level s. Without backorder, therefore, the fill rate can be calculated by $P(D=0)$. More generally, a service level can be defined by $P(D \leq \beta) \geq \alpha$, where β is a critical (allowable) delay-time, and α is a target service level.

The following theorem shows the relationship between the equilibrium delay D and equilibrium response time W in the fork-join processing system, and it also can be applied to the general arrival time and production time.

Theorem 2.1 : *For the assembly production system defined above with base-stock inventory policy of level s,*

$$D_n \xrightarrow{d} D$$

where $$F_D(t) = \int_0^\infty F_W(t+u)\, dF_{A_s}(u).$$

Proof : since $A_n + W_n$ is the completion time (the time epoch of the nth demand's arrival time plus the nth assembled product's response time which is triggered by the nth demand), it follows that

$$D_{n+s} = (A_n + W_n - A_{n+s})^+ =^d (W_n - A_s)^+$$

where in the last term, A_s is independent of W_n. Then the assertion follows by letting $n \to \infty$ in the preceding expression.

In [6], we already established an accurate approximation for $F_W(t)$ in two-node $M/M/1$-type fork-join networks. Using their results, we can obtain the following result.

Approximation 2.2 : *If the demand process is a Poisson process with rate* λ, *and service times* V_n^i *for* $i=1, 2$, *are exponentially distributed with rate* μ_i, *then the following approximations are very accurate:*

$$F_D(t) \approx 1 - \rho_1^s e^{-\gamma_1 t} - (1 - \frac{\rho_1}{4})[\rho_2^s e^{-\gamma_2 t} - (\frac{\lambda}{\gamma_1+\gamma_2+\lambda})^s e^{-(\gamma_1+\gamma_2)t}]$$

where $t \geq 0$, $\gamma_i = \mu_i - \lambda$, *and* $\rho_i = \lambda/\mu_i$. *Also*

$$ED \approx \rho_1^s \frac{1}{\gamma_1} + (1 - \frac{\rho_1}{4})[\rho_2^s \frac{1}{\gamma_2} - (\frac{\lambda}{\gamma_1+\gamma_2+\lambda})^s \frac{1}{\gamma_1+\gamma_2}]$$

Justification From Theorem 2.1,

$$F_D(t) = \int_0^\infty F_W(t+u)\, d F_{A_s}(u) \approx \int_0^\infty \widetilde{F}_W(t+u) \lambda \frac{(\lambda u)^{s-1}}{(s-1)!} e^{-\lambda u} du$$

Here $\widetilde{F}_W(t)$ is the approximation for $F_W(t)$ introduced in [6], and A_s is the Erlang random variable with parameter λ and s since the demand process is Poisson process with rate λ. An evaluation of the integral yields the approximation of a delay distribution.

To show that accuracy of our approximation for $F_D(t)$ is as good as that of the approximation for $F_W(t)$, we use a sup norm distance between two distributions F and G defined as

$$d(F, G) \equiv sup_{\ t \geq 0} |F(t) - G(t)|$$

Using the sup norm, the approximation for $F_D(t)$ is better than for $F_W(t)$, since

$$|F_D(t) - \widetilde{F}_D(t)| \leq \int_0^\infty |F_W(t+u) - \widetilde{F}_W(t+u)| dF_{A_s}(u) \leq d(F_W, \widetilde{F}_W)$$

In other words, the approximation error for $F_D(t)$ is always smaller than or equal to that for $F_W(t)$, which are negligible (see [6]).

Let D, B, and IO be an equilibrium delay, the amount of backorders and an on-hand inventory position, respectively. Hence, because the assembly production system is an overtaking-free queue, the distributional Little's law (see Haji and Newell [5]) can be applied to obtain the following relationship of D and B. Here we let $N^*(t)$ denote the number of arrivals up to time t for the equilibrium process (where the time of the first inter-arrival time is distributed as the forward recurrence time of the arrival process). That is, $N^*(t)$ is the stationary version of the arrival process.

Lemma 2.3 : *(Distributional Little's Law for Delay and Backorder)*

$$B =^d N^*(D)$$

From this fact, one can compute the fill rate from the probability of $P(B=0)$. Next, we investigate more details for performance measures. First of all, from the definition of $Q(t)$, the so-called work-in process (WIP or pipeline) inventory at time t, we can describe the total number of products in the system at time t by $IO(t)+Q(t)$. It can be also represented as $B(t)+s$ because of the definition of the base stock policy. So, we have

$$IO(t)+Q(t)=B(t)+s.$$

Then, $I(t)$, the inventory position at time t, is written as $I(t)=s-Q(t)$ since $I(t)=IO(t)-B(t)$. Therefore, we can figure out the followings.

$$IO(t)=[I(t)]^{+} \text{ and } B(t)=[I(t)]^{-}$$

where $[x]^{+}\equiv max\ \{x,0\}$ and $[x]^{-}\equiv -min\ \{x,0\}$. Moreover, the on-hand inventory $IO(t)$ and the amount of backorders $B(t)$ can be written in terms of s and $Q(t)$ as follows,

$$IO(t)=[s-Q(t)]^{+} \text{ and } B(t)=[s-Q(t)]^{-}.$$

Then, taking $t\rightarrow\infty$ leads the equilibrium distributions of IO and B, and the probability distributions for them are computed as,

$$P(IO=k)=\begin{cases} P(Q\geq s) & if\ k=0 \\ P(Q=s-k) & if\ 1\leq k\leq s \\ 0 & otherwise \end{cases}$$
$$P(B=k)=\begin{cases} P(Q\leq s) & if\ k=0 \\ P(Q=s+k) & if\ k\geq 1 \end{cases}.$$

From the approximation for $P(Q\leq q)$ in [6], we are able to obtain good approximated distributions for B and IO in Markovian systems.

3. Optimal Base-Stock Level

In this section, we introduce an optimization problem as an application of our results. That determines the optimal base- stock levels s minimizing a expected total cost, which consists of an inventory cost and a backorder cost subject to a fill rate constraint. There are two types of inventory in the system : on-hand inventory (or finished goods) and WIP inventory. The independent property of WIP with respect to base-stock level s allows us to consider only on-hand inventory.

The average total cost per unit time can be defined as a function of s as follows,

$$C(s)=h\,IO+bB=h(s-Q)^{+}+b(s-Q)^{-}$$

where h is a holding cost for on-hand inventory per unit and b is a penalty cost for backorders per unit. When the distribution of Q is known, one can therefore obtain the distribution and mean of the costs as follows.

$$P(C(s)\leq c)=P(s-c/h\leq Q\leq s+c/h),$$

and

$$E[C(s)]=h(s-EQ)+(h+b)E[(s-Q)^{-}].$$

Proposition 3.1 : *The expected cost $E[C(s)]$ is minimized when the base-stock level s is*

$$s^{*}=min\ \{s|P(Q\geq s+1)\leq h/(h+b)\} \quad\cdots\cdots (1)$$

Proof since

$$E[(s-Q)^{-}]=\sum_{k>s}(k-s)P(Q=k),$$

it follows that

$$E[C(s+1)]-E[C(s)]=h-(h+b)P(Q\geq s+1).$$

It is clear that this is a nondecreasing function in s and $E[C(s)]$ is unimodal. Therefore, it attains its minimum at $min\ \{s|E[C(s+1)]\geq E[C(s)]\}$, which equals to (1).

4. Numerical Examples

An application of our optimal base-stock result requires knowledge of Q, the equilibrium distribution of the number of parts, in the fork-join network. We will consider a two-node Markovian system, in which the Poisson demand has rate λ, and exponential processing times have rates μ_1 and μ_2 ($\mu_1\leq\mu_2$).

In order to determine the optimal base-stock level (s^{*}), we use the approximation for Q in [6], which is

$$P(Q\geq s+1)\approx\rho_1^{s+1}+(1-\frac{\rho_1}{4})[\rho_2^{s+1}-(\frac{\lambda}{\gamma_1+\gamma_2+\lambda})^{s+1}].$$

Comparing the resulting s^{*} with its upper and lower bounds shows the accuracy of our results. These upper and lower bounds can be computed by the following probabilities of $P(Q\geq s+1)$.

$$P_L(Q\geq s+1)=\rho_1^{s+1},$$

<Table 1> $\rho_1 = 0.9$

μ_2	b	s^*	s_L	s_U
1.11	2	14	10	15
1.11	4	20	15	21
1.11	6	23	18	24
1.11	8	25	20	27
1.11	10	27	22	29
1.67	2	10	10	10
1.67	4	15	15	15
1.67	6	18	18	18
1.67	8	20	20	20
1.67	10	22	22	22
2.22	2	10	10	10
2.22	4	15	15	15
2.22	6	18	18	18
2.22	8	20	20	20
2.22	10	22	22	22

<Table 2> $\rho_1 = 0.5$

μ_2	b	s^*	s_L	s_U
2.00	2	2	2	1
2.00	4	3	3	2
2.00	6	3	3	2
2.00	8	3	4	3
2.00	10	4	4	3
3.00	2	1	1	1
3.00	4	2	2	2
3.00	6	3	3	2
3.00	8	3	3	3
3.00	10	3	3	3
4.00	2	1	1	1
4.00	4	2	2	2
4.00	6	2	2	2
4.00	8	3	3	3
4.00	10	3	3	3

<Table 3> $\rho_1 = 0.1$

μ_2	b	s^*	s_L	s_U
10.00	2	0	0	0
10.00	4	0	0	0
10.00	6	1	1	0
10.00	8	1	1	0
10.00	10	1	1	1
15.00	2	0	0	0
15.00	4	0	0	0
15.00	6	0	0	0
15.00	8	1	1	0
15.00	10	1	1	1
20.00	2	0	0	0
20.00	4	0	0	0
20.00	6	0	0	0
20.00	8	1	1	0
20.00	10	1	1	1

and

$$P_U(Q \geq s+1) = \rho_1^{s+1} + \rho_2^{s+1} - \left(\frac{\lambda}{\gamma_1+\gamma_2+\lambda}\right)^{s+1}$$

where P_L and P_U denote lower and upper bounds for probability $P(Q \geq s+1)$, respectively.

Without loss of generality, we assume that $\lambda = 1$, and h = 1. <Table 1>~<Talbe 3> show the optimal base-stock level s^*, and lower and upper bounds s_U and s_L for the several scenarios. From the numerical results shown in <Table 1>~<Talbe 3>, one can see that s^* decreases as the μ_i's increase, and s^* increases as b increases, but the marginal increasing rate is decreasing. Moreover, it show that the tightness of the bounds increases as the ratio of service times μ_2/μ_1 increases. The suggested optimal base-stock level s^* seems to be exact for large μ_2/μ_1.

5. Conclusion

In this study, we consider a simple assembly line operated under a base-stock policy. We obtain a closed form formula of approximation for delay time or lead time distribution of a demand when a base-stock level is s. Moreover, it can be applied to the optimal base-stock level which minimizes the total inventory cost.

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