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Multiplicative reasoning in fractional contexts: Employing domain analysis and taxonomic analysis

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This study presents the results of a case study that investigated a seventh grader's fractional reasoning related to multiplicative reasoning. In addition, by employing domain analysis and taxonomic analysis for analyzing qualitative data, I show how a qualitative methodology was used for the data collected by teaching experiment methodology. The study identifies three distinct issues that emerged as the student engaged in solving fraction problems: a view of fractions as operations vs. results, the issue of units, and mixed numbers vs. improper fractions. These three issues have instructional implications in that each of them is critical in developing multiplicative reasoning and investigating how they relate to each other suggests a way to improve multiplicative reasoning in fractional contexts.

I. Introduction

Many students in middle school are accustomed to using a standard algorithm $a \div b = a \times \frac{1}{b}$. The equivalent relationship between division and multiplication clearly displays that dividing by a whole number is the same as multiplying its reciprocal. However, conceptual understanding of the equivalent relationship is challenging for teachers as well as students. One of the reasons students find this difficult to understand conceptually is because it involves both fractional reasoning and multiplicative reasoning, which have been considered as daunting in improving mathematical reasoning. There is a body of research that described various aspects of fractions and multiplicative structures. However, how these two daunting areas are related has been relatively less highlighted.

School mathematics tends to introduce multiplication as repeated addition. This tendency reinforces students' additive way of thinking without a notion of multiplicative reasoning even in the contexts requiring high-order multiplicative thinking[1]. Fractions are fundamentally based on multiplicative structures (Thompson & Saldanha, 2003; Vergnaud, 1988), whereas whole numbers are generated by additive reasoning through "one more operation." Therefore, thinking of multiplication and division within fractional contexts becomes a big challenge for students.

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1) Multiplication is not just a fast way of repeated addition but an operation that requires higher-order multiplicative thinking (Piaget, 1987).

This study is grounded in Steffe's perspective that multiplying schemes are constructed using number sequences (1994) and fractional schemes are constructed through the reorganization of numerical counting schemes (2002). By considering fractional schemes and mathematical concepts related to multiplicative reasoning, the study investigates how fractional reasoning relates with multiplicative reasoning by asking the following questions:

1) What kinds of fractional concepts or schemes are crucial in developing multiplicative reasoning?
2) In what ways do the concepts and schemes involved in multiplicative reasoning impact developing such reasoning?

II. Theoretical Framework

Multiplicative reasoning is one aspect of quantitative reasoning, which is thinking about a situation by conceiving of it in terms of quantities, or measurable qualities of an object (Thompson, 1994). By noticing that the arithmetic notations we ordinarily encounter serve a double function, that is, as quantities and as numerical values, this study investigates how a student conceives of a reciprocal relationship between the *quantities that emerge from a situation involving* division and multiplication.

The study, however, is not interested in the student's interpretation or performance of a standard algorithm "inverting and multiplying"; rather, I am interested in the student's way of thinking while he solved mathematical problems or created a situation that made sense to him. Students must find their own ways of operating to achieve their goals. Concerning instructional issues, Steffe (1994) emphasized seeking students' schemes and studying how students modify the schemes in a new situation rather than exploring how students interpret and perform standard algorithms. He also valued fundamental and crucial tasks guiding students to reach their goals. Such concerns are based on two basic principles of a constructivist perspective (von Glasersfeld, 1989); the first principle is "knowledge is not passively received but actively constructed by cognizing subjects," and the second one is "the function of cognition is adaptive and serves in the organization of the experiential world rather than in the discovery of ontological reality" (p. 162). This study builds on Steffe's instructional concern based on radical constructivism.

Confrey (1994) was worried about the exclusive repeated addition approach to multiplication. Multiplication as repeated addition was considered an intuitive model (Fischbein, Deri, Nello, & Marino, 1985) leading to the intuitive rules such as a multiplier must be a whole number, and multiplication makes bigger (Harel, Behr, Post, & Lesh, 1994). However, such intuitive rules need to be modified in fractional contexts. Responding positively to Confrey's concern, Steffe (1994) argued that "repeated addition should not be the only model for multiplication" (p. 24) because children may implement the activity of a units-coordinating scheme even though they seem to employ repeated addition for multiplication.

Regarding multiplicative structures, it is important to consider Steffe's units-coordinating scheme (1994). He considered a units-coordinating scheme as an assimilatory structure to promote multiplying schemes. Units-coordinating means "the mental operation of distributing a composite unit across the elements of another composite unit" (Steffe, 2002, p. 279). He argued that coordinating two composite units in this way is necessary to establish a situation as multiplicative. According to his idea, multiplicative structures are attained as a units-coordinating scheme associates with an interiorized concept of an iterable unit.

According to the above discussion, multiplication requires children to re-present a unit previously constructed in their mind by considering various levels of units. Children make comparisons of a single, non-repeated object before developing the idea of an iterable unit (Piaget, Inhelder, & Szeminska, 1960). However, as they are developing counting schemes[2], children become aware of composite unit items and finally establish an abstract composite unit item with respect to each element of a number sequence by considering an abstract unit item "one" as an iterable unit (Olive, 2001). Such establishment of abstract composite unit items encourages children to develop insight into units of units when they encounter a situation for multiplication. We are, though, likely to assume only a singleton unit for all quantities because the arithmetic of numbers is based on the hidden assumption that all the numbers represent quantities of the same unit of one (Behr, Harel, Post, & Lesh, 1994). This hidden assumption is criticized due to the negative impacts of extending whole number arithmetic toward rational number concepts and operations (Behr et al., 1994; Steffe & Cobb, 1988), representing and manipulating quantities in various unit types (Behr et al., 1994; Steffe, 1988), and experiencing units of units (Behr et al., 1994).

In the above paragraph, I discussed the notion of units related to multiplication. In the following discussion, I investigate how two interpretations of whole number division can be related to the concept of units. Fischbein et al. (1985) suggested partitive division and quotitive division as intuitive models of division; partitive division is based on the idea of equal sharing in which the dividend represents the number of objects and the divisor refers to the number of equivalent subcollections; quotitive division relies on the idea of measurement by questioning how many times a given quantity is contained in another quantity. However, these division models face constraints in fractional contexts (Harel et al., 1994): the equal sharing idea suggests that the divisor must be a whole number, the divisor must be smaller than the dividend, and the quotient must be smaller than the dividend; the measurement model suggests that the divisor must be smaller than the dividend.

We can think of the two intuitive models mentioned above while reflecting on children's counting schemes and the notion of units.

2) Steffe and Cobb (1988) developed three distinctly different number sequences from their extensive teaching experiments: the Initial Number Sequence (INS), the Tacitly Nested Number Sequence (TNS) and the Explicitly Nested Number Sequence (ENS). Olive (2001) extended the number sequences to a Generalized Number Sequences (GNS) and subsequently to the Rational Numbers of Arithmetic (RNA).

Because measuring out by a given sized group is easier for children than sharing among a given number of groups (Confrey, 1995), children tend to interpret division in terms of a chunk of a given size so they attempt to solve a partitive problem by estimating a divisor and segmenting the dividend using the estimate. Such a tendency is supported by research on how children develop their counting schemes: quotitive division is more easily constructed than partitive division (Olive, 2001). Such research argues that children who can coordinate unit items at two levels, or count recursively, can construct the concept of quotitive division, whereas in order to reason division in a partitive way, children should be able to deal with a hypothetical unit as well as to count recursively. Based on the above concern about interpretations of division and counting schemes, I investigated the notion of units.

III. Research Design

Methodology for data collection

This study used part of the data collected over one semester for a project whose purpose was to specify school algebra based on the productive and creative thinking of students of middle school age. The project employed teaching experiment methodology (Steffe & Thompson, 2000) for its research design.

The primary goal of teaching experiment methodology is for researchers to experience students' mathematical learning and reasoning by engaging in exploratory teaching so that they can build models of students' mathematics[3]. A teaching experiment involves a series of teaching episodes over extended periods, in which a teacher-researcher teaches one or more students while being observed by a witness. In addition, it is more than a clinical interview in that it explores students' mathematics while considering the ways of influencing their mathematical knowledge and promoting the greatest progress in participating students. Therefore, researchers are required to have research hypotheses to guide their overall intentions, and by teaching they have students test the hypotheses. However, what is crucial while teaching is that researchers should forget the hypotheses because the teaching experiment focuses on what actually happens while teaching. It means the hypotheses brought in to the teaching experiment should always be ready to be modified and regenerated as the researchers adapt to the constraints they experience while interacting with the students.

For gathering data, one professor and four doctoral students including me were involved, and seven students in a middle school participated. All of the students who participated were in the seventh grade; the data used in the study came from one of the students who was in his second year in the project. Each student was taught twice a week for thirty minutes by a teacher-researcher

3) Students' mathematics means whatever constitutes students' first-order mathematical knowledge; mathematics of students means our second-order models of students' mathematics; and mathematics for students means mathematical concepts and operations that students might learn, that is, what we have observed students actually learn (Steffe, in press).

in pairs or alone. Every teaching episode was videotaped using two video cameras; one for capturing interactions between the student(s) and the teacher-researcher, and the other for capturing the computer screen that contained the results of the student(s)'s solving fractional tasks using JAVA: Bars, a computer software program designed especially for engaging in fractional activity (Biddlecomb & Olive, 2000). By using the software, students can make rectangular regions called bars and partition the bars into parts, the parts into subparts, etc. The researchers met once a week to discuss problems for the next teaching episode and the appropriateness of the problems on the basis of the previous teaching episodes.

Participant

Mike, one of the students in the project, provided the data source for this study. He worked with three teacher-researchers including me during the period; he was paired with another seventh grader named Jenny four times, and for the rest of the episodes, he worked alone. I taught him five times and participated in a meeting to plan his teaching once a week; on the occasions when I was not his teacher, I usually took detailed field notes on what he was doing and how he responded to the questions asked by his teacher.

Mike was asked how to relate his operating with producing a fraction in a situation involving a times-statement. In addition, the researchers investigated how multiplicative reasoning emerges in fractional contexts. Throughout the period, he had difficulty understanding what his products meant in a problem context that he engaged. Even when he had no problem engaging a question, he had difficulty elaborating and articulating what he did mathematically. That is, he seemed fluent in using procedural knowledge, but he sometimes expressed confusion while trying to relate his procedural knowledge with what he did at the conceptual level. He had a sense of fractions as repeated amounts of a unit fraction; however, he revealed some limitations related to the concept of an iterable unit fraction. In other words, even though he verbalized "eight-thirds" as he repeated an one-third of one inch bar eight times, he had difficulty figuring out what the fraction referred to.

Methods for data analysis

For data analysis, I employed domain analysis and taxonomic analysis, which are acceptable methods for conducting ethnographic analysis. The goal of ethnography is to reconstruct participants' reality with the researcher's sensitivity to the fact that "researcher-constructed descriptions of reality may be quite different from the meanings that participants use to construct their reality" (LeCompte & Preissle, 1993, p. 235). As long as we are concerned with experiential realities rather than ontological realities, a notion of others' experiential world becomes crucial in understanding participants' activities in a research environment (von Glasersfeld, 1985). Spradley (1979), in particular, argued that symbols and meaning systems play a crucial role in understanding the process of construction for the following reasons: all the responses participants make in responding to researchers' questions are symbols; meaning is created by using symbols; and "the meaning of any symbol is its relationship to other symbols" (p. 97).

The purpose of domain analysis is "to identify native categories of thought and to gain a preliminary overview of the cultural sense" (p. 117) by identifying domains, cover terms and included terms (Spradley, 1979); the goal of taxonomy is to investigate the relationships of domains to the whole culture on the basis of the interrelationships among the included terms identified (Spradley, 1980). A domain means an abstracted category, and a taxonomy refers to a set of categories. Spradley recommended several useful steps for producing a set of tools to conduct domain analysis (1979) and taxonomic analysis (1980). The following steps are for domain analysis:

1) selecting a single semantic relationship;
2) preparing a domain analysis worksheet;
3) selecting a sample of informant statements;
4) searching for possible cover terms and included terms;
5) formulating structural questions for each domain; and
6) making a list of all hypothesized domains.

The steps for taxonomic analysis are as follows:

1) selecting a domain for taxonomic analysis;
2) looking for similarities based on the same semantic relationship;
3) looking for additional included terms;
4) searching for larger, more inclusive domains that might include as a subset the domain you are analyzing;
5) constructing a tentative taxonomy; and
6) making focused observations to check out your analysis.

First, I transcribed a thirty-minute long videotape. While transcribing, I strongly felt the need of collecting a general sense of the mathematical problems posed and the probing questions that emerged through the interaction between Mike and the teacher-researcher. So, as soon as I was finished transcribing, I started to give serial numbers to the problems and questions asked such as Q1, Q2, Q2-1, Q3, and so on as I wrote "memos" (Charmaz, 2002; Strauss & Corbin, 1990) for my thoughts, interpretations and concerns about the problems and students' responses (Table 1).

Table 1 Samples of memos

Memos: One-third of two vs. two-thirds of one (January 19, 2005) Mike seemed to consider two-thirds as a result that he produced by dividing each foot of two foot long sub sandwich into three parts and pulling two parts out of the sub divided into six parts, but I was wondering how he knew it is two-thirds, that is, how he knew that dividing two into three parts results in two-thirds. I also wondered how he used the quantity two for his production two-thirds. In other words, I questioned how he coordinated one-third of two with two-thirds of one, which is two one-thirds according to his conception of fractions. Such concerns of his reasoning remind me of quotitive and partitive models of division. I think repeating two-thirds three times leads to a quotitive way of thinking to get one-third of two, and two-thirds of one as two units of one-third is corresponded to a partitive way of thinking. Memos: (January 24, 2005) Mike seems to remember "times" context familiar to him. In his answer, one-tenth refers to one-tenth dollar and ten refers to the number of times that he should repeat to get one dollar from one-tenth dollar. The reason he seems confused about the question appears that he considers the question as one-tenth dollar "times" one dollar. However, the teacher seems to make sure of the equivalent relationship between one-tenth times one dollar and one-tenth of a dollar by asking him about the meaning of one dollar divided by ten in terms of "times one-tenth". That is, he seems to consider one-tenth in the question as the result of his dividing, one-tenth dollar.

For the fist step of research-based analysis, I did "line by line coding" to build a systematic account of what has occurred during teaching (Ezzy, 2002). I picked the words related to what the participants were doing, saying or feeling, and I underlined them, such as dollar, one-tenth, divide, pieces, pull out, times, split into, doing times, confusing, puzzle and so on. The words I selected through line by line coding were so closely related to each other, and they were also very specific to a particular problem. I created four kinds of areas to assign the words while considering their attributes as follows: objects/results, actions, mathematical terms, and feelings (cf. Table 2). I then wrote the selected words down in the margin of the transcript as I put them into one of the four areas: for example, "one-tenth" was assigned to both areas of results and mathematical terms, and "performing a times-question" was assigned to the area of actions. As soon as I produced a list of words in each area according to the above procedure, I started "initial coding" (Charmaz, 2002) to have a brief sense of the following things (Table 2): what Mike produced, why he produced the results, what activities helped him to solve the posed problems, why he made an activity, and what kind of mathematical terms were created by him or the teacher. I generated categories in each area produced from "line by line coding." For instance, the area "objects/ results" was divided into two categories, one for objects referring to whole numbers and the other for objects referring to fractions; and the area "actions" was divided into four categories such as making sure, making clear, making an answer, and making an expression.

However, since the initial coding did not provide enough information about a particular mathematics problem posed or about a specific

Table 2. Initial coding

Objects/results	Actions	Mathematical terms	Feeling
Money (whole numbers) ▪ Dollars ▪ Dimes ▪ Cents ▪ Two and sixty six cents Money (fractions) ▪ Pieces ▪ One tenth dollar ▪ One fiftieth dollar ▪ Two and two-thirds dollar	Making sure ▪ showing ▪ aligning ▪ comparing ▪ breaking ▪ fitting ▪ overlapping ▪ stacking Making clear ▪ erasing ▪ coloring ▪ keeping separate Making an answer ▪ pulling out ▪ splitting ▪ taking out of ▪ copying ▪ dialing ▪ multiplying in head ▪ repeating ▪ dividing Making expression ▪ agreeing ▪ nodding	Performing a times-question ▪ repeating ▪ counting by ▪ a unit fraction ▪ a unit fraction times Fractions ▪ five out of ten ▪ eight-twenty fourths ▪ improper fraction ▪ mixed number Functional meaning ▪ one-third of ▪ two-fifths of ▪ eight-thirds of Prove Equal Multiply	Doesn't make sense Confusing Puzzle Nodding Sure Not sure Hesitate Easy Laughing Sick Happy

topic that emerged, I implemented some of the steps for domain analysis to investigate how the above areas were interrelated in the data. First, I *selected a semantic relationship* "is a way to" and searched for the statements involving the relationship. For example,

- Dividing one dollar into ten parts and taking one out of them "is a way to" produce one-tenth of a dollar.
- One-tenth times ten "is a way to" perform one-tenth times a dollar.
- Thinking about an amount, ten times which equals a dollar, "is a way to" prove one-tenth times a dollar.
- Thinking of money "is a way to" solve one half times one-tenth dollar.
- Splitting into a half "is a way to" know that five cents is a half times a dime.
- Five out of ten "is a way to" know that five cents is a half times a dime.
- Dividing a tenth dollar into five "is a way to" produce one-fifth times a tenth dollar.
- Dividing ten cents into five "is a way to" produce one-fifth times a tenth dollar.
- Counting by two and then counting by ten "is a way to" prove that a piece is one-fiftieth dollar.

In the above statements, all the phrases preceding the selected semantic relationship are the actions to perform "a unit fraction times something." Thus, I gave a cover term "performing a times-question involving a fraction" to the statements. From the cover term and included terms (Figure 1), a domain "times-operation in fractional contexts" was generated based on the semantic relationship "is a way to." Similarly, I also produced another domain "times-operation in division contexts."

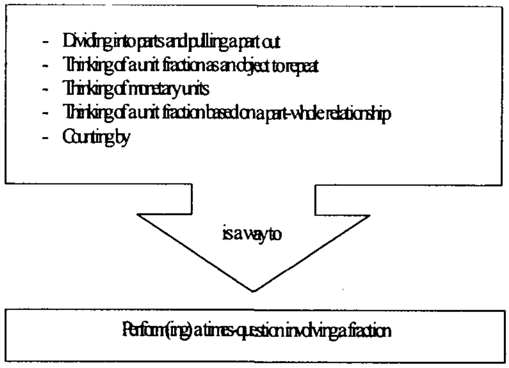

Figure 1. A domain "times-operation in fractional contexts"

In order to investigate a relationship among the included terms in the domain "times-operation in fractional contexts," I employed taxonomic analysis along the steps Spradley (1980) suggested. I first selected a domain "Performing one-tenth times one dollar" that can be a subcategory of a domain "times-operation in fractional contexts."

Stage 1: Making a one dollar bar;
Stage 2: *Dividing the bar into ten parts;*
Stage 3: Pulling one of them out;
Stage 4: Aligning the bar pulled out with the one dollar bar;
Stage 5: Comparing two bars;
Stage 6: Saying one-tenth times ten;
Stage 7: Speaking aloud that the bar he made times ten equals a dollar;
Stage 8: Breaking the one dollar bar and calling one of them a dime.

The above stages were organized based on the student's goal of an activity as follows:

Stage 1: Making a one-tenth dollar.

Stage 1.1: Making a one dollar bar;

Stage 1.2: Dividing the bar into ten parts;

Stage 1.3: Pulling one of them out.

Stage 2: Thinking of one tenth times in terms of what he produced.

Stage 2.1: Aligning the bar pulled out with the one dollar bar;

Stage 2.2: Comparing two bars.

Stage 3: Reflecting on the meaning of a fraction.

Stage 3.1: Saying one-tenth times ten;

Stage 3.2: Proving his answer: the bar he made times ten equals a dollar;

Stage 3.3: Checking one-tenth of one dollar by breaking one dollar into ten parts.

I searched for other domains in order to look for additional included terms for a domain "times-operation in fractional contexts," thereby adding some subcategories and category to the above stages as follows:

Stage 2.3: Converting to the whole number context.

Stage 2.3.1: Thinking of one dime or two cents; rather than one-tenth or one-fiftieth of a dollar

Stage 2.3.2: Saying sixty-six cents rather than two-thirds of a dime.

Stage 3.4: A part-whole concept of fractions.

Stage 3.4.1: Saying five out of ten with respect to the problem that five cents is a half times a dime.

From the categories, I produced a taxonomy of "times-operation in fractional contexts" (Figure 2).

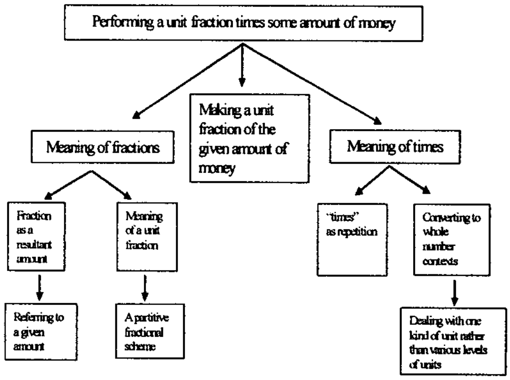

Figure 2. A taxonomy of "times-operation in fractional contexts"

IV. Findings

In the process of producing taxonomies and domains, interesting issues emerged around Mike's concepts of fractions, multiplication, division and units. I organized my findings around the following three issues.

Issue 1. Fractions as results vs. operations

I found that Mike considered a fraction as a resultant amount that depends on a predetermined whole, and he had difficulty developing a view of fractions as operations. In the protocol, "T" stands for "teacher" and "M" for "Mike."

Protocol I: Making one-tenth of a dollar

(Responding to the problem "Make a tenth of a dollar," Mike drew a bar and divided it into ten parts, and then pulled one of them out, which is called a 1/10-bar.)

T: Why is that one-tenth of a dollar?

M: 'cause split into ten and take out one.

T: So, one-tenth times a dollar, how much?

M: One-tenth (placing the 1/10-bar below the one dollar bar, so the 1/10-bar became aligned with the one dollar bar along the left side).

T: One-tenth? Yeah. And you said split into ten, so split into ten is the same as doing one-tenth times?

M: (Looks at the screen for a while and leans back on his chair while laughing.)

T: Is that what you are saying? If you, you said split into ten, right? Is that the same as doing one-tenth times something--split into ten?

M: Yeah (moving the one dollar bar beside the 1/10-bar. He seems unconvinced). Let's see. One-tenth times ten (putting the one dollar bar below the 1/10-bar).

T: So, I am just wondering if you got this (pointing at the 1/10-bar) by splitting into ten, (Mike moves the 1/10-bar so it is overlapped over the one dollar bar.) Right? But I asked you one-tenth times, so is that the same?

M: No (taking the part off the one dollar bar) Let's see, one-tenth times ten (leaning back on his chair).

T: Are you not sure?

M: (Shakes his head.)

T: No? But you think this (pointing to the 1/10-bar) is one-tenth times a dollar, right? The little piece, do you mean (Mike frowns as he moves the 1/10-bar below the one dollar bar) How could you prove that is one-tenth times a dollar?

M: Ten times equals a dollar.

Mike produced a 1/10-bar by dividing a one dollar bar into ten parts and pulling one of them out. He then answered "one-tenth" with respect to the question "How much is one-tenth times a dollar?"; however, while answering the times-question, he seemed distracted. He seemed to focus on the previous question, making a tenth of one dollar, thereby showing some confusion about the probing questions that followed the times-question.

In order to probe his activity of "dividing into parts and pulling one of them out" along with a notion of multiplicative reasoning, the teacher asked him whether the activity meant performing a times-question involving one-tenth. Responding to the question, he focused on a result of his dividing and pulling out rather than his operating to get the result: he compared the sizes of the 1/10-bar and the one dollar bar by aligning them horizontally and vertically. He mentioned "one-tenth times ten" several times as he tried to figure out the question "one-tenth times one." That indicates he deduced a fraction in terms of a reciprocal relationship. However, I doubt he considered the resultant amount, a 1/10-bar, with a notion of partitioning, "a psychological structure that included both operations of breaking a continuous unit into equal sized parts and iterating any of the parts to reconstitute the whole" (Steffe, 2002, p. 272), because dividing and repeating seemed separated in his mind when he conceptualized the fraction one-tenth.

Mike seemed to consider dividing as a way to produce a resultant amount without the notion of operating. Repeating activity seemed to confirm his concept of one-tenth based on a part-whole comparison. If he associated dividing with repeating at the conceptual level, he would be able to consider one-tenth as operating or a result of the operating rather than a resultant amount based on a predetermined whole. As a result, he had yet to implement partitioning. However, Mike was asked questions that were intended to provoke a view of fractions as operating. Therefore, I infer that his perspective of fractions as resultant amounts, without a notion of partitioning, was conflicting with a view of fractions as operating in his mind; therefore, he did not resolve this conflict at the conceptual level.

According to my domain analysis (Figure 1), Mike implemented dividing into parts as a way to perform a times-question involving a unit fraction. However, Mike seemed to consider dividing only as a way to produce a part of a given amount without any concern for a times-question asked. That is, dividing led him to just focus on a resultant amount 1/10-bar, one part, and the number of parts produced, ten parts. In addition, since the repeating activity was considered separate from the dividing activity, his dividing activity did not accompany a notion of ten-tenths for the one dollar bar, the original unit, which is necessary to achieve "one-tenth times one." Therefore, "dividing into ten parts" seemed yet to involve "one-tenth" as operating.

Regarding the question "one-tenth times one," he identified one-tenth in terms of a 1/10-bar, and then he identified one with the original one dollar bar. If he had thought of one-tenth as a resultant amount and one as the number of repetitions, or one-tenth as an operation and one as one dollar, the question "one-tenth times one" would have most likely made sense to him. However, by saying "one-tenth times ten," Mike tried to figure out "one-tenth times" in terms of what he produced by dividing while thinking of one as a one dollar bar. Figure 3 presents a possible explanation for Mike's inability to answer the question, "Is splitting into ten the same as doing one-tenth times?"

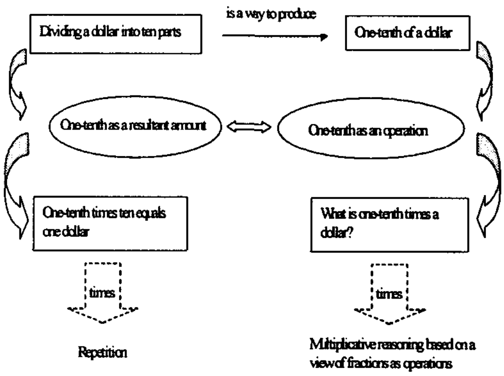

Figure 3. Fractions as results vs. operations

Issue 2. Dealing with various levels of units vs. maintaining one level of unit

I found that Mike tended to maintain one kind of unit rather than dealing with various levels of units throughout a fractional context. In particular, his tendency to convert an amount of money from a fractional form into whole numbers restricted him from developing an ability to deal with various levels of units in fractional contexts. Noticing various levels of units is very critical in developing multiplicative reasoning as well as fractional reasoning in that multiplicative reasoning is based on coordinating various levels of units (Steffe, 1994), and fractional reasoning involves multiplicative reasoning (Thompson & Saldanha, 2003). In the protocol below, "J" stands for another seventh grader "Jenny" paired with Mike.

Protocol II: One-fifth times one-tenth.

T: How much would be one-fifth times that (pointing at a 1/10-bar produced by dividing a one dollar bar into ten parts and pulling one part out)?

M: One-fifth?

T: Uh-huh. One-fifth times that (the 1/10-bar), think about how much of a dollar would be? What would it look like? What would you do to make it? (pause) you got it? (looking at Mike).

M: I think I do.

T: Did you do--what do you think, Jenny?

J: (Shakes her head.)

T: (Looking at Jenny) Not sure? What do you think, Mike?

M: Divided into five.

T: Divided into five? So, is taking one-fifth times something same as dividing into five?

M: Yes. (Jenny is also nodding.)

T: Oh, okay. If you divide that (pointing at the 1/10-bar) into five, go, why don't you make it?

M: (Dials to five and dividing the 1/10-bar into five parts.)

T: Jenny, do you know that how much money that's gonna be, one of those little tiny pieces?

J: One-fiftieth of the top bar (the one dollar bar).

T: Yeah, one-fiftieth of the top bar, one-fiftieth of a dollar.

M: Two cents.

T: Two cents? How did you get that's two cents?

M: Since these are ten and then divided into five, five divided by ten equals two.

Mike positively answered the question, "Is taking one-fifth times something the same as dividing into five?" However, such a response does not necessarily mean that he considered various levels of units involved in one-fifth. Responding to the question asking about the amount produced by dividing a 1/10-bar into five parts, Jenny answered as she simultaneously considered two levels of units, one tenth and one: "one fiftieth of the top bar (the original one dollar bar)." On the other hand, Mike showed that he focused only on one kind of unit, cents. He thought of one tenth as dividing 100 cents by ten and one fifth as dividing 10 cents by five. It is implicit that he considered 10 cents as a result of a tenth of a dollar, on which he would operate a fifth. In other words, he might not be able to think of two cents as one-fiftieth rather than one-fifth because the result, two cents, would not explicitly involve fractional reasoning of one-tenth. Awareness of two cents as one-fiftieth requires him to recognize a structure of the units involved. A unit of a dollar is ten units of a tenth of a dollar; a tenth of a dollar is five units of a fifth of a tenth of a dollar. Without noticing such a structure of units, his dividing activity would be considered a numerical operation rather than a quantitative operation (Thompson, 1994).

As soon as he noticed the relationship between a times-question and dividing by a whole number, he applied the relationship to his understanding of the times-questions by employing various kinds of monetary units, so he could deal with various levels of units involving whole numbers. For Mike, thinking of money was a way to solve one half times one-tenth dollar (cf. Figure 1). So, as soon as he was asked about one-fifth times a tenth, he converted a tenth into ten cents, and then he arrived at two cents by dividing ten by five. That is, he did not need fractional reasoning in terms of one-fifth or one-tenth in the converted situation. Figure 4 shows the idea of two levels of units was not reflected by his reasoning due to his tendency to understand a situation in terms of one kind of unit.

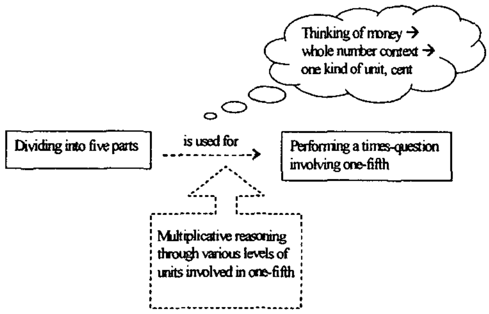

Figure 4. One level of unit vs. various levels of units

Issue 3. Mixed numbers vs. improper fractions

I found that for Mike, eight-thirds had a different meaning than two and two-thirds, and eight divided by three was not equivalent to eight times one-third at his conceptual level. In other words, a mixed number was considered an iterable composite unit, whereas an improper fraction was not considered an iterable unit fraction of a composite unit.

Protocol III: How much money does Mike have if eight dollars is three times as much as Mike's?

T: Well, right now we are thinking about eight dollars, that is three times as much money as Mikes'. And you said, you wanna do divide by three (looking at Jenny). Why don't you try it?

J: (Produces an 8/3-bar by dividing each of the eight bars into three parts, pulling one part (a 1/3-bar) out and repeating the part eight times.)

T: Oh, so that's Mike's money, huh? (Both Mike and Jenny are nodding.) How much money does Mike have?

J: Eight-twenty fourths.

M: Two dollars and of (pause) I don't know.

(An observer asked him to give a fraction for the mixed number.)

M: Two, two-thirds.

T: Two and two-thirds? Mike has two and two thirds. Jenny, you were saying eight something, right, because you got eight pieces. What were you saying? Eight what?

J: Eight dollars twenty-fourths?

T: Well, that's true! Because you have eight dollars twenty four pieces. Let's see. We wanna know that in terms of a dollar, right?

J: That's two dollars and two-thirds?

. . . . .

T: Give me just a fraction for two and two-thirds.

M: Two-twentieths? No, two-twenty fourths.

T: Really? For two and two-thirds?

M: I mean eight-twenty fourths.

T: Oh, eight-twenty fourths because there are twenty four pieces? How much is each one of those little guys (pointing at one part in the 8/3-bar), pieces of a dollar?

M: One third.

T: Yeah, so how much does Mike have?

M: Eight-thirds.

T: Oh, eight-thirds. Is that the same as two and two-thirds? (Both Mike and Jenny are nodding.) So, another way we can say is eight-thirds, right?

O: Eight-thirds of what?

M: Eight-thirds of the whole eight dollars?

T: Yeah, eight-thirds of whole eight dollars is what you are saying?

M: (Nods.)

In responding to the question, "Jenny has eight dollars, that's three times as much as Mike has. How much does Mike have?," Mike answered "two and two-thirds." To probe his answer, the teacher asked him to give a fractional answer. Mike changed his answer from two and two-thirds to eight-twenty fourths. He considered eight parts comprising an 8/3-bar as belonging to twenty-four parts which resulted from dividing each of eight one dollar bars into three parts. That is, he considered that two and two-thirds is equivalent to eight-twenty fourths because only the partitive fractional scheme, which is a scheme for producing fractions no greater than the fractional whole, was available to him. His scheme prevented him from considering and coordinating various levels of units (Lee, 2006a).

On the same day, right before this protocol, he articulated two and one-third is a third of seven dollars; however, he could not figure out a 7/3-bar consisting of seven of 1/3-bars as one-third of seven dollars. That is, he constructed a mixed number as a unit fraction referring to the given amount, seven dollars; whereas he constructed an improper fraction only based on a unit of one dollar without considering the given amount as a composite unit.

Dealing with various levels of units is necessary to construct improper fractions (Steffe, 2002), and especially three levels of units is required (Hackenberg, 2007; Lee, 2006b). However, as I have discussed in Figure 2 and Issue 2, Mike tended to conceive of a situation in terms of one kind of unit rather than various levels of units. That is, in Protocol III, he considered a unit of one dollar and a unit of one-third of one dollar for the 8/3-bar without considering a unit of eight units of one dollar. So, an 8/3-bar was not related with eight dollars at his conceptual level, and he had difficulty producing a multiplicative relationship between the 8/3-bar and eight dollars. Figure 5 shows Mike's conceptual variations along his various answers to the questions regarding the 8/3-bar.

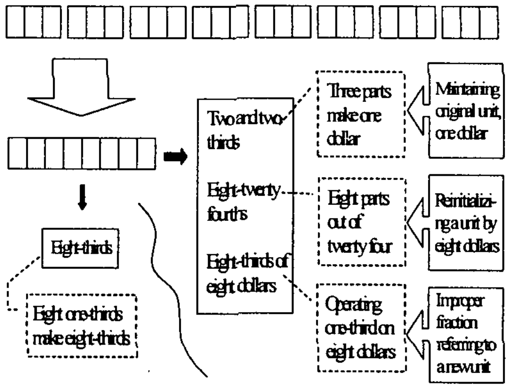

Figure 5. Possible conceptions of units involved in Mike's various answers related to an 8/3-bar

The following protocol, which is the continuation of Protocol III, presents such difficulty. In the protocol, "O" stands for "observer."

Protocol III: (Cont.) What do you multiply 8 by to get 8/3?

T: What can you multiply Jenny's money by, eight dollars, to get the eight-thirds?

M: Three.

T: Three? Multiply by three?

M: Yeah, wait. Get Jenny's money from my money.

T: Oh, if you start with Jenny's money, you will be right, no you mean, if you start with your money, you will be right. Eight-thirds times three is gonna give you Jenny's money?

M: Yeah.

T: What about the other way?

O: Can you explain that?

T: Mr. Hope wants you to explain why that is.

M: Because if you do times, I mean, eight-thirds times three then multiply eight times three, twenty four and then multiply one (inaudible).

T: That's okay, Mike. Your explaining is in terms of thinking how you write it out, how you calculate it. Can you explain it in terms of the picture (pointing her finger at the eight one dollar bars and the 8/3-bar consisting of eight 1/3-parts on the screen)? Why is that if you do three times your money to get Jenny's money?

M: (Shakes his head.)

When asked to find a way to produce the 8/3-bar from 8 dollars, he reminded himself of the fact that the 8/3-bar was an answer of the problem in Protocol III, "eight dollars, that is three times as much money as Mike's," and he answered "Three." Mike previously watched Jenny make the 8/3-bar by repeating a 1/3-bar eight times. However, since his concept of multiplication as repetition was dominant, the inverse way in which he was supposed to engage was beyond his mathematical understanding. In addition, his view of eight-thirds as a resultant amount would prevent him from reflecting on his operating to produce the 8/3-bar as well as from considering various levels of units involved in the construction of the 8/3-bar. Furthermore, he had difficulty elaborating eight-thirds times three using the bars, such as eight one dollar bars--each of which divided into three parts and the 8/3-bar consisting of eight of the 1/3-bars. His difficulty explicitly shows that he did not yet develop a units-coordinating scheme, so he could not operate one-third on eight units of one dollar.

V. Discussion and Conclusion

Regarding multiplicative structures, three main perspectives are considered in whole number contexts: repeated addition (Fischbein et al., 1985), splitting (Confrey, 1994) and a units-coordinating scheme (Steffe, 1994). The repeated addition model is defined in terms of two quantities, a multiplier and a multiplicand, where multiplier refers to the number of equivalent collections and multiplicand to the size of each collection. Splitting is defined as "a primitive operation that requires only recognition of the type of split and the requirement that the parts are equal" (Confrey, 1994, p.300). A units-coordinating scheme enables one to coordinate two composite units, which means distributing a composite unit over the elements of the other composite unit.

I am concerned with how to extend the ideas of multiplicative structure in whole numbers to fractional contexts along with a view of fractions. First, multiplying a whole number "N" and a fraction "F". According to the idea of repeated addition, N times F produces a result of repeating F as many times as N; the fraction is considered as a definite amount or a result. On the other hand, given the idea of splitting, N times F produces a result of replacing each unit of one comprising N with F; the fraction is considered as operating on a unit of one comprising the whole number or a definite amount. Finally, according to the idea of units-coordinating scheme, N times F is accomplished through coordinating a composite unit of N and a fractional whole of F. A units-coordinating scheme involving fractions is distinguished from doing so in whole number contexts in that partitioning is involved in the fractional contexts. So, we need to consider a fractional whole of F as a partitioned whole divided into N parts.

There are two ways to coordinate two composite units. First, distributing the fractional whole over each unit of one comprising the composite unit of N; and second, distributing the composite unit of N over the parts of the partitioned fractional whole. Either way, the fraction is considered an operation. In case we apply the three ideas to multiplying fractions in the same way as above, we can see the idea of a units-coordinating scheme is appropriate while the others ideas are not. This approach, which indicates a view of fractions as operations, is critical in developing fraction multiplication conceptually.

In conclusion, Mike developed a partitive unit fractional scheme, but he had yet to construct an iterable unit fraction in a generative manner. Such a conceptual level of fractional reasoning was related to his view of fractions as results, and this view prevented him from developing a units-coordinating scheme involving partitioning and further conceptualizing improper fractions. This conclusion is consistent with Tzur (1999)'s argument "the iterative fraction scheme involves a rudimentary form of multiplicative reasoning about fractions" (p. 393). Therefore, I infer a view of fractions as operations is critical in developing multiplicative reasoning in fractional contexts by impacting a construction of a units-coordinating scheme involving partitioning.

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분수맥락에서의 곱셈 추론: Domain Analysis and Taxonomic Analysis를 적용하여

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이 논문은 7학년 학생의 곱셈추론과 관련된 분수추론을 연구한 사례연구이다. 이 논문은 또한 교수실험방법론에 의해 거둬진 질적 자료를 domain analysis와 taxonomic analysis라는 질적 분석법에 따라 분석함으로써 특정 질적 분석법들의 실제 적용사례를 자세히 보여준다. 자료 분석을 통해 세 가지 이슈가 부각되었다: 분수에 대한 시각 (조작으로써 대 결과물로써), 단위 문제, 대분수와 가분수의 관계이다. 그러한 이슈들은 첫째, 각 이슈들이 곱셈추론의 발달에 중요하며, 둘째 그 이슈들 사이의 관계가 분수추론을 통한 곱셈추론의 한 방법을 제시해 준다는 점에서 교육적인 의미를 갖는다.

* key words : fractions (분수), multiplicative reasoning (곱셈추론), fractions as results (결과물로써의 분수), fractions as operations (조작으로써의 분수), units (단위), improper fractions (가분수), domain analysis (도메인분석), taxonomic analysis (분류분석).

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