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論 文
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A New Stability Criterion of a Class of Neutral Differential Equations

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Abstract - In this letter, the problem for a class of neutral differential equation is considered. Based on the Lyapunov method, a stability criterion, which is delay-dependent on both τ and σ, is derived in terms of linear matrix inequality (LMI). Two numerical examples are carried out to support the effectiveness of the proposed method.

Key Words : Neutral differential equation, asympototic stability, Lyapunov method, LMI

1. Introduction

Since time delays are frequently occurred in many industrial system such as large-scale system, power system, chemical process, and network control systems, many researchers have been focused on the stability analysis of time-delay systems. For references, see [1-10] and references there-in.

Recently, the stability analysis of the neutral differential equations of the form

$$\frac{d}{dt}[x(t)+px(t-\tau)]=-ax(t)+b\tanh x(t-\sigma), \quad t\geq 0, (1)$$

has been investigated in [6-8]. This equation is special case of the neural network model which has been extensively investigated for the asymptotic stability and can be applied to various systems such as pattern classification, image processing, signal processing, and fixed-point computation, and so on [11-12]. Here, a, τ and σ are positive real scalar, b, and p are real numbers, and $|p|<1$. It is assumed that the delays τ and σ are bounded as $0\leq\tau\leq\overline{\tau}$ and $0\leq\sigma\leq\overline{\sigma}$. With each solution of Eq. (1), the initial condition is defined as:

$$x(s)=\phi(s), \quad s\in[-\delta,0],$$

where $\delta=\max(\overline{\tau},\overline{\sigma})$, $\phi(s)\in C([-\delta,0],R)$.

More recently, delay-dependent asymptotic stability for Eq. (1), which gives a less conservative result than delay-independent one when the size of delay is small, have been studied in [9-10].

In this paper, we propose a new stability criterion for asymptotic stability of Eq. (1). The proposed criterion is delay-dependent on both τ and σ. In order to derive a less conservative results, an integral inequality lemma is introduced and a new Lyapunov-krasovskii functional with descriptor form are utilized in deriving the stability criterion for Eq. (1). Unlike the methods in [9-10], we do not use the model transformation technique, which leads an additional dynamics. The proposed stability criterion is represented in terms of an LMI which can be solved efficiently by various convex optimization algorithms. Two numerical examples are given to show the effectiveness of the proposed method. Throughout this paper, ★ represents the elements below the main diagonal of a symmetric matrix. The notation $X>Y$, where X and Y are matrices of same dimensions, means that the matrix $X-Y$ is positive definite. $C([0\ \infty),R^n)$ denotes the Banach space of continuous vector functions from $[0\ \infty)$ to R^n. $diag\{\cdots\}$ denotes the block diagonal matrix.

2. Main result

Before deriving the main result, we need the following facts and lemma.

Fact 1. (Schur Complement) Given constant symmetric matrices $\Sigma_1,\Sigma_2,\Sigma_3$ where $\Sigma_1=\Sigma_1^T$ and $0<\Sigma_2=\Sigma_2^T$, then $\Sigma_1+\Sigma_3^T\Sigma_2^{-1}\Sigma_3<0$ if and only if

$$\begin{bmatrix}\Sigma_1 & \Sigma_3^T\\ \Sigma_3 & -\Sigma_2\end{bmatrix}<0, \text{ or } \begin{bmatrix}-\Sigma_2 & \Sigma_3\\ \Sigma_3^T & -\Sigma_1\end{bmatrix}<0.$$

Fact 2. For any real vectors a, b and any matrix $Q>0$ with appropriate dimension, the following inequality

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$$2a^Tb \le a^TQa + b^TQ^{-1}b$$

is always satisfied.

To derive a less conservative stability criterion, let us introduce a new integral inequality which will be used to derive the upper bound of Lyapunov function.

Lemma 1. For any scalar $\gamma>0$, and η, the following inequality holds:

$$-\gamma\int_{t-\tau}^{t}\dot{x}^2(s)ds \le \zeta^T(t)\mathcal{F}\zeta(t) + \gamma^{-1}\bar{\tau}\zeta^T(t)\bar{F}^T\bar{F}\zeta(t) \quad (2)$$

where

$$\zeta^T(t) = \begin{bmatrix} x(t) \\ x(t-\tau) \\ \dot{x}(t) \\ \dot{x}(t-\tau) \\ \int_{t-\tau}^{t}\dot{x}(s)ds \\ \tanh x(t) \\ \tanh x(t-\sigma) \\ \int_{t-\sigma}^{t}\tanh x(s)ds \end{bmatrix}^T, \quad (3)$$

$$\bar{F} = [0\ 0\ 0\ 0\ \eta\ 0\ 0\ 0], \quad (4)$$

$$\mathcal{F} = diag\{0,0,0,0,2\eta,0,0,0\}. \quad (5)$$

Proof. By utilizing Fact 2, we have

$$\begin{aligned}-\gamma\int_{t-\tau}^{t}\dot{x}^2(s)ds \le\ & 2\int_{t-\tau}^{t}\dot{x}(s)\bar{F}\zeta(t)ds \\ & + \gamma^{-1}\int_{t-\tau}^{t}\zeta^T(t)\bar{F}^T\bar{F}\zeta(t)ds \\ \le\ & 2\zeta^T(t)\bar{I}^T\bar{F}\zeta(t) + \gamma^{-1}\bar{\tau}\zeta^T(t)\bar{F}^T\bar{F}\zeta(t) \\ =\ & \zeta^T(t)\mathcal{F}\zeta(t) + \gamma^{-1}\bar{\tau}\bar{F}^T\bar{F}\zeta(t),\end{aligned} \quad (6)$$

where $\bar{I} = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 0]$. This completes the proof. ■

For simplicity of matrix dimension, let us define the matrix Σ in Appendix. Then, we have the following theorem for asymptotic stability of Eq. (1).

Theorem 1. For given $\bar{\tau}>0$, and $\bar{\sigma}>0$, every solution $x(t)$ of Eq. (2) satisfies $x(t)\to 0$ as $t\to\infty$, if there exist positive scalar $\gamma_i (i=1,\dots,5), \rho_1, \theta$, and any scalar $\eta_1, \eta_2, \rho_i (i=2,\dots,17)$ such that the following LMI holds

$$\begin{bmatrix} \Sigma & \bar{\tau}F_1^T & \bar{\sigma}F_2^T \\ \star & -\bar{\tau}\gamma_4 & 0 \\ \star & \star & -\bar{\sigma}\gamma_5 \end{bmatrix} < 0, \quad (7)$$

where Σ is defined in Appendix and

$$F_1 = [0\ 0\ 0\ 0\ \eta_1\ 0\ 0\ 0],$$

$$F_2 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ \eta_2].$$

Proof. For positive scalar $\gamma_i (i=1,\dots,5), \rho_1$, and any scalar $\rho_i (i=2,\dots,17)$, let us consider the following Lyapunov function defined by

$$V = V_1 + V_2 + V_3 \quad (8)$$

where

$$V_1 = \zeta^T(t)EP\zeta(t),$$

$$V_2 = \gamma_1\int_{t-\tau}^{t}x^2(s)ds + \gamma_2\int_{t-\tau}^{t}\dot{x}^2(s)ds + \gamma_3\int_{t-\tau}^{t}\tanh x^2(s)ds,$$

$$V_3 = \gamma_4\int_{t-\tau}^{t}\int_{s}^{t}\dot{x}^2(u)duds + \gamma_5\int_{t-\sigma}^{t}\int_{s}^{t}\tanh^2 x(u)duds,$$

$$E^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} \rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_7 & \rho_8 & \rho_9 \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} & \rho_{17} \end{bmatrix}, \quad (9)$$

and $\zeta(t)$ is defined in (3) of Lemma 1.

First, the time derivative of V_1 along the solution of Eq. (1) is obtained by

$$\begin{aligned}\dot{V}_1 &= 2\zeta^T(t)P^TE^T\dot{\zeta}(t) \\ &= 2\zeta^T(t)P^T\begin{bmatrix}\dot{x}(t) \\ 0 \\ 0\end{bmatrix} \\ &= \zeta^T(t)P^T\begin{bmatrix}\dot{x}(t) \\ -\dot{x}(t) - ax(t) + b\tanh x(t-\sigma) - p\dot{x}(t-\tau) \\ x(t) - x(t-\tau) - \int_{t-\tau}^{t}\dot{x}(s)ds\end{bmatrix} \\ &= \zeta^T(t)(P^TG + G^TP)\zeta(t),\end{aligned} \quad (10)$$

where

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & -1 & -p & 0 & 0 & b & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Second, differentiating V_2 leads to

$$\begin{aligned}\dot{V}_2 =\ & \gamma_1x^2(t) - \gamma_1x^2(t-\tau) + \gamma_2\dot{x}^2(t) - \gamma_2\dot{x}^2(t-\tau) \\ & + \gamma_3\tanh^2x(t) - \gamma_3\tanh^2x(t-\sigma).\end{aligned} \quad (11)$$

Last, $\dot{V}_3$ can be obtained as

$$\begin{aligned}\dot{V}_3 =\ & \gamma_4\tau\dot{x}^2(t) - \gamma_4\int_{t-\tau}^{t}\dot{x}^2(s)ds \\ & + \gamma_5\sigma\tanh^2x(t) - \gamma_5\int_{t-\sigma}^{t}\tanh^2x(s)ds \\ \le\ & \gamma_4\bar{\tau}\dot{x}^2(t) - \gamma_4\int_{t-\tau}^{t}\dot{x}^2(s)ds \\ & + \gamma_5\bar{\sigma}\tanh^2x(t) - \gamma_5\int_{t-\sigma}^{t}\tanh^2x(s)ds.\end{aligned} \quad (12)$$

By utilizing Lemma 1, we have a new upper bound of $\dot{V}_3$ as

$$\begin{aligned}\dot{V}_3 \le\ & \gamma_4\bar{\tau}\dot{x}^2(t) + \gamma_5\bar{\sigma}\tanh^2x(t) \\ & + \zeta^T(t)(\mathcal{F}_1 + \gamma_4^{-1}\bar{\tau}F_1^TF_1 + \mathcal{F}_2 + \gamma_5^{-1}\bar{\sigma}F_2^TF_2)\zeta(t)\end{aligned} \quad (13)$$

where

$$\mathcal{F}_1 = diag\{0,0,0,0,2\eta_1,0,0,0\},$$
$$\mathcal{F}_2 = diag\{0,0,0,0,0,0,0,2\eta_2\},$$

and F_1 and F_2 are defined in Theorem 1.

By utilizing the relation $\tanh^2x(t) \le x^2(t)$, we have

$$-\theta x^2(t) \le -\theta(\tanh x(t))^2, \quad (14)$$

where θ is a positive constant to be chosen later.

From Eqs. (10)-(14), we have the following upper bound of $\dot{V}$ as

$$
\begin{aligned}
\dot{V} \le\ & \zeta^T(t)(P^TG+G^TP)\zeta(t)+\gamma_1x^2(t)-\gamma_1x^2(t-\tau) \\
& +\gamma_2\dot{x}^2(t)-\gamma_2\dot{x}^2(t-\tau)+\gamma_3\tanh^2x(t)-\gamma_3\tanh^2x(t-\sigma) \\
& +\gamma_4\bar{\tau}\dot{x}^2(t)+\zeta^T(t)\overline{F}_1\zeta(t)+\gamma_4^{-1}\bar{\tau}\zeta^T(t)F_1^TF_1\zeta(t) \\
& +\gamma_5\bar{\sigma}\tanh^2x(t)+\zeta^T(t)\overline{F}_2\zeta(t)+\gamma_5^{-1}\bar{\sigma}\zeta^T(t)F_2^TF_2\zeta(t) \\
& +\Theta x^2(t)-\Theta\tanh^2x(t) \\
=\ & \zeta^T(t)(\Sigma+\gamma_4^{-1}\bar{\tau}F_1^TF_1+\gamma_5^{-1}\bar{\sigma}F_2^TF_2)\zeta(t).
\end{aligned} \tag{15}
$$

By Fact 1, $\Sigma+\gamma_4^{-1}\bar{\tau}F_1^TF_1+\gamma_5^{-1}\bar{\sigma}F_2^TF_2<0$ is equivalent to the LMI (7). Therefore, if LMI (7) holds, $\dot{V}$ is negative. This means that $\dot{V}\le -k\|x(t)\|^2$ for sufficiently small $k>0$. According to the Theorem 9.8.1 in [1], we conclude that if $|p|<1$ and the inequality (7) holds, then, system (1) is asymptotically stable. This completes the proof. ■

Remark 1. The problem for solving LMI(7) in Theorem 1 is to determine whether the problem is feasible or not. It is called the feasibility problem. The solutions of the LMI (7) can be found by solving eigenvalue problem with respect to γ_i, ρ_i, η_i, and Θ, which is a convex optimization problem [13]. Various efficient convex optimization algorithms can be used to check the feasibility of the LMI in Theorem 1. In this letter, in order to solve the LMI of Theorem 1, we utilize Matlab's LMI Control Toolbox [14], which implements state-of-the-art interior-point algorithms, which is significantly faster than classical convex optimization algorithm [13]. Therefore, all solutions γ_i, ρ_i, η_i, and Θ, can be obtained simultaneously.

Remark 2. The criterion of Theorem 1 is delay-dependent on both τ and σ.

3. Numerical Examples

To demonstrate the effectiveness of the proposed criterion, we give the following two examples.

Example 1. Consider the equation in [10]:

$$\frac{d}{dt}[x(t)+0.35x(t-0.5)]=-1.5x(t)+b\tanh x(t-0.5). \tag{16}$$

For this system, the maximum allowable bound of b is given in Table 1. From Table 1, one can see our result gives a larger upper bound of b for guaranteeing the asymptotc stability of (16) than those in other literature.

Example 2. Consider the neutral differential equation in [10]:

$$\frac{d}{dt}[x(t)+0.2x(t-\tau)]=-0.6x(t)+0.3\tanh x(t-\sigma). \tag{17}$$

The maximum allowable delay bound for stability in Park and Kwon [10] with $\tau=0.1$ was 1.902. However, by applying Theorem 1 to the system (17), one can see that the system (17) is asymptotically stable for any $\sigma\ge0$ and $\tau\ge0$. For reference, the LMI solution of Theorem 1 with

표 1 예제 1에서 안정성을 보장하는 b의 상한 값

Table 1 Upper bounds of b for guaranteeing stability in Example 1.

	Upper bounds of b
Agrawal and Grace [8]	0.318
El-Morshedy and Gopalsamy [6]	0.423
Park [9]	0.423
Park and Kwon [10]	0.669
Theorem 1	1.40

$\bar{\sigma}=\bar{\tau}=10^9$ can be obtained as

$$
\begin{aligned}
\gamma_1 &= 554.2395,\ \gamma_2=1.8001\times10^3,\ \gamma_3=1.0271\times10^3, \\
\gamma_4 &= 1.3224\times10^{-6},\ \gamma_5=4.0204\times10^{-7},\ \Theta=1.8079\times10^2, \\
\eta_1 &= 3.0646\times10^{-15},\ \eta_2=-2.2595\times10^{-8}, \\
\rho_1 &= 3.8021\times10^{3},\ \rho_2=2.0876\times10^3,\ \rho_3=594.2890, \\
\rho_4 &= 2.0728\times10^3,\ \rho_5=-467.6535,\ \rho_6=675.2639, \\
\rho_7 &= 0,\ \rho_8=576,1082,\ \rho_9=0, \rho_{10}=-375.8661, \\
\rho_{11} &= 103.4981,\ \rho_{12}=-580.1084,\ \rho_{13}=-39.0418, \\
\rho_{14} &= 304.3665,\ \rho_{15}=0,\ \rho_{16}=33.3565,\ \rho_{17}=0.
\end{aligned}
$$

Thus, Theorem 1 in this paper provides a much less conservative result than the one in Park and Kwon [10].

4. Conclusions

In this paper, a new delay-dependent stability criterion for a class of neutral differential equations is proposed. To obtain a less conservative result, a new Lyapunov - krasovskii functional of descriptor form which includes zero equations is proposed in deriving the stability criterion of Eq. (1). And an integral inequality lemma, which includes free variables, is utilized in obtaining an upper bound of the integral term. Through two numerical examples, the effectiveness of the proposed stability criterion is shown.

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Appendix

$\Sigma=(\Sigma_{ij}), i,j=1,\ldots,8,\ \Sigma_{11}=-2a\rho_2+2\rho_{10}+\gamma_1+\Theta,$

$\Sigma_{12}=-a\rho_3+\rho_{11}-\rho_{10},\ \Sigma_{13}=-a\rho_4+\rho_{12}+\rho_1-\rho_2,$

$\Sigma_{14}=-a\rho_5+\rho_{13}-p\rho_2,\ \Sigma_{15}=-a\rho_6+\rho_{14}-\rho_{10},$

$\Sigma_{16}=-a\rho_7+\rho_{15},\ \Sigma_{17}=-a\rho_8+\rho_{16}+b\rho_2,$

$\Sigma_{18}=-a\rho_9+\rho_{17},\ \Sigma_{22}=-2\rho_{11}-\gamma_1,\ \Sigma_{23}=-\rho_{12}-\rho_3,$

$\Sigma_{24}=-\rho_{13}-p\rho_3,\ \Sigma_{25}=-\rho_{14}-\rho_{11},\ \Sigma_{26}=-\rho_{15},$

$\Sigma_{27}=-\rho_{16}+b\rho_3,\ \Sigma_{28}=-\rho_{17},\ \Sigma_{33}=-2\rho_4+\gamma_2+\bar{\tau}\gamma_4,$

$\Sigma_{34}=-\rho_5-p\rho_4,\ \Sigma_{35}=-\rho_6-\rho_{12},\ \Sigma_{36}=-\rho_7,$

$\Sigma_{37}=-\rho_8+b\rho_4$, $\Sigma_{38}=-\rho_9$, $\Sigma_{44}=-2p\rho_5-\gamma_2$,
$\Sigma_{45}=-p\rho_6-\rho_{13}$, $\Sigma_{46}=-p\rho_7$, $\Sigma_{47}=-p\rho_8+b\rho_5$,
$\Sigma_{48}=-p\rho_9$, $\Sigma_{55}=-2\rho_{14}+2\eta_1$, $\Sigma_{56}=-\rho_{15}$,
$\Sigma_{57}=-\rho_{16}+b\rho_6$, $\Sigma_{58}=-\rho_{17}$, $\Sigma_{66}=\gamma_3+\bar{\sigma}\gamma_5-\Theta$,
$\Sigma_{67}=b\rho_7$, $\Sigma_{68}=0$, $\Sigma_{77}=2b\rho_8-\gamma_3$, $\Sigma_{78}=b\rho_9$,
$\Sigma_{88}=2\eta_2$.

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