Gains Achieved by Symbol-by-Symbol Rate Adaptation on Error-Constrained Data Throughput over Fading Channels

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Abstract: **Methods for symbol-by-symbol channel feedback and adaptation of symbol durations have been recently proposed. In this paper, we quantitatively analyze the gain in error-constrained data throughput due to such an extremely rapid adaptation of symbol durations to fast-time-varying channels. The results show that a symbol-by-symbol adaptation can achieve a throughput gain by orders of magnitude over a frame-by-frame adaptation.**

Index Terms: **Fading, rate adaptation, throughput**

I. INTRODUCTION

In this paper we compare the data rates of the two ideal adaptive systems that give the same symbol error probability (SEP). Both adaptive systems adjust the symbol duration to the channel's instantaneous gain (or attenuation). One system can rapidly adjust every symbol's duration, so is referred to as the symbol-by-symbol (SBS) adaptive system. The other system can change the symbol duration only at the beginning of each frame and keeps constant the duration of the symbols transmitted in the frame. This latter system is referred to as the frame-by-frame (FBF) adaptive system. An example of the FBF rate adaptation would be the UMTS-2000 WCDMA System, in which the transmitter can change the spreading gain for every data frame of 10 ms in some channels. SBS adaptation was recently proposed in the CDMA system in the form of spreading gain adaptation [1]. Then, in [2] we proposed improved protocols and the entire architecture for rapid channel status feedback that enable the transmitter to rapidly adapt to the fast time-varying channel condition by changing the spreading gain symbol-by-symbol. A purpose of the present paper is to quantitatively estimate the benefits that such a rapid adaptation can provide. The contribution of the present paper will be to show that the SBS adaptation can improve by orders of magnitude the average data rate for a given SEP requirement.

II. ANALYSIS

We first derive the symbol-by-symbol rate (symbol duration) adaptation policy that maximizes the average symbol rate under the constraint of the required symbol error probability, ε (error-constrained data throughput). Our analysis is conducted under conditions similar to those used in [3], where channel's instantaneous conditions are assumed to be known by the system's transmitter and receiver. The transmitter decides the duration of each symbol on the basis of the instantaneous channel condition, and we assume that the receiver knows the symbol duration chosen by the transmitter. Thus, the correct duration of each symbol is used by the receiver to process the incoming signal and detect the content of the symbol.

In this study, we apply the wide-sense stationary uncorrelated scattering channel model. We denote by $a(t)$ a normalized ergodic complex-valued random process that models complex, base-band fading channel gain. We define this gain $a(t)$ in such a way that $P\left|a(t)\right|^2$ is the receive power at time t, where P is the average receive power, i.e., $a(t)$ is normalized such that $E\left[\left|a\left(t\right)\right|^2\right] = 1$. If we are given a transmission power P_T and an unnormalized channel gain process $\tilde{a}(t)$, then we can normalize it as

$$a(t) = \tilde{a}(t) \Big/ \sqrt{E\left[\left|\tilde{a}(t)\right|^2\right]}$$

where $P = P_T E\left[\left|\tilde{a}(t)\right|^2\right]$ is the average receive power of the signal.

Assuming that the receive power stays constant during the symbol duration, the instantaneous signal to noise ratio per symbol is $P\left|a(t)\right|^2 T \big/ N_0$, where T is the symbol duration and N_0 is the spectral density of the additive noise power at the receiver.

We represent by the function $r(|a|)$ of the normalized channel amplitude gain's magnitude $|a|$, the rate adaptation policy. The value $r(|a|)$ represents the instantaneous symbol rate chosen in accordance with the policy when the channel amplitude gain is $|a|$. Applying policy $r\left(\left|a\left(t\right)\right|\right)$ to the symbol-by-symbol adaptation of the symbol duration would be to use as the duration of the ith symbol $1/r\left(\left|a\left(t_i\right)\right|\right)$ (the reciprocal of the rate), where t_i is the beginning of the ith symbol interval. That is, the channel gain at the beginning of each symbol interval is used as the 'representative' channel gain for that symbol. This application would coincide with policy $r\left(\left|a\left(t\right)\right|\right)$ under the assumption that the channel gain stays constant during each symbol interval, which is commonly assumed for such studies as this [3]. We will often use set notation $\{r(\alpha)|\alpha > 0\}$ to represent a rate adaptation policy. Policy $\{r(\alpha)|\alpha > 0\}$ results in average throughput, $E\left[r\left(|a|\right)\right] = \int r\left(\alpha\right) f_{|a|}\left(\alpha\right) d\alpha$, where $f_{|a|}$denotes the probability density function (pdf) of the normalized channel amplitude gain $\left|a\left(t\right)\right|$. (We assume that the time-varying channel amplitude gain is a mean-ergodic process.) Thus, the maximum-throughput policy is derived through the following maximiza-

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tion:

$$\max_{\{r(\alpha)\}} \quad \int r(\alpha) f_{|a|}(\alpha)\, d\alpha$$
$$\text{subject to} \quad \frac{\int r(\alpha) h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha}{\int r(\alpha) f_{|a|}(\alpha)\, d\alpha} \le \varepsilon \quad (1)$$

where the left-hand side of the inequality constraint is the average SEP, P is the average receive power, N_0 is spectral density of the additive noise power at the receiver, and function $h(x)$ is the SEP corresponding to the instantaneous signal-to-noise ratio, x, per symbol. Different modulation schemes have different SEP functions, $h(\cdot)$. We assume that $h(\cdot)$ is a monotonically decreasing and convex function, which is valid for all useful modulation schemes. Obviously, there is an implicit constraint, $r(\alpha) > 0,\ \forall\alpha > 0$ on policy $\{r(\alpha)|\alpha > 0\}$.

It is intuitive and can also be proven that the maximum policy in criterion (1) satisfies the constraint with equality. Therefore, the maximum-throughput policy can be derived through the following maximization:

$$\max_{\{r(\alpha)\}} \quad \int r(\alpha) f_{|a|}(\alpha)\, d\alpha \quad (2)$$
$$\text{subject to} \quad \int r(\alpha) h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha = \varepsilon \int r(\alpha) f_{|a|}(\alpha)\, d\alpha.$$

We consider the Lagrangian associated with (2),

$$L(\{r(\alpha)|\alpha > 0\}, \lambda) = \int r(\alpha) f_{|a|}(\alpha)\, d\alpha + \lambda \left[\begin{array}{l} \varepsilon \int r(\alpha) f_{|a|}(\alpha)\, d\alpha - \\ \int r(\alpha) h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha \end{array} \right] \quad (3)$$

Using convexity of $h(\cdot)$, we can prove that Lagrangian $L(\{r(\alpha)|\alpha > 0\}, \lambda)$ is concave in the space of feasible policies for each non-negative λ. Therefore, for a given non-negative Lagrange multiplier λ, a feasible policy $\{r(\alpha)|\alpha > 0\}$ that satisfies

$$\frac{\partial L}{\partial r(\alpha)}(\{r(\alpha)|\alpha > 0\}, \lambda) = 0, \ \forall\alpha > 0 \quad (4)$$

maximizes Lagrangian L in (3). (All partial derivatives of a concave function vanish.) From (3) we can derive, for each α,

$$\begin{aligned} &\frac{\partial L}{\partial r(\alpha)}(\{r(\alpha)|\alpha > 0\}, \lambda) \quad (5)\\ &= f_{|a|}(\alpha) + \lambda \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha) - \\ r(\alpha) h'\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) \left\{\frac{\partial}{\partial r(\alpha)}\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right)\right\} f_{|a|}(\alpha) \end{array} \right] \\ &= f_{|a|}(\alpha) + \lambda \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha) \\ + h'\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) \frac{\alpha^2 P}{N_0 r(\alpha)} f_{|a|}(\alpha) \end{array} \right]. \end{aligned}$$

Therefore, (4) is equivalent to:

$$\begin{aligned} &\frac{\partial L}{\partial r(\alpha)}(\{r(\alpha)|\alpha > 0\}, \lambda) \\ &= f_{|a|}(\alpha) + \lambda \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha) \\ + h'\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) \frac{\alpha^2 P}{N_0 r(\alpha)} f_{|a|}(\alpha) \end{array} \right] \\ &= 0, \qquad \forall\alpha > 0 \end{aligned}$$

or equivalently,

$$\frac{1}{\lambda} + \varepsilon = h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) - h'\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) \frac{\alpha^2 P}{N_0 r(\alpha)}, \ \forall\alpha > 0 \quad (6)$$

$1/\lambda + \varepsilon$ on the left-hand side of (6) is constant over α, so we can observe that the policy $\{r(\alpha)|\alpha > 0\}$, to satisfy (6), must be such that the right-hand side of (6) is constant over α. We also note that policy $\{r(\alpha) = q\alpha^2 | \alpha > 0\}$, for any value of q, makes the right-hand side of (6) constant over α, as seen in the following:

$$\begin{aligned} &h\left(\frac{\alpha^2 P}{N_0 q\alpha^2}\right) - h'\left(\frac{\alpha^2 P}{N_0 q\alpha^2}\right) \frac{\alpha^2 P}{N_0 q\alpha^2} \quad (7)\\ &= h\left(\frac{P}{N_0 q}\right) - h'\left(\frac{P}{N_0 q}\right) \frac{P}{N_0 q}. \qquad \forall\alpha > 0 \end{aligned}$$

Thus, a policy in the form of $\{r(\alpha) = q\alpha^2 | \alpha > 0\}$ (a policy that keeps instantaneous symbol rate proportional to the power gain of the channel), is of interest to us because in light of (6) and (7), if we choose the value of q such that

$$\frac{1}{\lambda} + \varepsilon = h\left(\frac{P}{N_0 q}\right) - h'\left(\frac{P}{N_0 q}\right) \frac{P}{N_0 q}, \ \forall\alpha > 0, \quad (8)$$

then the policy $\{r(\alpha) = q\alpha^2 \mid \alpha > 0\}$ maximizes $L(\{r(\alpha)| \alpha > 0\}, \lambda)$. Now, among the polices in the form of $\{r(\alpha) = q\alpha^2 | \alpha > 0\}$ with some value of q, we consider the policy with $q = q^*$ such that $h(P/N_0 q^*) = \varepsilon$, i.e.,

$$q^* = \frac{P}{N_0 h^{-1}(\varepsilon)} \quad . \quad (9)$$

This policy $\{r^*(\alpha) \equiv q^*\alpha^2 | \alpha > 0\}$ also satisfies the equality constraint in (2), as seen in the following:

$$\begin{aligned} &\int r(\alpha) h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha) d\alpha \Big|_{r(\alpha) = q^*\alpha^2} \\ &= \int q^* \alpha^2 h\left(\frac{P}{N_0 q^*}\right) f_{|a|}(\alpha) d\alpha \\ &\underset{\text{from (9)}}{=} \int \frac{P}{N_0 h^{-1}(\varepsilon)} \alpha^2 h\left(h^{-1}(\varepsilon)\right) f_{|a|}(\alpha)\, d\alpha \\ &= \int \frac{P}{N_0 h^{-1}(\varepsilon)} \alpha^2 \varepsilon f_{|a|}(\alpha)\, d\alpha \\ &= \varepsilon \int r(\alpha) f_{|a|}(\alpha)\, d\alpha \Big|_{r(\alpha) = q^*\alpha^2}. \quad (10) \end{aligned}$$

Therefore, from (3) and (10) we have

$$L(\{r^*(\alpha)|\alpha > 0\}, \lambda) = \int r^*(\alpha) f_{|a|}(\alpha)\, d\alpha, \ \forall\lambda. \quad (11)$$

Now, we pick Lagrange multiplier

$$\lambda = -\frac{N_0 q^*}{P} \Big/ h'\left(\frac{P}{N_0 q^*}\right) \equiv \lambda^* \quad (12)$$

and consider Lagrangian (3) for this multiplier λ^*:

$$L(\{r(\alpha)|\alpha > 0\}, \lambda^*) = \int r(\alpha) f_{|a|}(\alpha)\, d\alpha + \lambda^* \left[\begin{array}{l} \varepsilon \int r(\alpha) f_{|a|}(\alpha)\, d\alpha - \\ \int r(\alpha) h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha \end{array} \right]$$

Note that $\lambda^* \geq 0$ because SEP function $h(\cdot)$ is monotonically decreasing, so $L(\{r(\alpha)|\alpha > 0\}, \lambda^*)$ is a concave function of vector $\{r(\alpha)|\alpha > 0\}$, which follows from convexity of $h(\cdot)$. As in (5), we have

$$\frac{\partial L}{\partial r(\alpha)}(\{r(\alpha)|\alpha > 0\}, \lambda^*) = f_{|a|}(\alpha) + \lambda^* \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha) + \\ h'\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) \frac{\alpha^2 P}{N_0 r(\alpha)} f_{|a|}(\alpha) \end{array} \right]$$

and thus

$$\begin{aligned} &\frac{\partial L}{\partial r(\alpha)}(\{r(\alpha)|\alpha > 0\}, \lambda^*)|_{\{r(\alpha)|\alpha>0\}=\{r^*(\alpha)|\alpha>0\}} \\ &= f_{|a|}(\alpha) + \lambda^* \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(\frac{\alpha^2 P}{N_0 q^* \alpha^2}\right) f_{|a|}(\alpha) \\ +h'\left(\frac{\alpha^2 P}{N_0 q^* \alpha^2}\right) \frac{\alpha^2 P}{N_0 q^* \alpha^2} f_{|a|}(\alpha) \end{array} \right] \\ &= f_{|a|}(\alpha) + \lambda^* \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(\frac{P}{N_0 q^*}\right) f_{|a|}(\alpha) \\ +h'\left(\frac{P}{N_0 q^*}\right) \frac{P}{N_0 q^*} f_{|a|}(\alpha) \end{array} \right] \\ &=_{\text{from (9)}} f_{|a|}(\alpha) + \lambda^* \left[\begin{array}{l} \varepsilon f_{|a|}(\alpha) - h\left(h^{-1}(\varepsilon)\right) f_{|a|}(\alpha) \\ +h'\left(\frac{P}{N_0 q^*}\right) \frac{P}{N_0 q^*} f_{|a|}(\alpha) \end{array} \right] \\ &= f_{|a|}(\alpha) + \lambda^* h'\left(\frac{P}{N_0 q^*}\right) \frac{P}{N_0 q^*} f_{|a|}(\alpha) \\ &=_{\text{from (12)}} f_{|a|}(\alpha) - f_{|a|}(\alpha) \\ &= 0, \qquad \forall \alpha > 0. \end{aligned}$$

Because partial derivatives $\frac{\partial L}{\partial r(\alpha)}(\{r(\alpha)|\alpha > 0\}, \lambda^*)$ vanish at vector $\{r^*(\alpha) \equiv q^* \alpha^2 |\alpha > 0\}$ and $L(\{r(\alpha)|\alpha > 0\}, \lambda^*)$ is concave of vector variable $\{r(\alpha)|\alpha > 0\}$, we can see that policy $\{r^*(\alpha) \equiv q^* \alpha^2 |\alpha > 0\}$ maximizes Lagrangian

$$L(\{r(\alpha)|\alpha > 0\}, \lambda^*) = \int r(\alpha) f_{|a|}(\alpha)\, d\alpha + \lambda^* \left[\begin{array}{l} \varepsilon \int r(\alpha) f_{|a|}(\alpha)\, d\alpha - \\ \int r(\alpha) h\left(\frac{\alpha^2 P}{N_0 r(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha \end{array} \right].$$

That is,

$$\begin{aligned} &L(\{r^*(\alpha)|\alpha > 0\}, \lambda^*) \\ &\geq L(\{r(\alpha)|\alpha > 0\}, \lambda^*), \forall \text{policy } \{r(\alpha)|\alpha > 0\} \in \pi_a \end{aligned} \tag{13}$$

where π_a denotes the set of all policies. The set of policies that satisfy the equality constraint (2) is a subset of π_a, that is, $\pi_a \supset \pi_e$, where

$$\pi_e \equiv \left\{ \{r_e(\alpha)|\alpha > 0\} \;\middle|\; \begin{array}{l} \int r_e(\alpha) h\left(\frac{\alpha^2 P}{N_0 r_e(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha \\ = \varepsilon \int r_e(\alpha) f_{|a|}(\alpha)\, d\alpha \end{array} \right\}.$$

Thus, from (13) and the fact $\pi_a \supset \pi_e$, we have

$$\begin{aligned} &L(\{r^*(\alpha)|\alpha > 0\}, \lambda^*) \\ &\geq L(\{r_e(\alpha)|\alpha > 0\}, \lambda^*), \forall \text{policy } \{r_e(\alpha)|\alpha > 0\} \in \pi_e. \end{aligned} \tag{14}$$

For an arbitrary policy $\{r_e(\alpha)|\alpha > 0\} \in \pi_e$, we have

$$\begin{aligned} &L(\{r_e(\alpha)|\alpha > 0\}, \lambda^*) \\ &= \int r_e(\alpha) f_{|a|}(\alpha)\, d\alpha + \lambda^* \left[\varepsilon \int r_e(\alpha) f_{|a|}(\alpha)\, d\alpha - \right. \\ &\left. \int r_e(\alpha) h\left(\frac{\alpha^2 P}{N_0 r_e(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha \right] = \int r_e(\alpha) f_{|a|}(\alpha)\, d\alpha. \end{aligned} \tag{15}$$

The last equality is because $\{r_e(\alpha)|\alpha > 0\}$ satisfies the equality constraint in (2):

$$\int r_e(\alpha) h\left(\frac{\alpha^2 P}{N_0 r_e(\alpha)}\right) f_{|a|}(\alpha)\, d\alpha = \varepsilon \int r_e(\alpha) f_{|a|}(\alpha)\, d\alpha.$$

As mentioned before, policy $\{r^*(\alpha)|\alpha > 0\}$ satisfies the inequality constraint in (2), so from (3) and (15) we have

$$\begin{aligned} &L(\{r^*(\alpha)|\alpha > 0\}, \lambda^*) \\ &= \int r^*(\alpha) f_{|a|}(\alpha) d\alpha + \lambda^* \left[\varepsilon \int r^*(\alpha) f_{|a|}(\alpha) d\alpha \right. \\ &\qquad \left. - \int r^*(\alpha) h\left(\frac{\alpha^2 P}{N_0 r^*(\alpha)}\right) f_{|a|}(\alpha) d\alpha \right] \\ &= \int r^*(\alpha) f_{|a|}(\alpha)\, d\alpha. \end{aligned} \tag{16}$$

Relations (14), (15), and (16) lead to

$$\begin{aligned} &\int r^*(\alpha) f_{|a|}(\alpha)\, d\alpha \\ &\geq \int r^e(\alpha) f_{|a|}(\alpha)\, d\alpha, \ \forall \{r^e(\alpha)|\alpha > 0\} \in \pi_e. \end{aligned} \tag{17}$$

Relation (17) states that policy $\{r^*(\alpha) = q^* \alpha^2 |\alpha > 0\}$, which satisfies the equality constraint in (2), is a solution to maximization (2). In words, the optimal instantaneous symbol rate is directly proportional to the channel's power gain with slope q^*, where q^* can be obtained from (9). The maximal average throughput resulting from this policy is then

$$\begin{aligned} \bar{r}_S &\equiv \int r^*(\alpha) f_{|a|}(\alpha)\, d\alpha = \int q^* \alpha^2 f_{|a|}(\alpha)\, d\alpha \\ &= E\left[q^* |a(t)|^2\right] = q^* =_{\text{from (9)}} \frac{P}{N_0 h^{-1}(\varepsilon)}. \end{aligned} \tag{18}$$

(Recall that gain $a(t)$ by definition is normalized so that $E\left[|a(t)|^2\right] = 1$.)

Now we consider the maximal average symbol rate for frame-by-frame adaptation. Unlike symbol-by-symbol adaptation, the symbol rate is fixed during each frame in the case of frame-by-frame adaptation. The rate decision, r, for a frame will have an associated error probability of symbols transmitted within that frame. For the sake of simplicity, we assume that the adaptive system is operating over channels highly fluctuating in time relative to the frame duration, that is, to such an extent that the empirical distribution of the fading channel gain in each data frame can be approximated by the ensemble distribution of the channel gain. Under this assumption, each frame's symbol error probability associated with a particular symbol rate decision r for that frame can be approximated by the symbol error probability for the entire time horizon when the symbol rate is fixed at r at all times. (In other words, the frame-by-frame rate adaptation to the channel gain in this case is not useful for adapting to fading, although it can be useful for slower time-varying channel fluctuations such as shadowing or path loss due to distance.) Therefore, the frame-by-frame adaptive system's maximal average rate for a given SEP constraint can be approximated by the maximal fixed (non-adaptive) rate for the same constraint. In the fixed-symbol-rate system, the symbol duration is fixed all the time, so the problem of maximizing the error-constrained throughput can be viewed as the constrained optimization problem obtained by adding an additional constraint $r(\alpha) = r, \ \forall\alpha$ to problem (1), that is,

$$\max_r \quad r$$
$$\text{subject to} \quad \int h\left(\frac{\alpha^2 P}{N_0 r}\right) f_{|a|}(\alpha) d\alpha \le \varepsilon.$$

Therefore, the maximum throughput is obtained by finding the maximum value of r, say $\bar{r}_F$, that satisfies

$$\int h\left(\alpha^2 P / N_0 \bar{r}_F\right) f_{|a|}(\alpha)\, d\alpha = \varepsilon. \tag{19}$$

In this paper, our main interest is the ratio $\mathcal{R} \equiv \bar{r}_S / \bar{r}_F$, which we call "throughput gain". Now, we analyze this ratio. From (9) and (18), we have

$$\bar{r}_S \equiv E\left[q^* |a(t)|^2\right] = \frac{P}{N_0 h^{-1}(\varepsilon)} E\left[|a(t)|^2\right] = \frac{P}{N_0 h^{-1}(\varepsilon)}. \tag{20}$$

For analyzing $\bar{r}_F$, we define a function

$$H(z) \equiv \int_0^\infty h(z\xi) f_{|a|^2}(\xi)\, d\xi \tag{21}$$

where $f_{|a|^2}(\xi)$ is the probability density function of the normalized power gain $|a(t)|^2$. Then, from (19) and (21), we have that $\bar{r}_F$ satisfies

$$\varepsilon = \int h\left(\frac{\xi P}{N_0 \bar{r}_F}\right) f_{|a|^2}(\xi)\, d\xi = H\left(\frac{P}{N_0 \bar{r}_F}\right). \tag{22}$$

Because that $h(\cdot)$ is a monotonically decreasing, $H(z)$ is also monotonically decreasing with z, and thus we can define the inverse function, $H^{-1}(\cdot)$, of $H(\cdot)$. From (22) we have

$$\bar{r}_F = \frac{P}{N_0 H^{-1}(\varepsilon)}. \tag{23}$$

Then, from (20) and (23), we have

$$\mathcal{R} \equiv \bar{r}_S / \bar{r}_F = \frac{H^{-1}(\varepsilon)}{h^{-1}(\varepsilon)} \tag{24}$$

III. NUMERICAL STUDY OF IDEAL ADAPTATION

In this paper, our main interest is the ratio $\mathcal{R} \equiv \bar{r}_S / \bar{r}_F$, which we call "throughput gain". In this section, we numerically study the ideal throughput gain (24) for different modulation schemes. The numerical study was conducted for Rayleigh fading channels. Thus, the pdf of the normalized fading power gain $|a(t)|^2$ is

$$f_{|a|^2}(\xi) = \exp(-\xi), \ \xi > 0. \tag{25}$$

A. Example: Noncoherent binary FSK, Rayleigh

In a simple example of the noncoherent binary FSK (frequency shift keying), we have bit error probability curve,

$$h(x) = \frac{1}{2}\exp\left(-\frac{x}{2}\right) \tag{26}$$

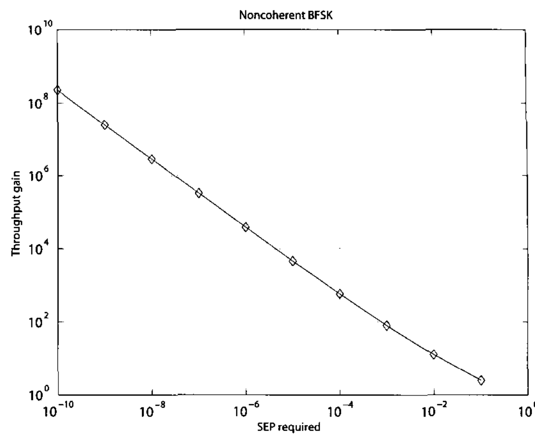

Fig. 1. $\mathcal{R}$ (throughput gain) vs. SEP requirement for BFSK.

so we have

$$h^{-1}(\varepsilon) = -2\ln(2\varepsilon) \tag{27}$$

$$H(z) = \int_0^\infty \frac{1}{2}\exp\left(-\frac{z\xi}{2}\right)\exp(-\xi)\, d\xi = \frac{1}{z+2} \tag{28}$$

$$H^{-1}(\varepsilon) = \frac{1}{\varepsilon} - 2 \tag{29}$$

$$\mathcal{R} \equiv \bar{r}_S / \bar{r}_F = \frac{H^{-1}(\varepsilon)}{h^{-1}(\varepsilon)} = \frac{1 - 2\varepsilon}{-2\varepsilon \ln(2\varepsilon)}. \tag{30}$$

Fig. 1 plots throughput gain $\mathcal{R}$ against the required SEP, ε.

B. Example: MPAM, Rayleigh

Consider the M-ary pulse amplitude modulation with M one-dimensional signal points with value

$$(2m - 1 - M)\sqrt{\frac{3P}{(M^2-1)\, r}}, \ m = 1, 2, \cdots, M \tag{31}$$

for symbol duration $1/r$. We note that the average power of this symbol is

$$\frac{r}{M}\sum_{m=1}^{M}(2m - 1 - M)^2 \frac{3P}{(M^2-1)\, r} = P \tag{32}$$

for this symbol, assuming that each of M symbols is equally likely. For such *MPAM,* from [4] we have

$$h(x) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6x}{M^2-1}}\right) \tag{33}$$

where x is the average received signal to noise ratio per symbol. Then,

$$h^{-1}(\varepsilon) = \frac{(M^2-1)}{3}\left[erfcinv\left(\frac{\varepsilon M}{M-1}\right)\right]^2 \tag{34}$$

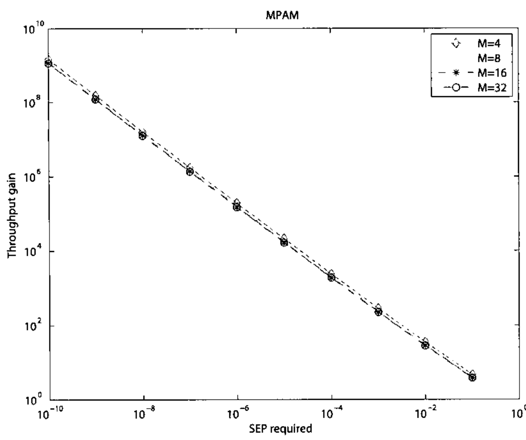

Fig. 2. $\mathcal{R}$ (throughput gain) vs. SEP requirement for MPAM.

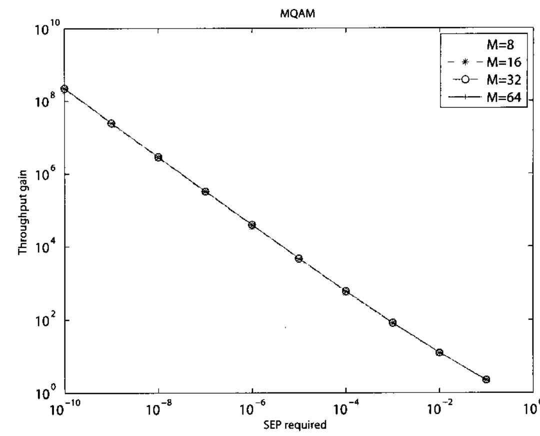

Fig. 3. $\mathcal{R}$ (throughput gain) vs. SEP requirement for MQAM.

where we denote by *erfcinv* the inverse function of $erfc(y) \equiv 2Q(\sqrt{2}y)$. Also,

$$
\begin{aligned}
&H(z)\\
&=\int_0^\infty \frac{(M-1)}{M} erfc\left(\sqrt{\frac{3z}{M^2-1}}\sqrt{\alpha}\right)\exp\left(-\alpha\right)d\alpha\\
&=_{(\text{from [5, 7.4.19]})} \frac{(M-1)}{M} - \frac{(M-1)}{M}\frac{\sqrt{3z}}{\sqrt{3z+M^2-1}}
\end{aligned}
$$

$$
H^{-1}(\varepsilon) = \frac{(M^2-1)}{3}\frac{(M-1-M\varepsilon)^2}{M\varepsilon\left\{2(M-1)-M\varepsilon\right\}}. \quad (35)
$$

Therefore,

$$
\mathcal{R} \equiv \frac{\bar{r}_S}{\bar{r}_F} = \frac{H^{-1}(\varepsilon)}{h^{-1}(\varepsilon)} = \frac{\left(\frac{M-1}{M}-\varepsilon\right)^2}{\varepsilon\left(2\frac{M-1}{M}-\varepsilon\right)}\frac{1}{\left[erfcinv\left(\frac{\varepsilon M}{M-1}\right)\right]^2}.
$$

Fig. 2 plots throughput gain $\mathcal{R}$ against the required SEP, ε. We note that the throughput gain curves are similar for 4PAM, 8PAM, 16PAM, and 32PAM.

C. Example: MQAM, $M = 2^k$, even k, Rayleigh

The SEP of *MQAM* for $M = 2^k$ [4, p. 276] is

$$
h(x) = 1 - \left[1 - \frac{\left(\sqrt{M}-1\right)}{\sqrt{M}} erfc\left(\sqrt{\frac{3x}{2(M-1)}}\right)\right]^2 \quad (36)
$$

so

$$
h^{-1}(\varepsilon) = \frac{2(M-1)}{3}\left[erfcinv\left(\frac{\sqrt{M}\left(1-\sqrt{1-\varepsilon}\right)}{\sqrt{M}-1}\right)\right]^2.
$$

Also, from [6, 7] we have

$$
\begin{aligned}
h(x) &= \tfrac{4}{\pi}\left(1-\tfrac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\exp\left[-\tfrac{3}{2(M-1)\sin^2\theta}x\right]d\theta\\
&\quad -\tfrac{4}{\pi}\left(1-\tfrac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\exp\left[-\tfrac{3}{2(M-1)\sin^2\theta}x\right]d\theta.
\end{aligned} \quad (37)
$$

From this, we derive

$$
\begin{aligned}
H(z) &= \int_0^\infty h(z\xi)\exp\left(-\xi\right)d\xi \quad (38)\\
&= \frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)\int_0^{\pi/2}\frac{\sin^2(\theta)}{\sin^2(\theta)+c(z)}d\theta\\
&\quad -\frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2\int_0^{\pi/4}\frac{\sin^2(\theta)}{\sin^2(\theta)+c(z)}d\theta
\end{aligned}
$$

where $c(z) = 3z/\{2(M-1)\}$. Now, [7, 5A.8, 5A.4a, 5A.13] provides closed forms of the integrals in (38), and thus we have

$$
\begin{aligned}
H(z) &= 2\left(1-\frac{1}{\sqrt{M}}\right)\left(1-\sqrt{\frac{3z}{3z+2(M-1)}}\right)\\
&\quad -\left(1-\frac{1}{\sqrt{M}}\right)^2 \quad (39)\\
&\quad \left[1-\sqrt{\frac{3z}{3z+2(M-1)}}\left(\frac{4}{\pi}\arctan\sqrt{1+\frac{2(M-1)}{3z}}\right)\right].
\end{aligned}
$$

In order to compute $H^{-1}(\varepsilon)$ numerically, we found the zero of function $H(z)-\varepsilon$ by using a zero-finding routine. Fig. 3 plots throughput gain $\mathcal{R}$ against the required SEP, ε. We note that the throughput gain curves for M=8, 16, 32, and 64 are almost identical.

IV. CONCLUSIONS

Figs. 1–3 show numerical results of P for a few modulation schemes for different values of the SEP (fidelity) requirement ε. It is noteworthy that Figs. 1–3 are all similar. Also, we have numerically computed $\mathcal{R} \equiv \bar{r}_S/\bar{r}_F$ values in the same range of required SEPs for other modulation schemes, MPSK, DPSK, NC-FSK, and MSK. The throughput gain curves for these modulations are very close to Figs. 1–3. Thus, we conclude that the throughput 'gains' due to SBS adaptation for different modulations are very close to one another, even though the error-constrained throughput may differ from modulation to modulation because of different SEP functions $h(\cdot)$. Thus, in a wide range of communication systems and SEP requirements, Figs. 1–3 indicate that **SBS adaptation can achieve a throughput gain by orders of magnitude**. We also observe that the throughput gain $\mathcal{R}$ is larger for the smaller ε.

Appendix

The numerical study in section III provides good insights into the potential benefit of the fast adaptation. However, the SBS adaptation policy $\{r^*(\alpha) = q^*\alpha^2 \,|\alpha > 0\}$ is ideal in two respects. First, the fading gain in the Rayleigh model is unbounded, so the instantaneous rate $r^*(\alpha) = q^*\alpha^2$ is unbounded. This is impractical because the symbol rate is bounded due to a fixed and finite signal bandwidth. Also, $r^*(\alpha) = q^*\alpha^2$ can become arbitrary small because α^2 can be arbitrarily close to zero, that is, the symbol duration $1/r^*(\alpha)$ can become arbitrarily long. For practical purpose, the rate adaptation policy must have additional constraint that the symbol durations must be in a specific range, that is,

$$r_l \le r^*(\alpha) \le r_u \tag{40}$$

for some r_l and r_u. Under this additional constraint, an optimal policy $\{r(\alpha)\}$ can be numerically computed. However, for simplicity of analysis and good insight, we now consider a suboptimal policy

$$r(\alpha) = \begin{cases} 0, & \alpha < a_l \\ q^*\alpha^2, & a_l \le \alpha \le a_u \\ q^*\alpha_u^2, & \alpha > a_u \end{cases} \tag{41}$$

which satisfies both constraint (1) and (40). Under this suboptimal policy, the average throughput of the SBS adaptation is

$$\int_{a_l^2}^{a_u^2} q^*\xi f_{|a|^2}(\xi)d\xi + \int_{a_u^2}^{\infty} q^* a_u^2 f_{|a|^2}(\xi)d\xi \tag{42}$$
$$= \frac{P}{N_0 h^{-1}(\varepsilon)}\left[a_l^2 \exp\left(-a_l^2\right) + \exp\left(-a_l^2\right) - \exp\left(-a_u^2\right)\right].$$

From (23) and (42), the throughput ratio for this suboptimal policy is

$$\frac{H^{-1}(\varepsilon)}{h^{-1}(\varepsilon)}\left[a_l^2 \exp\left(-a_l^2\right) + \exp\left(-a_l^2\right) - \exp\left(-a_u^2\right)\right].$$

For $a_l^2 = 0.01$ and $a_u^2 = 100$, the factor $a_l^2 \exp\left(-a_l^2\right) + \exp\left(-a_l^2\right) - \exp\left(-a_u^2\right) \approx 1.0$. For $a_l^2 = 0.1$ and $a_u^2 = 100$, we have

$$a_l^2 \exp\left(-a_l^2\right) + \exp\left(-a_l^2\right) - \exp\left(-a_u^2\right) = 0.9953.$$

For $a_l^2 = 1.0$ and $a_u^2 = 100$, we have

$$a_l^2 \exp\left(-a_l^2\right) + \exp\left(-a_l^2\right) - \exp\left(-a_u^2\right) = 0.7358.$$

Therefore, in comparison with (24), for a reasonable range of rate adaptability, we can conclude that the throughput ratio for even suboptimal policy (41) yields a huge the throughput gain, considering the huge gains exhibited in Figs. 1–3. Therefore, even with the constraint (40), SBS adaptation can achieve a throughput gain by orders of magnitude.

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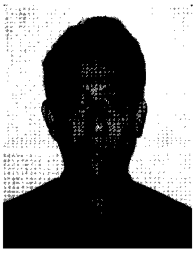

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