Journal of the Korean Statistical Society (2007), **36**: *3, pp 411–421*

NEW EXPRESSIONS FOR REPEATED LOWER TAIL INTEGRALS OF THE NORMAL DISTRIBUTION

CHRISTOPHER S. WITHERS[1] AND SARALEES NADARAJAH[2]

ABSTRACT

The recent work by the authors (see, Withers, 1999; Withers and McGavin, 2006; Withers and Nadarajah, 2006) provided new expressions for repeated upper tail integrals of the univariate normal density and so also for the general Hermite function. Here we derive new expressions for repeated lower tail integrals of the same. The calculations involve the use of Moran's L-function and the Airy function. In particular, the Hermite functions are expressed in terms of Moran's L-function and vice versa.

AMS 2000 subject classifications. Primary 60E10; Secondary 60F99.
Keywords. Airy functions, Hermite functions, Hermite polynomials, Mills' ratio, normal distribution, repeated integrals.

1. INTRODUCTION

Let $\phi(x)$ and $\Phi(x)$ be the density and distribution of a unit normal random variable $N \sim \mathcal{N}(0,1)$:

$$\phi(x) = (2\pi)^{-1/2} e^{-x^2/2},\ \Phi(x) = P(N \le x) = \int_{-\infty}^{x} \phi(y)dy.$$

Denote by D the differential operator and by $(-D)^{-1}$ the upper integral operator

$$(-D)^{-1} f(x) = \int_{x}^{\infty} f(x)dx.$$

For an integer n, set

$$H_n = H_n(x) = e^{x^2/2}(-D)^n e^{-x^2/2}.$$

Received May 2006; accepted May 2007.
[1]Applied Mathematics Group, Industrial Research Limited Lower Hutt, New Zealand (e-mail: c.withers@irl.cri.nz)
[2]Corresponding author. School of Mathematics, University of Manchester, Manchester M13 9PL, UK (e-mail: saralees.nadarajah@manchester.ac.uk)

So for $n = 1, 2, \ldots,$

$$H_{-n}(x) = \phi(x)^{-1} \int_x^\infty dx_1 \int_{x_1}^\infty dx_2 \cdots \int_{x_{n-1}}^\infty dx_n \phi(x_n),$$
$$H_{-n}(-x) = \phi(x)^{-1} \int_{-\infty}^x dx_1 \int_{-\infty}^{x_1} dx_2 \cdots \int_{-\infty}^{x_{n-1}} dx_n \phi(x_n).$$

Here H_n is known as the *Hermite polynomial* for $n \geq 0$ and as the *Hermite function* for $n < 0$. H_{-1} is known as *Mills' ratio*: $H_{-1}(x) = \Phi(-x)/\phi(x)$. These repeated integrals are directly useful in many applications. Three statistical applications given by Fisher (1931) are:

1. in the calculation of moments of truncated normal distribution,
2. in the expression of the non–central t density and
3. in the posterior distribution of a Poisson variate with chi–squared prior for the squared mean parameter of the Poisson variate.

Recently, Goodall and Mardia (1991) and Mardia (1998) have shown that the repeated integrals occur also in the calculation of shape distributions.

Withers (2000) showed that the Hermite polynomials are the moments of $x + iN$ where $i = \sqrt{-1}$:

$$H_n(x) = E(x + iN)^n \text{ for } n = 0, 1, 2, \ldots, \tag{1.1}$$

giving the well known result

$$H_n = \sum_{0 \leq k \leq n/2} (-1)^k h_{nk} x^{n-2k}$$

for $h_{nk} = n! 2^{-k} / \{k!(n-2k)!\}$, where k is an integer. In particular, $H_0 = 1$, $H_1 = x$, $H_2 = x^2 - 1$, $H_3 = x^3 - 3x$ and $H_4 = x^4 - 6x^2 + 3$. One can view (1.1) as a linear transformation of x^n to $H_n(x)$, say

$$H_n(x) = \mathcal{K}_0 x^n, \text{ where } \mathcal{K}_0 f(x) = Ef(x + iN).$$

The inverse linear transformation of $H_n(x)$ to x^n is

$$x^n = \mathcal{K}_1 H_n(x) \text{ for } n = 0, 1, 2, \ldots, \text{ where } \mathcal{K}_1 f(x) = Ef(x + N).$$

(This follows by replacing (x, N) in (1.1) by $(x + N_1, N_2)$, where N_1, N_2 are independent $\mathcal{N}(0, 1)$, and using the well known result

$$EN_c^n = 0 \text{ for } n = 1, 2, \ldots,$$

where $N_c = N_1 + iN_2$ is known as the complex normal $\mathcal{CN}(0, 2)$, see, Miller, 1974, 1975). So $\mathcal{K}_0, \mathcal{K}_1$ take a polynomial of degree n to others of degree n and a sufficiently convergent power series to other power series. For example

$$\mathcal{K}_0 e^{tx} = e^{tx-t^2/2}, \ \mathcal{K}_1 e^{tx} = e^{tx+t^2/2}.$$

Note that $\mathcal{K}_1$ is a linear integral operator with kernel $\phi(x-y)$ not in L_2: $\int\int \phi(x-y)^2 dxdy = \infty$, so Fredholm theory does not apply.

Withers and McGavin (2006) showed that the Hermite functions are the negative moments of $x + iN$:

$$H_{-n}(x) = \mathcal{K}_0 x^{-n} = E(x + iN)^{-n} \text{ for } x > 0, \ n = 1, 2, \ldots.$$

That is, (1.1) also holds when n is a *negative* integer and x is real and positive. So

$$E\ (x + iN)^{-n} = \text{sign}(x)^n \ H_{-n}(|x|) \text{ for real } x \neq 0, \ n = 1, 2, \ldots.$$

However, $\mathcal{K}_1 x^{-n}$ does not exist for $n = 1, 2, \ldots$, Withers and McGavin (2006) also showed that for $x > 0$ and $n \geq 0$,

$$\begin{aligned} n! H_{-n-1}(x) &= (2\pi)^{1/2} E I(N > 0) N^n e^{-xN} \\ &= (-D)^n H_{-1}(x) \\ &= (-1)^n (H_n^*(x) H_{-1}(x) - P_{n-1}(x)), \end{aligned} \tag{1.2}$$

where $I(A)$ is the indicator function, $P_{-1} = 0$ and for $n \geq 0$, $H_n^* = H_n^*(x)$ and $P_n = P_n(x)$ are polynomials of degree n:

$$H_n^*(x) = e^{-x^2/2} D^n e^{x^2/2} = \mathcal{K}_1 x^n = E\ (x + N)^n = \sum_{0 \leq k \leq n/2} h_{nk} x^{n-2k},$$

$$P_n(x) = \sum_{0 \leq k \leq n/2} P_{nk} x^{n-2k}/(n-2k)! \text{ for } P_{nk} = \sum_{j=0}^{k} (n-k+j)! 2^{-j}/j!,$$

$$H_0^* = 1, \ H_1^* = x, \ H_2^* = x^2 + 1, \ H_3^* = x^3 + 3x, \ H_4^* = x^4 + 6x^2 + 3, \ldots,$$

$$P_0 = 1, \ P_1 = x, \ P_2 = x^2 + 2, \ P_3 = x^3 + 5x, \ P_4 = x^4 + 9x^2 + 8, \ldots.$$

Here $H_n^*(x)$ is known as the *modified Hermite polynomial* and was introduced by Fisher (1931). Since $\mathcal{K}_0$ is the inverse of $\mathcal{K}_1$,

$$x^n = \mathcal{K}_0 H_n^*(x) \text{ for } n = 0, 1, \ldots.$$

Note that

$$H_n(x) = i^n E(-ix + N)^n = i^n H_n^*(-ix).$$

However, Fisher (1931) did not give H_{-n} for negative argument. Withers and Nadarajah (2006) showed that for $x > 0$ and $n = 0, 1, \ldots$,

$$\begin{aligned} n! H_{-n-1}(-x) &= n!(-1)^{n+1} H_{-n-1}(x) + H_n^*(x)/\phi(x) \\ &= H_n^*(x) H_{-1}(-x) + P_{n-1}(x) \end{aligned}$$

and

$$H_{-n}(0) = 2^{n/2}\Gamma(n/2+1)/n! \text{ for } n = 1, 2, \ldots. \tag{1.3}$$

Withers and Nadarajah (2006) also established alternative expressions for the Hermite functions $H_{-n}(x)$ given by

$$H_{-n}(x) = x^n \sum_{0 \le j \le n/2} (-x^2)^{-j} a_{nj} J_{n-j}(x) \text{ for } x > 0 \text{ and } n \ge 0,$$

where

$$J_n = J_n(x) = E(x^2 + N^2)^{-n}$$

and a_{nj} is some constant.

In this paper, we derive new expressions for repeated lower tail integrals of the normal distribution. Two functions that will be of help with this investigation are *Moran's L-function* (Moran, 1983) and the *Airy function* (Airy, 1931) defined by

$$Hh_n(x) = (-D)^{-n-1} e^{-x^2/2} = H_{-n-1}(x) e^{-x^2/2}. \tag{1.4}$$

See Fisher (1931) in his introduction to the tables of Hh_n by Airy (1931). Also, Fisher (1931) provided applications of

$$I_n(x) = k_{n-1}(x) = (-D)^{-n-1}\phi(x) = \phi(0) Hh_n(x) = \phi(x) H_{-n-1}(x)$$

to the Student's t, truncated normal and modified Poisson distributions; see Note 1.1 of Withers and McGavin (2006) for some errors.

Our results are organized as follows. In Section 2, we express the Hermite functions in terms of Moran's L-functions and vice versa. Section 3 derives expressions for the repeated lower integral $I^n e^{\pm x^2/2}$, where $If(x) = \int_0^x f(y)dy$.

2. HERMITE FUNCTIONS IN TERMS OF MORAN'S L-FUNCTIONS

Here we express $H_{-n}(x)$ in terms of *Moran's L-functions* and vice versa. We also derive several recurrence relations for the latter and show how they lead to a known result.

Expanding the equation before (A.1) of Withers and McGavin (2006), for $m \geq 1$,

$$H_{-n}(x) = H_{-n} = \sum_{k=0}^{n-1} L_k(x)(-x)^{n-1-k}/\{(n-1-k)!k!\},$$

where, for any k,

$$L_k = L_k(x) = \phi(x)^{-1} \int_x^\infty \phi(y)y^k dy.$$

Conversely expanding $(y-x+x)^k$ gives

$$L_k = \sum_{n=0}^{k} (k)_n x^{k-n} H_{-n-1} \quad \text{for} \quad k \geq 0.$$

Moran's expansion for the distribution of the multivariate normal is also written in terms of these L_k functions (see Moran, 1983; Kotz *et al.*, 2000, p. 141). They satisfy

$$L_0 = H_{-1}, \ L_1 = 1$$

and integrating by parts

$$L_k = x^{k-1} + (k-1)L_{k-2}. \tag{2.1}$$

So for $r = 2, 4, \ldots$,

$$\begin{aligned} L_{r+1} &= x^r + rx^{r-2} + r(r-2)x^{r-4} + \cdots \\ &\quad + r(r-2)(r-4)\cdots 2 \end{aligned}$$

and for $r = 1, 3, 5, \ldots$,

$$\begin{aligned} L_{r+1} &= x^r + rx^{r-2} + r(r-2)x^{r-4} + \cdots \\ &\quad + r(r-2)\cdots 3\cdot 1x + r(r-2)\cdots 3\cdot 1H_{-1}. \end{aligned}$$

Similarly we can express L_{-2k} in terms of H_{-1} and L_{-2k-1} in terms of L_{-1}:

$$\begin{aligned} L_{-2} &= x^{-1} - H_{-1}, \\ L_{-3} &= (x^{-2} - L_{-1})/2, \end{aligned}$$

$$\begin{aligned} L_{-4} &= (x^{-3} - x^{-1} + H_{-1})/3, \\ L_{-5} &= x^{-4}/4 - (x^{-2} - L_{-1})/4 \cdot 2, \\ L_{-6} &= x^{-5}/5 - x^{-3}/5 \cdot 3 + (x^{-1} - H_{-1})/5 \cdot 3 \cdot 1, \\ L_{-7} &= x^{-6}/6 - x^{-4}/6 \cdot 4 + (x^{-2} - L_{-1})/6 \cdot 4 \cdot 2 \end{aligned}$$

and so on. Similarly putting $k = 0, 2, \ldots$ in (2.1), we obtain (26.2.12) of Abramowitz and Stegun (1964), which we write as

$$H_{-1} = \sum_{j=0}^{n} (-1)^j x^{-2j-1} EN^{2j} + (-1)^{n+1} L_{-2n-2}(x) EN^{2n+2} \quad \text{for } \ n \geq 0.$$

3. Repeated Finite Integrals of $e^{\pm x^2/2}$

Analogous to (1.4) let us define the *complementary Airy function* as the repeated lower tail integral

$$G_n(x) = I^n e^{-x^2/2} \ \text{ for } \ If(x) = \int_0^x f(y)\, dy, \ n \geq 0. \tag{3.1}$$

In this section, we derive two expressions for computing (3.1), one based on Taylor series expansion and the other based on numerical integration. A third expression is given by expressing (3.1) in terms of Hh_n and vice versa. We also introduce some modified versions of (3.1) and Hh_n and discuss their computation with extensions for negative n.

One can easily calculate G_n in (3.1) from its Taylor series as follows. For x_0 in R, if $g_0 : R \to R$ is any function with a Taylor series expansion about x_0 then

$$g_n(x) = \left(\int_{x_0}^x dx \right)^n g_0(x) \ \text{ for } n \geq 0$$

has Taylor series

$$g_n(x) = \sum_{j=0}^{\infty} g_{0j} (x - x_0)^{j+n} / (j+n)!,$$

where

$$g_{0j} = g_0^{(j)}(x_0).$$

Applying this with $x_0 = 0$, $G_0(x) = e^{-x^2/2}$, $G_{0,2k+1} = 0$ and

$$G_{0,2k} = (-2)^{-k}(2k)!/k! = (-1)^k EN^{2k},$$

we get the Taylor series of G_n about 0,

$$G_n(x) = \sum_{k=0}^{\infty} G_{0,2k} x^{2k+n}/(2k+n)! \tag{3.2}$$

$$= EN^{-n} K_n(Nx) \tag{3.3}$$

for

$$K_n(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k+n}/(2k+n)! = D^{-1}K_{n-1}(x) = D^{-n}\cos x.$$

So for $k \geq 0$,

$$\begin{aligned} K_{2k}(x) &= (-1)^k \left\{ \cos x - \sum_{j=0}^{k-1} (-x^2)^j/(2j)! \right\} = (-1)^k \left\{ \sum_{j=k}^{\infty} (-x^2)^j/(2j)! \right\}, \\ K_{2k+1}(x) &= (-1)^k \left\{ \sin x - \sum_{j=0}^{k-1} (-1)^j x^{2j+1}/(2j+1)! \right\} \\ &= (-1)^k \left\{ \sum_{j=k}^{\infty} (-1)^j x^{2j+1}/(2j+1)! \right\}. \end{aligned}$$

So (3.2) gives a power series method for computing G_n, while (3.3) gives a numerical integration method. For $n = 0, 1, \ldots, 6$, (3.3) gives

$$\begin{aligned} e^{-x^2/2} &= G_0(x) = EK_0(Nx) = E\cos Nx, \\ (2\pi)^{1/2}\{\phi(x) - 1/2\} &= \int_0^x e^{-y^2/2} dy = G_1(x) \\ &= EN^{-1}K_1(x) = EN^{-1}\sin Nx, \\ D^{-2}e^{-x^2/2} &= EN^{-2}(1 - \cos Nx), \\ D^{-3}e^{-x^2/2} &= EN^{-3}(Nx - \sin Nx), \\ D^{-4}e^{-x^2/2} &= EN^{-4}\{\cos Nx - 1 + (Nx)^2/2!\}, \\ D^{-5}e^{-x^2/2} &= EN^{-5}\{\sin Nx - Nx + (Nx)^3/3!\} \quad \text{and} \\ D^{-6}e^{-x^2/2} &= EN^{-6}\{-\cos Nx + 1 - (Nx)^2/2! + (Nx)^4/4!\}. \end{aligned}$$

These expressions would not seem a good way of calculating their LHS as the functions on the RHS vary rapidly. Also

$$g_n(x) = \left(\int_{x_0}^x dx \right)^n g_0 = \int_{x_0}^x (x-y)^{n-1} g_0(y) dy/(n-1)!.$$

To express G_n in terms of Hh_{n-1}, we first express g_n with $x_0 = 0$ in terms of

$$h_n = h_n(x) = (-D)^n g_0.$$

Here g_0 is any function such that h_n exists for $x \geq 0$, $n \geq 0$. Since $\int_0^x = \int_0^\infty - \int_x^\infty$,

$$\begin{aligned} g_1(x) &= \int_0^x g_0 = h_1(0) - h_1(x), \\ g_2(x) &= \int_0^x g_1 = h_1(0)x - h_2(0) + h_2(x) \quad \text{and} \\ g_n(x) &= (-1)^n \left\{ h_n(x) - \sum_{j=0}^{n-1} h_{n-j}(0)(-x)^j/j! \right\}. \end{aligned}$$

Taking $g_0 = e^{-x^2/2}$ gives

$$(-1)^n G_n(x) = H_{-n}(x)e^{-x^2/2} - \sum_{j=0}^{n-1} H_{-n+j}(0)(-x)^j/j!,$$

where $H_{-n}(0)$ is given by (1.3). Conversely $Hh_{n-1}(x) = H_{-n}(x)e^{-x^2/2}$ is given in terms of $G_n(x)$ by

$$H_{-n}(x)e^{-x^2/2} = (-1)^n G_n(x) + \sum_{j=0}^{n-1} H_{-n+j}(0)(-x)^j/j!.$$

We now introduce the modified version of G_n of (3.1) given by

$$\begin{aligned} G_n^* &= G_n^*(x) = D^{-n}e^{x^2/2} \quad \text{for } n \geq 0 \\ &= \int_0^x (x-y)^{n-1} g_0(y)dy/(n-1)! \quad \text{for } n \geq 1. \end{aligned} \tag{3.4}$$

Alternatively let us define the *modified Airy function*, for $n \geq 0$ as

$$Hh_n^*(x) = D^{-n-1}e^{x^2/2}.$$

So $G_n^* = Hh_{n-1}^*$. By the method above we have

$$G_n^*(x) = \sum_{k=0}^{\infty} |G_{0,2k}| x^{2k+n}/(2k+n)! \tag{3.5}$$

$$= EN^{-n}K_n^*(Nx) \tag{3.6}$$

for

$$K_n^*(x) = \sum_{k=0}^{\infty} x^{2k+n}/(2k+n)! = D^{-1}K_{n-1}^*(x) = D^{-n}\cosh x.$$

So for $k \geq 0$,

$$K_{2k}^*(x) = \left\{\cosh x - \sum_{n=0}^{k-1} x^{2n}/(2n)!\right\} = \sum_{n=k}^{\infty} x^{2n}/(2n)!,$$

$$K_{2k+1}^*(x) = \left\{\sinh x - \sum_{n=0}^{k-1} x^{2n+1}/(2n+1)!\right\} = \sum_{n=k}^{\infty} x^{2n+1}/(2n+1)!.$$

So (3.5) gives a power series method for computing G_n^*, while (3.6) gives a numerical integration method. For $n = 0, 1, \ldots, 6$, (3.6) gives

$$\begin{aligned}
e^{x^2/2} &= E\cosh Nx,\\
D^{-1}e^{x^2/2} &= EN^{-1}\sinh Nx,\\
D^{-2}e^{x^2/2} &= EN^{-2}(\cosh Nx - 1),\\
D^{-3}e^{x^2/2} &= EN^{-3}(\sinh Nx - Nx),\\
D^{-4}e^{x^2/2} &= EN^{-4}\{\cosh Nx - 1 - (Nx)^2/2!\},\\
D^{-5}e^{x^2/2} &= EN^{-5}\{\sinh Nx - Nx - (Nx)^3/3!\} \text{ and}\\
D^{-6}e^{x^2/2} &= EN^{-6}\{\cosh Nx - 1 - (Nx)^2/2! - (Nx)^4/4!\}.
\end{aligned}$$

Unfortunately we do not have a proper extension of H_n^* to negative n, as for real x, $E(x+N)^n$ does not exist for negative n and $(-D)^{-1}e^{x^2/2} = \infty$. However a near miss is given by the function

$$T_n = T_n(x) = e^{-x^2/2}G_n^* = e^{-x^2/2}D^{-n}e^{x^2/2}, \; n \geq 0,$$

in that it satisfies

$$T_{n-1} = (D+x)T_n,$$

as compared with

$$H_{n+1}^* = (D+x)H_n^*.$$

We can view T_1 as a modified Mills' ratio. From $DT_1 = 1 - xT_1$, we obtain the analog of (1.2) given by

$$(-D)^n T_1 = H_n^*\, T_1 + P_{n-1}^*,$$

where

$$\begin{aligned}
P_n^* &= (-i)^n P_n(ix),\\
P_0^* &= 1,\\
P_1^* &= x,\\
P_2^* &= x^2-2,\\
P_3^* &= x^3-5x,\\
P_4^* &= x^4-9x^2+8,\\
P_5^* &= x^5-14x^3+33x,\\
P_6^* &= x^6-20x^4+87x^2-48,\\
P_7^* &= x^7-27x^5+185x^3-279x,\\
P_8^* &= x^8-35x^6+345x^4-975x^2+384
\end{aligned}$$

and so on. We now show that T_n can also be expressed as linear in T_1. By (3.4) for $n \geq 1$

$$T_n = \sum_{k=0}^{n} x^{n-1-k}(-1)^k L_k^*/\{(n-1-k)!k!\}, \tag{3.7}$$

where for $k > -1$,

$$L_k^* = e^{-x^2/2}\int_0^x y^k e^{y^2/2}dy.$$

They can be written in terms of

$$L_0^* = T_1,\ L_1^* = 1-e^{-x^2/2},$$

using the recurrence relation (integrating by parts),

$$L_k^* = x^{k-1}-(k-1)L_{k-2}^* \quad \text{for} \quad k>1.$$

So

$$\begin{aligned}
L_2^* &= x - L_0^*,\ L_3^* = x^2-2L_1^*,\\
L_4^* &= x^3-3x+3L_0^*,\ L_5^* = x^4-4x^2+4\cdot 2L_1^*
\end{aligned}$$

and for $r = 2, 4, 6, \ldots,$

$$\begin{aligned}
L_{r+1}^* &= x^r - rx^{r-2} + r(r-2)x^{r-4} - \cdots\\
&\quad + (-1)^{r/2+1}r(r-2)(r-4)\cdots 4(x^2-2L_1^*)\\
&= L'_{r+1} + (-1)^{r/2+1}2^{r/2}(r/2)!e^{-x^2/2},
\end{aligned}$$

where L'_r can be written in terms of Moran's L-function L_r as

$$L'_{r+1} = i^{-r} L_{r+1}(x/i)$$

and for $r = 3, 5, \dots,$

$$L^*_{r+1} = x^r - rx^{r-2} + r(r-2)x^{r-4} - \cdots + (-1)^{(r-1)/2} r(r-2)\cdots 3(x - L^*_0).$$

A converse of (3.7) is obtained by expanding $(y - x + x)^k$, giving

$$L^*_k = \sum_{n=0}^{k} (-1)^n (k)_n x^{k-n} T_{n+1} \quad \text{for} \quad k \geq 0.$$

ACKNOWLEDGEMENTS

The authors would like to thank the editor and the two referees for carefully reading the paper and for their comments which greatly improved the paper.

REFERENCES

ABRAMOWITZ, M. AND STEGUN, I. A. (1964). *Handbook of Mathematical Functions*, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, **55**.

AIRY, J. R. (1931). *Table of Hh Functions*, British Association for the Advancement of Science, London.

FISHER, R. A. (1931). "Introduction of table of Hh functions", In *Table of Hh Functions* (Airy, J. R., eds.), British Association for the Advancement of Science, London.

GOODALL, C. R. AND MARDIA, K. V. (1991). "A geometrical derivation of the shape density", *Advances in Applied Probability,* **23**, 496–514.

KOTZ, S., BALAKRISHNAN, N. AND JOHNSON, N. L. (2000). *Continuous Multivariate Distributions, Volume 1: Models and Applications,* 2nd ed., John Wiley & Sons, New York.

MARDIA, K. V. (1998). "Fisher's repeated normal integral function and shape distributions", *Journal of Applied Statistics,* **25**, 231–235.

MILLER, K. S. (1974). *Complex Stochastic Processes: An Introduction to Theory and Application,* Addison-Wesley Publishing Company, London.

MILLER, K. S. (1975). *Multivariate Distributions,* Robert E. Krieger Publishing Company, Huntington, New York.

MORAN, P. A. P. (1983). "A new expansion for the multivariate normal distribution", *The Australian Journal of Statistics,* **25**, 339–344.

WITHERS, C. S. (2000). "A simple expression for the multivariate Hermite polynomials", *Statistics & Probability Letters,* **47**, 165–169.

WITHERS, C. S. AND MCGAVIN, P. N. (2006). "Expressions for the normal distribution and repeated normal integrals", *Statistics & Probability Letters,* **76**, 479–487.

WITHERS, C. S. AND NADARAJAH, S. (2006). "New expressions for repeated upper tail integrals for the normal distribution", preprint.