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SOME PROPERTIES OF BIVARIATE GENERALIZED HYPERGEOMETRIC PROBABILITY DISTRIBUTIONS

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ABSTRACT

In this paper we study some important properties of the bivariate generalized hypergeometric probability (BGHP) distribution by establishing the existence of all the moments of the distribution and by deriving recurrence relations for raw moments. It is shown that certain mixtures of BGHP distributions are again BGHP distributions and a limiting case of the distribution is considered.

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1. INTRODUCTION

Moothathu and Kumar (1997) introduced a family of bivariate discrete probability distributions, called the bivariate generalized hypergeometric probability (BGHP) distribution, through the following probability generating function (*p.g.f.*).

$$H(t_1, t_2) = \frac{{}_pF_q(\underline{a}; \underline{b}; \theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2)}{{}_pF_q(\underline{a}; \underline{b}; \theta_1 + \theta_2 + \theta_3)}, \tag{1.1}$$

where ${}_pF_q(\underline{a}; \underline{b}; z)$ is the generalized hypergeometric series (Mathai and Saxena, 1973; Slater, 1966), in which $a's, b's$ and z are assumed to be appropriate real numbers such that the ${}_pF_q(\cdot)$ remains positive; $\theta_1 > 0, \theta_2 > 0, \theta_3 \geq 0$ or $\theta_1 < 0, \theta_2 < 0$, $\theta_3 \leq 0$ and $\theta_1 + \theta_2 + \theta_3$ is an element of Θ according as Θ is a subset of $(0, \infty)$ or $(-\infty, 0)$ respectively. Well-known bivariate versions of distributions such as bivariate Bernoulli, bivariate binomial, bivariate geometric, bivariate

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negative binomial and bivariate Poisson distributions discussed in Kocherlakota and Kocherlakota (1992) are special cases of BGHP distribution. Note that BGHP distribution is a bivariate version of generalized hypergeometric probability (GHP) distribution due to Kemp (1968). Thus BGHP distribution includes bivariate versions of several other discrete distributions such as hyper-Poisson, displaced-Poisson, hypergeometric, inverse hypergeometric, negative hypergeometric, Waring, Yule, Naor and Feller. For a detailed account of GHP distribution see Johnson *et al.* (1992). Kumar (2002) introduced and studied a further generalized form of GHP distribution.

Here in Section 2 we show that all the moments of the BGHP distribution exists finitely. Two recurrence relations for raw moments of BGHP distribution are also established in the same section. In Section 3 it is shown that each of the two different types of mixtures of certain BGHP distributions is a BGHP distribution and that the limit of a certain BGHP distribution is another BGHP distribution. The results presented in this paper provide a unified approach to the derivation of some theoretical results regarding several types of bivariate discrete distributions.

2. Recurrence Relations for Moments

LEMMA 2.1. (Moothathu and Kumar, 1997). *Consider the random sum* $U_{rM} = \sum_{j=1}^{M} Z_{rj}$, $r = 1, 2$ *of independent and identically distributed bivariate random vectors, where* Z_j *and* M *are independent,* $Z_j = (Z_{1j}, Z_{2j})$ *has the p.g.f.*

$$P(t_1, t_2) = \frac{\theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2}{\theta_1 + \theta_2 + \theta_3}$$

and M *has* GHP *distribution with p.g.f.*

$$A(t) = \frac{{}_pF_q[\underline{a}; \underline{b}; (\theta_1 + \theta_2 + \theta_3)t]}{{}_pF_q(\underline{a}; \underline{b}; \theta_1 + \theta_2 + \theta_3)}. \tag{2.1}$$

Then the distribution of $U_M = (U_{1M}, U_{2M})$ *is* BGHP *distribution with p.g.f. (1.1).*

Lemma 2.1 shows that BGHP distribution can be obtained as the distribution of a random sum of certain independent and identically distributed bivariate Bernoulli random variables. Consequently, corresponding to every special case of GHP distribution it is possible to derive a bivariate version of the distribution. Thus BGHP distribution includes bivariate versions of several well-known discrete distributions such as binomial, Poisson, hyper-Poisson, displaced-Poisson, negative binomial, hypergeometric, inverse hypergeometric, negative hypergeometric, Waring, Yule, Naor and Feller.

LEMMA 2.2. *If M has* GHP *distribution with p.g.f. (2.1), then $E(M^r) < \infty$ for every positive integer r.*

Proof of this lemma is quite simple and hence omitted.

RESULT 2.1. If $W = (W_1, W_2)$ has BGHP distribution with *p.g.f.* (1.1) then for any positive integers $r, s, \mu_{r,s} = E(W_1^r, W_2^s)$ is finite.

PROOF. (W_1, W_2) and (U_{1M}, U_{2M}) of Lemma 2.1 are identically distributed. For any positive integer m, from the definition of U_{rM} it is obvious that $P(0 \leq U_{rM} \leq m) = 1$, for $r = 1, 2$ which implies that $E(U_{1M}^r, U_{2M}^s) \leq m^{r+s}$. Hence

$$\begin{aligned} 0 \leq \mu_{r,s} &= E(U_{1M}^r, U_{2M}^s) \\ &= \sum_{m=0}^{\infty} E(U_{1M}^r, U_{2M}^s | M = m) P(M = m) \\ &\leq \sum_{m=0}^{\infty} m^{r+s} P(M = m) \\ &= E(M^{r+s}) < \infty \end{aligned}$$

by Lemma 2.2. □

The characteristic function $\varphi(t_1, t_2)$ of the BGHP distribution with *p.g.f.* (1.1) is the following. For (t_1, t_2) in R^2

$$\begin{aligned} \varphi(t_1, t_2) &= H(e^{it_1}, e^{it_2}) \\ &= R_0^{-1} {}_pF_q[\underline{a}; \underline{b}; \gamma(t_1, t_2; \underline{\theta})], \end{aligned} \tag{2.2}$$

where $i = \sqrt{-1}$, $\underline{\theta} = (\theta_1, \theta_2, \theta_3), \gamma(t_1, t_2; \underline{\theta}) = \theta_1 e^{it_1} + \theta_2 e^{it_2} + \theta_3 e^{i(t_1+t_2)}$ and $R_0 =_p F_q(\underline{a}, \underline{b}; \theta_1 + \theta_2 + \theta_3)$.

In Result 2.1 we proved that for non-negative integers r, s, the $(r, s)^{th}$ moment $\mu_{r,s}$ of the BGHP distribution exists finitely. Now we shall denote $\mu_{r,s}$ by $\mu_{r,s}(a, b)$. Hence $\varphi(t_1, t_2)$ given in (2.2) has the following series representation.

$$\begin{aligned} \varphi(t_1, t_2) &= R_0^{-1} {}_pF_q[\underline{a}; \underline{b}; \gamma(t_1, t_2; \underline{\theta})] \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \mu_{r,s}(\underline{a}; \underline{b}) \frac{(it_1)^r (it_2)^s}{r! s!}. \end{aligned} \tag{2.3}$$

RESULT 2.2. Two recurrence relations for the moments of the BGHP distribution are

$$\mu_{r+1,s}(\underline{a};\underline{b}) = \frac{D_0R_1}{R_0}\left[\theta_1\sum_{j=0}^{r}\binom{r}{j}\mu_{r-j,s}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\right.$$
$$\left.+\theta_3\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j,s-k}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\right], \tag{2.4}$$

$$\mu_{r,s+1}(\underline{a};\underline{b}) = \frac{D_0R_1}{R_0}\left[\theta_2\sum_{k=0}^{s}\binom{s}{k}\mu_{r,s-k}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\right.$$
$$\left.+\theta_3\sum_{j=0}^{r}\sum_{k=0}^{s}\binom{r}{j}\binom{s}{k}\mu_{r-j,s-k}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\right], \tag{2.5}$$

where

$$D_0 = \frac{\prod_{n=1}^{p}(a_n)}{\prod_{m=1}^{q}(b_m)}, R_1 = {}_pF_q(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q;\theta_1+\theta_2+\theta_3)$$

and R_0 is as mentioned in (2.2).

PROOF. On differentiating (2.3) with respect to t_1, we get

$$D_0R_0^{-1}{}_pF_q[\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q;\gamma(t_1,t_2;\underline{\theta})]\{i(\theta_1+\theta_3e^{it_2})e^{it_1}\}$$
$$=\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}i\mu_{r,s}(\underline{a},\underline{b})\frac{(it_1)^{r-1}(it_2)^s}{(r-1)!s!}. \tag{2.6}$$

On replacing $\underline{a}$, $\underline{b}$ by $\underline{a}+\underline{1}_p$, $\underline{b}+\underline{1}_q$ respectively in (2.3), one has

$${}_pF_q[\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q;\gamma(t_1,t_2;\underline{\theta})]$$
$$=R_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{r,s}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\frac{(it_1)^r(it_2)^s}{r!s!}. \tag{2.7}$$

From (2.6) and (2.7), we have the following:

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_{r+1,s}(\underline{a};\underline{b})\frac{(it_1)^r(it_2)^s}{r!s!}$$
$$=D_0R_0^{-1}R_1\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\frac{(it_1)^r(it_2)^s}{r!s!}$$
$$\times\mu_{r,s}(\underline{a}+\underline{1}_p;\underline{b}+\underline{1}_q)\frac{(it_1)^j}{j!}\left\{\theta_1+\theta_3\sum_{k=0}^{\infty}\frac{(it_2)^k}{k!}\right\} \tag{2.8}$$

$$= D_0 R_0^{-1} R_1 \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(it_1)^r (it_2)^s}{r!s!} \left\{ \theta_1 \sum_{j=0}^{r} \binom{r}{j} \mu_{r-j,s}(\underline{a}+\underline{1}_p; \underline{b}+\underline{1}_q) \right.$$

$$\left. +\theta_3 \sum_{j=0}^{r}\sum_{k=0}^{s} \binom{r}{j}\binom{s}{k} \mu_{r-j,s-k}(\underline{a}+\underline{1}_p; \underline{b}+\underline{1}_q) \right\}, \tag{2.9}$$

in the light of the following.

$$\sum_{r=0}^{\infty}\sum_{i=0}^{\infty} B(i,r) = \sum_{r=0}^{\infty}\sum_{i=0}^{r} B(i, r-i)$$

On equating coefficients of $\{(it_1)^r (it_2)^s\}/r!s!$ on both sides of (2.9) we get the relation (2.4). Similarly one can prove (2.5) by differentiating (2.3) with respect to t_2 and equating the coefficients. □

3. Mixtures and a Limiting Case of BGHP Distribution

THEOREM 3.1. *Assume that λ has the following probability density function, in which $c > 0, d > 0, a's, b's$ and $\alpha's$ are real numbers such that $h(\lambda) > 0$ for $0 < \lambda < 1$.*

$$h(\lambda) = \frac{\lambda^{c-1}(1-\lambda)^{d-1}}{B(c,d)} \times \frac{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1+\alpha_2+\alpha_3)\lambda]}{{}_{p+1}F_{q+1}[\underline{a}, c; \underline{b}, c+d; \alpha_1+\alpha_2+\alpha_3]}.$$

Given λ, let the discrete random vector $\underline{X} = (X_1, X_2)$ have the BGHP *distribution with p.g.f.*

$$\frac{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2)\lambda]}{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1+\alpha_2+\alpha_3)\lambda]}.$$

Then the unconditional distribution of $\underline{X}$ is a BGHP *distribution with p.g.f. (3.1).*

PROOF. Let $G_{\underline{X}}(t_1, t_2)$ denote the *p.g.f.* of the unconditional distribution of $\underline{X} = (X_1, X_2)$. Now from the well-known properties of integrals of a power series we readily obtain the following.

$$\begin{aligned} G_{\underline{X}}(t_1,t_2) &= E[t_1^{X_1} t_2^{X_2}] = E_\lambda\{E[t_1^{X_1} t_2^{X_2}|\lambda]\} \\ &= \int_0^1 \frac{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2)\lambda]}{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1+\alpha_2+\alpha_3)\lambda]} h(\lambda) d\lambda \\ &= \frac{{}_{p+1}F_{q+1}[\underline{a}, c; \underline{b}, c+d; \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2]}{{}_{p+1}F_{q+1}[\underline{a}, c; \underline{b}, c+d; \alpha_1+\alpha_2+\alpha_3]}. \end{aligned} \tag{3.1}$$

□

THEOREM 3.2. *Assume that δ has the following probability density function, in which $p \leq q$, $w > 0$, $a's, b's$ and $\alpha's$ are real numbers such that $g(\delta) > 0$ for $\delta > 0$.*

$$g(\delta) = \frac{e^{-\delta}\delta^{w-1}}{\Gamma(w)} \times \frac{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1 + \alpha_2 + \alpha_3)\delta]}{{}_{p+1}F_q[\underline{a}, w; \underline{b}; \alpha_1 + \alpha_2 + \alpha_3]}.$$

Given δ, let the discrete random vector $\underline{Y} = (Y_1, Y_2)$ have the BGHP *distribution with p.g.f.*

$$\frac{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2)\delta]}{{}_pF_q[\underline{a}; \underline{b}; (\alpha_1 + \alpha_2 + \alpha_3)\delta]}.$$

Then the unconditional distribution of $\underline{Y}$ is a BGHP *distribution with p.g.f.*

$$G_{\underline{Y}}(t_1, t_2) = \frac{{}_{p+1}F_q[\underline{a}, w; \underline{b}; \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_1 t_2]}{{}_{p+1}F_q[\underline{a}, w; \underline{b}; \alpha_1 + \alpha_2 + \alpha_3]}.$$

We omit the proof of this theorem, as it is similar to that of Theorem 3.1. The following is a standard result for generalized hypergeometric functions.

LEMMA 3.1. *The generalized hypergeometric function ${}_pF_q(\underline{a}; \underline{b}; \theta)$ with $p \leq q$ can be obtained as the limiting form of the following function as $u \longrightarrow \infty$.*

$$F = {}_{p+1}F_q(\underline{a}, u; \underline{b}; u^{-1}\theta), \quad u > |\theta|.$$

The proof easily follows by observing that

$$\lim_{u \longrightarrow \infty} \frac{(u)_r}{u^r} = 1.$$

As a consequence of Lemma 3.1, we have the following result.

RESULT 3.1. The BGHP distribution with *p.g.f.* (1.1) is the limiting form as $c \to \infty$ of the distribution with the following *p.g.f.*, in which $c > 0$.

$$R(t_1, t_2) = \frac{{}_{p+1}F_q[\underline{a}, c; \underline{b}; c^{-1}(\theta_1 t_1 + \theta_2 t_2 + \theta_3 t_1 t_2)]}{{}_{p+1}F_q[\underline{a}, c; \underline{b}; c^{-1}(\theta_1 + \theta_2 + \theta_3)]}.$$

As a special case of Result 3.1, we obtain that the limit of bivariate negative binomial distribution is the bivariate Poisson distribution (a result proved in Kocherlakota and Kocherlakota, 1992, p. 146) and that the bivariate Poisson distribution is the limit of bivariate binomial distribution (a result due to Campbell, 1934).

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