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ORDER RESTRICTED TESTS FOR SYMMETRY AGAINST POSITIVE BIASEDNESS†

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ABSTRACT

Two new types of positive biasedness, which are closely related to Type III positive biasedness (Yanagimoto and Sibuya, 1972), are proposed. We call these near Type III positive biasedness. Though no implication between Type II and near Type III biasedness exists, near Type III seems to be less restrictive than Type II biasedness. Constrained maximum likelihood estimates of distribution functions under near Type III positive bisedness are obtained. The likelihood ratio tests of symmetry against new positive biasedness restrictions are proposed. A small simulation study is conducted to compare the performance of the tests.

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1. INTRODUCTION

One of the widely accepted assumptions in many statistical problems is that the underlying distribution is symmetric. Many statistical procedures, for example Wilcoxon's signed rank test, may result in low validity if this symmetry assumption is severely violated. Moreover, when the symmetry assumption is satisfied many statistical procedures based on normal theory such as t test can be applied for many problems with moderate or large sample sizes. In this sense the symmetry assumption is very crucial. Though a large number of nonparametric tests are available in the literature for this problem, very few procedures focus on the distributional structure when the symmetry assumption is rejected.

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Yanagimoto and Sibuya (1972) provided the various types of asymmetric distributional structure. They called them "*Positive biasedness*". Let F be the distribution function of random variable X. Note that $F(x-) = \Pr[X < x]$.

Type 0 $\quad 1 - F(0) \geq F(0-)$,

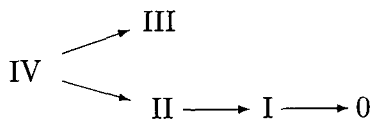
Type I $\quad F(x) + F((-x)-) \leq 1$ for any $x \geq 0$,

Type II $\quad F(x+y) - F(x) \geq F((-x)-) - F((-x-y)-)$ for any $x, y \geq 0$,

Type III $\quad (F(x+y) - F(y))/(F((-y)-) - F((-x-y)-))$ is nondecreasing in both $x, y > 0$ and

Type IV $\quad (F(x+y) - F(y))/(F((-y)-) - F((-x-y)-))$ is nondecreasing in both $x > 0$ and y.

The implication scheme of positive biasedness is depicted as follows. This figure is adapted from Yanagomoto and Sibuya (1972).



Among these, type I and II biases have received a substantial amount of interest because they are closely related to stochastic ordering. Specifically we say that X is positively type I biased if X is stochastically larger than $-X$. Dykstra *et al.* (1995b) studied likelihood ratio tests against Type I and II positive biasedness. To our best knowledge no test for symmetry against Type III or IV bias has given so far. This seems to be mainly due to that the restriction of Type III or IV positive biasedness is too strong.

In this paper we are going to consider the test of symmetry against several new positive biasedness restrictions which are stronger than Type I but weaker than Type III and hence IV. These new positive biasedness restrictions are closely related to the various types of stochastic ordering such as uniform stochastic ordering and likelihood ratio ordering. In Section 2 we discussed these new types of positive biasedness and their relationship to various type of stochastic orderings. In Section 3 estimation of distribution functions under new positive biasedness and likelihood ratio test for symmetry against new positive biasedness under discrete setting are discussed. A simulation study was conducted to compare the performance of the tests under new restrictions.

2. NEAR TYPE III POSITIVE BIAS

In this section we restrict the problem to discrete or grouped data. Let $\mathbf{p} = (p_{-k}, p_{-k+1}, \ldots, p_{-1}, p_0, p_1, \ldots, p_k)$ be $2k+1$ dimensional probability vector, *i.e.*, $p_i > 0$ and $\sum_{i=-k}^{k} p_i = 1$. We note that one can consider the case of even number of probability masses by letting $p_0 \geq 0$. The Type III bias can be expressed as

$$\frac{\sum_{j=i_1}^{i_2} p_j}{\sum_{j=i_1}^{i_2} p_{-j}} \text{ is nondecreasing in both } 0 < i_1 \leq i_2.$$

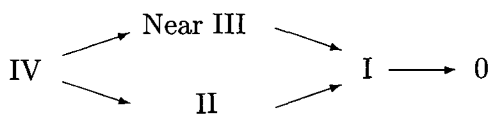
We note that the Type IV bias is expressed as

$$\frac{\sum_{j=i_1}^{i_2} p_j}{\sum_{j=i_1}^{i_2} p_{-j}} \text{ is nondecreasing in both } i_1 \leq i_2.$$

It is clear that Type III bias does not satisfy the restriction $\sum_{j=1}^{k} p_j / \sum_{j=1}^{k} p_{-j} \geq 1$ and hence does not imply Type II nor Type I bias. Now we add this restriction to Type III biasedness. Then we call it *Near Type III Positive Bias.* It is easy to show that the near Type III bias now imply Type I bias. Since $\sum_{j=i_1}^{k} p_j / \sum_{j=i_1}^{k} p_{-j}$ is nondecreasing in i_1 for Type IV bias, *i.e.*,

$$\sum_{j=i_1}^{k} p_j / \sum_{j=i_1}^{k} p_{-j} \;\geq\; 1 \text{ for } i_1 = -k, \ldots, k,$$

then the Type IV bias imply the near Type III bias. Now we have new implication scheme as shown below.



The near Type III restriction is, however, still too strong. Now we are going to pick up the two special cases from near Type III positive biasedness. First let

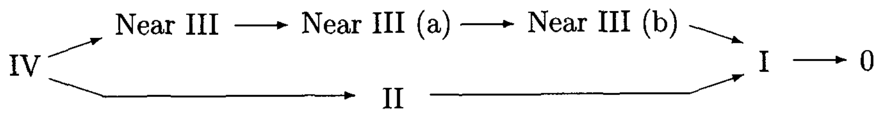
$i_1 > 0$ and $i_2 = i_1$. Then this restriction becomes

$$\frac{p_{i_1}}{p_{-i_1}} \text{ is nondecreasing in } i_1 = 1, \ldots, k, \text{ and } \sum_{j=1}^{k} p_j \geq \sum_{j=1}^{k} p_{-j}. \tag{2.1}$$

Note that (2.1) with an additional restriction $p_1 \geq p_{-1}$ imply Type II biasedness. Second let $i_1 > 0$ and $i_2 = k$. Then it becomes

$$\sum_{j=i_1}^{k} p_j / \sum_{j=i_1}^{k} p_{-j} \text{ is nondecreasing in } i_1 = 1, \ldots, k, \text{ and } \sum_{j=1}^{k} p_j \geq \sum_{j=1}^{k} p_{-j}. \tag{2.2}$$

For convenience we call the former restriction a near Type III (a) biasedness and the latter near Type III (b) biasedness. It is not difficult to show that we have new implication scheme as follows.



We note that theses two new restrictions are closely related to the well-known types of stochastic ordering. The near Type III (a) biasedness is related to the likelihood ratio ordering and the near Type III (b) biasedness is related to the uniform stochastic ordering.

Next we are going to find likelihood ratio tests of symmetry against near Type III (a) and (b) positive biasedness.

3. Likelihood Ratio Tests

For Type I and II biasedness, Dykstra *et al.* (1995b) proposed the likelihood ratio test statistic whose limiting null distribution is chi-bar squared distribution. In this section we consider the problem of testing the null hypothesis of symmetry about 0

$$H_0 \ : \ p_i = p_{-i} \text{ for } i = 1, \ldots, k$$

against the alternatives $H_1 - H_0$ and $H_2 - H_0$, where H_1 and H_2 postulate the restrictions (2.1) and (2.2) respectively.

3.1. *Test of H_0 vs. $H_1 - H_0$*

First consider Type III (a) restriction. We need to find the estimate of distribution function under Type III (a) restriction, which can be achieved by

$$\text{maximize } p_0^{n_0} \prod_{i=1}^{k} p_i^{n_i} p_{-i}^{n_{-i}} \tag{3.1}$$

subject to restriction (2.1), where n_i denote the the number of observations at $i = -k, \ldots, k$. We use a one-to-one transformation of parameter space. Let $A_1 = \sum_{i=1}^k p_i$, $A_{-1} = \sum_{i=1}^k p_{-i}$, $A_0 = p_0$, $a_i = p_i/A_1$, $a_{-i} = p_{-i}/A_{-1}$ for $i = 1, \ldots, k$, and $a_0 = 1$. Then the maximization problem becomes

$$\text{maximize } \prod_{i=1}^{k} a_i^{n_i} a_{-i}^{n_{-i}} \cdot A_1^{\sum_{i=1}^k n_i} A_{-1}^{\sum_{i=1}^k n_{-i}} A_0^{n_0} \tag{3.2}$$

subject to

$$\frac{a_i}{a_{-i}} \text{ is nondecreasing in } i = 1, \ldots, k, \text{ and } A_1 \geq A_{-1} \tag{3.3}$$

and $a_i > 0$, $\sum_{i=1}^k a_i = \sum_{i=1}^k a_{-i} = 1$ and $\sum_{i=-1}^1 A_i = 1$. We note that no restrictions relate a_i's and A_i's to each other. This means that we only need to maximize (3.2) by maximizing two parts separately under corresponding restrictions. The former part is likelihood ratio ordering problem which was studied extensively by Dykstra *et al.* (1995a). We use another one-to-one transformation again. Let $n_+ = \sum_{i=1}^k n_i$ and $n_- = \sum_{i=1}^k n_{-i}$. Let $\theta_i = n_+ a_i/(n_+ a_i + n_- a_{-i})$ and $\phi_i = n_+ a_i + n_- a_{-i}$. Then the maximization problem becomes

$$\text{maximize } \left(\frac{1}{n_+}\right)^{n_+} \left(\frac{1}{n_-}\right)^{n_-} \prod_{i=1}^{k} \theta_i^{n_i} (1-\theta_i)^{n_{-i}} \prod_{i=1}^{k} \phi_i^{n_i + n_{-i}} \tag{3.4}$$

subject to

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_k, \tag{3.5}$$

together with (a) $0 \leq \theta_i \leq 1$, $\phi_i \geq 0$ for $i = 1, \ldots, k$, (b) $\sum_{i=1}^k \phi_i = n_+ + n_-$ and (c) $\sum_{i=1}^k \theta_i \phi_i = n_+$. From Theorem 2.1 of Dykstra *et al.* (1995a) the constraint estimator of θ under (3.5) is given by

$$\boldsymbol{\theta}^* = E_{\mathbf{n}_+ + \mathbf{n}_-} \left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-} \Big| I \right),$$

where $\mathbf{n}_+ = (n_1, n_2, \ldots, n_k)$, $\mathbf{n}_- = (n_{-1}, n_{-2}, \ldots, n_{-k})$, $I = \{\mathbf{x} \in R^k : x_1 \leq x_2 \leq \cdots \leq x_k\}$, and $E_{\mathbf{w}}(\mathbf{x}|A)$ is the isotonic regression of $\mathbf{x}$ with respect to $\mathbf{w}$ onto

A. All the vector operations are componentwise. Hence we have

$$a_i^* = \frac{n_i + n_{-i}}{n_+} E_{\mathbf{n}_+ + \mathbf{n}_-}\left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-}\Big| I\right)_i,$$
$$a_{-i}^* = \frac{n_i + n_{-i}}{n_-} E_{\mathbf{n}_+ + \mathbf{n}_-}\left(\frac{\mathbf{n}_-}{\mathbf{n}_+ + \mathbf{n}_-}\Big| A\right)_i,$$

for $i = 1, \ldots, k$, where $A = \{\mathbf{x} \in R^k : -\mathbf{x} \in I\}$.

Next we find the constraint estimator of A_i's under (3.3). Let $D = \{\mathbf{x} \in R^3 : x_1 \le x_3\}$ and $\mathbf{n}_A = (\sum_{i=1}^k n_{-i}, n_0, \sum_{i=1}^k n_i)$. Then

$$\mathbf{A}^* = (A_{-1}^*, A_0^*, A_1^*) = E\left(\frac{\mathbf{n}_A}{n}\Big| D\right), \tag{3.6}$$

where $n = n_- + n_0 + n_+$.

THEOREM 3.1. *If $n_i > 0$, then the maximum likelihood estimate of $\mathbf{p}$ under (2.1) is given by $\mathbf{p}^*$ where $\mathbf{p}^* = (p_{-k}^*, \ldots, p_k^*)$ with*

$$p_i^* = \frac{n_i + n_{-i}}{n_+} E_{\mathbf{n}_+ + \mathbf{n}_-}\left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-}\Big| I\right)_i \cdot E\left(\frac{\mathbf{n}_A}{n}\Big| D\right)_1,$$
$$p_0^* = E\left(\frac{\mathbf{n}_A}{n}\Big| D\right)_0 = \frac{n_0}{n},$$
$$p_{-i}^* = \frac{n_i + n_{-i}}{n_-} E_{\mathbf{n}_+ + \mathbf{n}_-}\left(\frac{\mathbf{n}_-}{\mathbf{n}_+ + \mathbf{n}_-}\Big| A\right)_i \cdot E\left(\frac{\mathbf{n}_A}{n}\Big| D\right)_{-1},$$

for $i = 1, \ldots, k$.

The estimation of $\mathbf{p}$ under H_0 is easy but we state them in terms of transformed parameter. The null hypothesis restriction is

$$\frac{a_1}{a_{-1}} = \frac{a_2}{a_{-2}} = \cdots = \frac{a_k}{a_{-k}} \text{ and } A_1 = A_{-1}. \tag{3.7}$$

Let $\mathbf{p}^\circ$ denote the estimate under null hypothesis. Then

$$p_i^\circ = \frac{n_i + n_{-i}}{n_+} E_{\mathbf{n}_+ + \mathbf{n}_-}\left(\frac{\mathbf{n}_+}{\mathbf{n}_+ + \mathbf{n}_-}\Big| C\right)_i \cdot E\left(\frac{\mathbf{n}_A}{n}\Big| D^\circ\right)_1,$$
$$p_0^\circ = E\left(\frac{\mathbf{n}_A}{n}\Big| D^\circ\right)_0 = \frac{n_0}{n},$$
$$p_{-i}^\circ = \frac{n_i + n_{-i}}{n_-} E_{\mathbf{n}_+ + \mathbf{n}_-}\left(\frac{\mathbf{n}_-}{\mathbf{n}_+ + \mathbf{n}_-}\Big| C\right)_i \cdot E\left(\frac{\mathbf{n}_A}{n}\Big| D^\circ\right)_{-1},$$

for $i = 1, \ldots, k$, where $C = \{\mathbf{x} \in R^k : x_1 = x_2 = \cdots = x_k\}$, and $D^\circ = \{\mathbf{x} \in R^3 : x_1 = x_3\}$.

For discrete setting the restricted estimators are strongly consistent. This is due to the continuity property of isotonic regression with respect to weights and arguments. See Robertson *et al.* (1988). The consistency property for the case of arbitrary distribution will be discussed in later section.

The likelihood ratio test statistic is

$$\begin{aligned}\Lambda_{01} &= \frac{\prod_{i=-k}^{k}(p_i^\circ)^{n_i}}{\prod_{i=-k}^{k}(p_i^*)^{n_i}} \\ &= \frac{\prod_{i=1}^{k}(\theta_i^\circ)^{n_i}(1-\theta_i^\circ)^{n_{-i}}(A_{-1}^\circ)^{\sum_{i=1}^k n_{-i}}(A_0^\circ)^{n_0}(A_1^\circ)^{\sum_{i=1}^k n_i}}{\prod_{i=1}^{k}(\theta_i^*)^{n_i}(1-\theta_i^*)^{n_{-i}}(A_{-1}^*)^{\sum_{i=1}^k n_{-i}}(A_0^*)^{n_0}(A_1^*)^{\sum_{i=1}^k n_i}}.\end{aligned}$$

The test rejects H_0 in favor of H_1 for the large values of

$$\begin{aligned}T_{01} &= -2\ln\Lambda_{01} \\ &= 2\left[\sum_{i=1}^{k} n_i(\ln\theta_i^* - \ln\theta_i^\circ) + \sum_{i=1}^{k} n_{-i}(\ln(1-\theta_i^*) - \ln(1-\theta_i^\circ))\right] \\ &\quad + 2\left[\left(\sum_{i=1}^{k} n_{-i}\right)(\ln A_{-1}^* - \ln A_{-1}^\circ) + \left(\sum_{i=1}^{k} n_i\right)(\ln A_1^* - \ln A_1^\circ)\right].\end{aligned}$$

Note that $A_0^\circ = A_0^*$.

THEOREM 3.2. *If $p_i = p_{-i}$ for $i = i, \ldots, k$ and n goes to infinity, then for every $t > 0$,*

$$\begin{aligned}\lim_{n\to\infty}\Pr[T_{01} \geq t] &= \sum_{\ell=0}^{k}\frac{1}{2}\left[P(\ell,k;\mathbf{p}_+) + P(\ell+1,k;\mathbf{p}_+)\right]\Pr[\chi_\ell^2 \geq t], \\ &\leq \sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{2}\right)^k \Pr[\chi_\ell^2 \geq t],\end{aligned}$$

where $\mathbf{p}_+ = (p_1, \ldots, p_k)$ and $P(\ell, k; \mathbf{p}_+)$ for $\ell = 1, \ldots, k$ is the probability that $E_{\mathbf{p}_+}(\mathbf{X}|I)$ takes ℓ distinct values, where $\mathbf{X} = (X_1, \ldots, X_k)$ consists of independent random variables and X_i is $N(0, 1/p_i)$ and $P(0, k; \mathbf{p}_+) = P(k+1, k; \mathbf{p}_+) = 0$. Note that the upper bound for the probability is obtained by taking $\sup_{\mathbf{p}\in\mathbf{H}_0}\lim_{n\to\infty}\Pr[T_{01} \geq t]$. The distribution corresponding to the upper bound is called least favorable distribution.

PROOF. Let $T_{01} = T_{01}^1 + T_{01}^2$, say. Consider T_{01}^1 first. It is obvious that T_{01}^1 is the likelihood ratio test statistic for testing equality of two multinomial population against likelihood ratio ordering. It follows from Theorem 3.1 of Dykstra *et al.* (1995a) that for all $t > 0$

$$\begin{aligned}\lim_{n\to\infty} \Pr[T_{01}^1 \geq t] &= \sum_{\ell=1}^{k} P(\ell, k; \mathbf{a}_+)\Pr[\chi_{\ell-1}^2 \geq t],\\ &\leq \sum_{\ell=1}^{k} \binom{k-1}{\ell-1}\left(\frac{1}{2}\right)^{k-1} \Pr[\chi_{\ell-1}^2 \geq t],\end{aligned}$$

where $\mathbf{a}_+ = (a_1, \ldots, a_k)$.

For deriving the null distribution of T_{01}^2 we consider the test concerning trinomial parameter, $\mathbf{q}$, say. The null hypothesis is $H_0 : q_1 = q_3$ and the alternative $H_1 : q_1 \leq q_3$. Since neither null nor alternative specifies the value of q_2, the level sets in computing the H_1-constrained estimate are $\{(1,3),(2)\}$ and $\{(1),(2),(3)\}$. We note that $T_{01}^2 = 0$ for the case of two level sets and $T_{01}^2 > 0$ otherwise. We know that the probability of two level sets is $1/2$. Hence we can conclude that

$$\lim_{n\to\infty} \Pr[T_{01}^2 \geq t] = \frac{1}{2}\Pr[\chi_1^2 \geq t].$$

The result follows from the asymptotic independence of T_{01}^1 and T_{01}^2. Since the level probability $P(\ell, k; \mathbf{a})$ is scale invariant we can replace $\mathbf{a}_+$ by $\mathbf{p}_+ = (p_1, \ldots, p_k)$. This completes the proof. □

3.2. Test of H_0 vs. $H_2 - H_0$

To find maximum likelihood estimate of $\mathbf{p}$ we consider the maximization of (3.1) under (2.2). We use the same reparametrization scheme as in (3.2). Restriction (2.2) becomes

$$\frac{\sum_{j=i}^{k} a_j}{\sum_{j=i}^{k} a_{-j}} \text{ is nondecreasing in } i = 1, \ldots, k \text{ and } A_1 \geq A_{-1}. \tag{3.8}$$

The constraint on a_i and a_{-i} in (3.8) is uniform stochastic ordering. Dykstra *et al.* (1991) first proposed the statistical inference including likelihood ratio test for discrete distribution under uniform stochastic ordering. Using the same estimation procedure we can easily find the maximum likelihood estimate of a_I's under (3.8).

Let $\eta_i = \sum_{j=i+1}^k a_j / \sum_{j=i}^k a_j$ and $\eta_{-i} = \sum_{j=i+1}^k a_{-j} / \sum_{j=i}^k a_{-j}$. Then (3.8) becomes

$$\eta_{-i} \le \eta_i \text{ for } i = 1, \ldots, k-1, \text{ and } A_1 \ge A_{-1}. \tag{3.9}$$

Now we need to find η and A which maximize

$$\prod_{i=1}^{k-1} \eta_i^{\sum_{j=i+1}^k n_j} (1-\eta_i)^{n_i} \eta_{-i}^{\sum_{j=i+1}^k n_{-j}} (1-\eta_{-i})^{n_{-i}} A_1^{\sum_{i=1}^k n_i} A_{-1}^{\sum_{i=1}^k n_{-i}} A_0^{n_0}$$

subject to (3.9). Note that no restriction relate the pairs (η_i, η_{-i}) for different values of i for η's and A's. The constrained estimate of $\mathbf{A}$ is given in (3.6). Let $\mathbf{n}^{(i)} = (\sum_{j=i}^k n_j, \sum_{j=i}^k n_{-j})$. The constrained estimates of (η_i, η_{-i}) under (3.9) is given by $(\eta_i^\dagger, \eta_{-i}^\dagger)$ where, for $i = 1, \ldots, k-1$,

$$(\eta_i^\dagger, \eta_{-i}^\dagger) = E_{\mathbf{n}^{(i)}} \left(\frac{\mathbf{n}^{(i+1)}}{\mathbf{n}^{(i)}} \middle| I_2 \right),$$

where $I_2 = \{\mathbf{x} \in R^2 : x_1 \ge x_2\}$.

THEOREM 3.3. *If $n_i > 0$, then the maximum likelihood estimate of $\mathbf{p}$ under (2.2) is given by $\mathbf{p}^\dagger$ where $\mathbf{p}^\dagger = (p_{-k}^\dagger, \ldots, p_k^\dagger)$ with*

$$p_i^\dagger = \begin{cases} \left(1 - E_{\mathbf{n}^{(i)}} \left(\frac{\mathbf{n}^{(i+1)}}{\mathbf{n}^{(i)}} \middle| I_2 \right)_1 \right) E\left(\frac{\mathbf{n}_A}{n} \middle| D \right)_1, & \text{if } i = 1, \\ \prod_{j=1}^{i-1} E_{\mathbf{n}^{(j)}} \left(\frac{\mathbf{n}^{(j+1)}}{\mathbf{n}^{(j)}} \middle| I_2 \right)_1 \\ \quad \times \left(1 - E_{\mathbf{n}^{(i)}} \left(\frac{\mathbf{n}^{(i+1)}}{\mathbf{n}^{(i)}} \middle| I_2 \right)_1 \right) E\left(\frac{\mathbf{n}_A}{n} \middle| D \right)_1, & \text{if } i = 2, \ldots, k-1, \\ \prod_{j=1}^{k-1} E_{\mathbf{n}^{(j)}} \left(\frac{\mathbf{n}^{(j+1)}}{\mathbf{n}^{(j)}} \middle| I_2 \right)_1 E\left(\frac{\mathbf{n}_A}{n} \middle| D \right)_1, & \text{if } i = k, \end{cases}$$

$$p_0^\dagger = E\left(\frac{\mathbf{n}_A}{n} \middle| D \right)_0 = \frac{n_0}{n},$$

$$p^{\dagger}_{-i} = \begin{cases} \left(1 - E_{\mathbf{n}^{(i)}}\left(\frac{\mathbf{n}^{(i+1)}}{\mathbf{n}^{(i)}}\Big| I_2\right)_2\right) E\left(\frac{\mathbf{n}_A}{n}\Big| D\right)_{-1}, & \text{if } i = 1, \\ \prod_{j=1}^{i-1} E_{\mathbf{n}^{(j)}}\left(\frac{\mathbf{n}^{(j+1)}}{\mathbf{n}^{(j)}}\Big| I_2\right)_2 & \\ \quad \times \left(1 - E_{\mathbf{n}^{(i)}}\left(\frac{\mathbf{n}^{(i+1)}}{\mathbf{n}^{(i)}}\Big| I_2\right)_2\right) E\left(\frac{\mathbf{n}_A}{n}\Big| D\right)_{-1}, & \text{if } i = 2, \dots, k-1, \\ \prod_{j=1}^{k-1} E_{\mathbf{n}^{(j)}}\left(\frac{\mathbf{n}^{(j+1)}}{\mathbf{n}^{(j)}}\Big| I_2\right)_2 E\left(\frac{\mathbf{n}_A}{n}\Big| D\right)_{-1}, & \text{if } i = k. \end{cases}$$

The likelihood ratio test statistic is

$$\begin{aligned} \Lambda_{02} &= \frac{\prod_{i=-k}^{k} (p_i^{\circ})^{n_i}}{\prod_{i=-k}^{k} (p_i^{\dagger})^{n_i}} \\ &= \frac{\prod_{i=1}^{k-1} (\eta_i^{\circ})^{\sum_{j=i+1}^{k} n_j} (1 - \eta_i^{\circ})^{n_i} (\eta_{-i}^{\circ})^{\sum_{j=i+1}^{k} n_{-j}} (1 - \eta_{-i}^{\circ})^{n_{-i}}}{\prod_{i=1}^{k-1} (\eta_i^{\dagger})^{\sum_{j=i+1}^{k} n_j} (1 - \eta_i^{\dagger})^{n_i} (\eta_{-i}^{\dagger})^{\sum_{j=i+1}^{k} n_{-j}} (1 - \eta_{-i}^{\dagger})^{n_{-i}}} \\ &\quad \times \frac{(A_1^{\circ})^{\sum_{i=1}^{k} n_i} (A_{-1}^{\circ})^{\sum_{i=1}^{k} n_{-i}} A_0^{n_0}}{(A_1^{*})^{\sum_{i=1}^{k} n_i} (A_{-1}^{*})^{\sum_{i=1}^{k} n_{-i}} A_0^{n_0}}. \end{aligned}$$

The test rejects H_0 in favor of H_2 for large values of

$$\begin{aligned} T_{02} &= -2 \ln \Lambda_{02} \\ &= 2 \left[\sum_{i=1}^{k-1} \left(\sum_{j=i+1}^{k} n_j \right) (\ln \eta_i^{\dagger} - \ln \eta_i^{\circ}) + \sum_{i=1}^{k-1} n_i (\ln(1 - \eta_i^{\dagger}) - \ln(1 - \eta_i^{\circ})) \right. \\ &\quad \left. + \sum_{i=1}^{k-1} \left(\sum_{j=i+1}^{k} n_{-j} \right) (\ln \eta_{-i}^{\dagger} - \ln \eta_{-i}^{\circ}) + \sum_{i=1}^{k-1} n_{-i} (\ln(1 - \eta_{-i}^{\dagger}) - \ln(1 - \eta_{-i}^{\circ})) \right] \\ &\quad + 2 \left[\left(\sum_{i=1}^{k} n_{-i} \right) (\ln A_{-1}^{*} - \ln A_{-1}^{\circ}) + \left(\sum_{i=1}^{k} n_i \right) (\ln A_1^{*} - \ln A_1^{\circ}) \right] \\ &= \sum_{i=1}^{k-1} T_{02}^{1(i)} + T_{02}^{2}, \text{ say.} \end{aligned}$$

Suppose $n_i > 0$ for all $i = -k, \dots, k$. Under H_0, it is not difficult to show that for all $t > 0$,

$$\lim_{n\to\infty} P[T_{02}^{1(i)} \geq t] = \sum_{\ell=1}^{2} P(\ell, 2; \mathbf{n}^{(i)}) \Pr[\chi^2_{\ell-1} \geq t],$$

for each $i = 1, \ldots, k-1$. Note that $P(\ell, 2; \mathbf{n}^{(i)}) = 1/2$ for $\ell = 1, 2$. Since $T_{02}^{1(i)}$ are asymptotically independent to each other, we have

$$\lim_{n\to\infty} P\left[\sum_{i=1}^{k-1} T_{02}^{1(i)} \geq t\right] = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \left(\frac{1}{2}\right)^{k-1} \Pr[\chi^2_\ell \geq t].$$

Note that $T_{01}^2 = T_{02}^2$, and $\sum_{i=1}^{k-1} T_{02}^{1(i)}$ and T_{02}^2 are asymptotically independent. Hence we have the following Theorem.

THEOREM 3.4. *If $p_i = p_{-i}$ for $i = 1, \ldots, k$ and n goes to infinity, then for every $t > 0$,*

$$\lim_{n\to\infty} \Pr[T_{02} \geq t] = \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{1}{2}\right)^{k} \Pr[\chi^2_\ell \geq t].$$

4. A SIMULATION STUDY

It is easy to show that the following shifted binomial distribution satisfies the Type I, II, III (a) and (b) positive biasedness if $p > 0.5$.

$$p_j = \binom{2k}{j+k} p^{j+k}(1-p)^{k-j}, \ j = 0, \pm 1, \ldots, \pm k.$$

In this section we conduct a small simulation study to compare the performance of the proposed tests with other tests. In our study, the tests against four types of biasedness are compared for $k = 3, 4$ and 5 and the sample size $n = 100, 200, 250$ and 500. The replication size is 10,000. Since no particular differences has been detected for various values of k and sample sizes, we only list the result of the power comparison for $k = 3$, sample size $= 100$, significance level $= 0.05$ in Table 4.1. Since the asymptotic null distribution of test against Type III (a) depends upon unknown parameter $\mathbf{p} \in H_0$ we approximate the level probabilities $P(\ell, k; \mathbf{p}_+)$ by plugging in $\mathbf{p}_+^\circ$. This method is known to provide a good approximation in general. We use the similar method for the test against Type I.

TABLE 4.1 *Power comparison for $k = 3$, sample size = 100, significance level = 0.05*

	Test or Type of Positive Biasedness				
p	UMP	$III(a)$	$III(b)$	II	I
0.50	0.0448	0.0604	0.0613	0.0621	0.0464
0.51	0.1167	0.1195	0.1169	0.1231	0.0919
0.52	0.2500	0.2278	0.2239	0.2401	0.1842
0.53	0.4132	0.3579	0.3478	0.3725	0.2934
0.54	0.6117	0.5354	0.5227	0.5458	0.4600
0.55	0.7747	0.7041	0.6944	0.7146	0.6291
0.56	0.9029	0.8435	0.8356	0.8483	0.7864
0.57	0.9599	0.9231	0.9180	0.9266	0.8871
0.58	0.9883	0.9740	0.9707	0.9736	0.9583
0.59	0.9965	0.9923	0.9914	0.9923	0.9876
0.60	0.9992	0.9981	0.9978	0.9980	0.9962

Dykstra *et al.* (1995b) show that tests against Type I and II biasedness perform better than the unrestricted likelihood ratio test which is essentially the chi-square goodness-of-fit test for symmetry. We excluded the unrestricted likelihood ratio test in our simulation study. They also show that the power function of two tests approaches close to that of the uniformly most powerful test which rejects the symmetry hypothesis for the large value of the sample sum.

As shown in Dykstra *et al.* (1995b) the test against Type I is conservative while other three tests are little bit liberal. We, however, note that all four test are conservative for $k = 4$ and 5 and $n = 100$. The powers for Type III (a) and (b) are greater than that of Type I as we expected since Type III (a) and (b) are more restrictive than Type I. The test against Type II is more powerful than other three restricted tests. This reveals that Type II is slightly more restrictive than Type III (a) and (b).

5. REMARKS

It is quite interesting to observe that the asymptotic distribution for testing against H_2 is exactly the same as that of testing against Type II positive biasedness. Moreover, both asymptotic null distributions do not depend on the true distribution function itself.

In previous two sections we have dealt with discrete or grouped data. Although the test procedures discussed in the previous sections can not be applied directly to the case of arbitrary distribution, possibly continuous distribution cases, the estimation procedures can be easily extended with minor modification. The estimated distribution function under positive biasedness restriction is a nonparametric maximum likelihood estimate in Kiefer and Wolfowitz sense. In this case we might doubt the consistency of nonparametric estimation of distribution functions under near Type III (a) and (b) positive biasedness restriction. As we mentioned near Type III (a) and (b) positive biasedness are closely related to likelihood ratio ordering and uniform stochastic ordering, respectively. Dykstra *et al.* (1995a) show that the nonparametric MLE of two distribution functions under likelihood ratio ordering is strongly consistent. By applealing to this result we can easily show that the estimation under Type III (a) have strong consistency for arbitrary distribution. Though Rojo and Samaniego (1991) show that one-sample nonparametric ML estimation of distribution function under uniform stochastic ordering does not have even weak consistency property, the strong consistency property of nonparametric MLE of two or more distribution functions under uniform stochastic ordering is retained as discussed in Dykstra *et al.* (1991), Park *et al.* (1998). Moreover, for stochastic ordering and likelihood ratio ordering cases Dykstra *et al.* (1995a, 1995b) show that the uniform consistency property of nonparametric ML estimation of two distribution functions under the restriction is retained.

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