The Korean Communications in Statistics **Vol. 14** No. 2, 2007, pp. 431–442

Noninformative Priors for the Difference of Two Quantiles in Exponential Models

Sang Gil Kang[1] Dal Ho Kim[2] and Woo Dong Lee[3]

Abstract

In this paper, we develop the noninformative priors when the parameter of interest is the difference between quantiles of two exponential distributions. We want to develop the first and second order probability matching priors. But we prove that the second order probability matching prior does not exist. It turns out that Jeffreys' prior does not satisfy the first order matching criterion. The Bayesian credible intervals based on the first order probability matching prior meet the frequentist target coverage probabilities much better than the frequentist intervals of Jeffreys' prior. Some simulation and real example will be given.

Keywords: Difference of two quantiles; exponential models; probability matching prior.

1. Introduction

The exponential distribution plays an important role in the field of reliability. The usefulness of the exponential distribution in reliability applications can be found in the early work of Davis (1952), Epstein and Sobel (1953) and others. Further justification, in the form of theoretical arguments to support the use of the exponential distribution as the failure law of complex equipment, is presented in the book by Barlow and Proschan (1975) and Lawless (2003).

Comparison between two populations is an important problem in statistics and is commonly used in real fields. The populations are usually compared with

1) Associate Professor, Department of Computer and Data Information, Sangji University, Wonju 220-702, Korea.
Correspondence : sangkg@sangji.ac.kr
2) Professor, Department of Statistics, Kyungpook National University, Taegu 702-701, Korea.
E-mail : dalkim@knu.ac.kr
3) Professor, Department of Asset Management, Daegu Haany University, Kyungsan 712-715, Korea.
E-mail : wdlee@dhu.ac.kr

respect to their means to establish superiority of one population over the other or to check if the two populations are equivalent. For example, two drugs may be compared with respect to their mean effects to determine the better one. Even though, comparing two populations with respect to means is a common problem, there are situations where one needs to compare the quantiles instead of their means (see, Albers and Löhnberg, 1984; Huang and Johnson, 2006).

The present paper focuses on developing noninformative priors for the difference of two quantiles of exponential distributions. We consider Bayesian priors such that the resulting credible intervals for the difference of two quantiles have coverage probabilities equivalent to their frequentist counterparts. Although this matching can be justified only asymptotically, our simulation results indicate that this is indeed achieved for small or moderate sample sizes as well.

This matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), DiCiccio and Stern (1994), Datta and Ghosh (1995a, 1995b, 1996), Mukerjee and Ghosh (1997).

On the other hand, Ghosh and Mukerjee (1992), and Berger and Bernardo (1989, 1992a, 1992b) extended Bernardo's (1979) reference prior approach, giving a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier.

For comparison of two quantiles, Albers and Löhnberg (1984) presented a biomedical problem where comparison between the p^{th} quantiles of two populations arises. They provided an approximate distribution-free confidence interval for the difference of two quantiles. Bristol (1990) suggested a modification to Albers and Löhnberg's method. Guo and Krishnamoorthy (2005) proposed methods for interval estimation and testing the difference between the quantiles of two normal populations and two exponential populations. Their methods are based on the concepts of generalized p-value and generalized limit. On the other hand, Huang and Johnson (2006) derived confidence regions for the ratio of quantiles from two normal populations. They developed an exact confidence procedure when the ratio of variances is known. And when the ratio of variances is unknown, they obtained confidence intervals for the ratio of quantiles based on large sample methods. However there is a little work in this problem from the viewpoint of Bayesian framework.

The outline of the remaining sections is as follows. In section 2, we consider first order and second order probability matching priors for the difference of two quantiles in exponential models. We revealed that the second order matching prior does not exist. It turns out that Jeffreys' prior does not satisfy the first order matching criterion. We provide that the propriety of the posterior distribution for the first order matching prior and Jeffreys' prior in section 3. In section 4, simulated frequentist coverage probabilities under the proposed priors and a real example are given.

2. The Noninformative Priors

For a prior π, let $\theta_1^{1-\alpha}(\pi;\mathbf{X})$ denote the $(1-\alpha)^{th}$ percentile of the posterior distribution of θ_1, that is,

$$P^{\pi}[\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbf{X})|\mathbf{X}] = 1-\alpha, \tag{2.1}$$

where $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_t)^T$ and θ_1 is the parameter of interest. We want to find priors π for which

$$P[\theta_1 \le \theta_1^{1-\alpha}(\pi;\mathbb{X})|\boldsymbol{\theta}] = 1-\alpha+o(n^{-u}) \tag{2.2}$$

for some $u>0$, as n goes to infinity. Priors π satisfying (2.2) are called matching priors. If $u=1/2$, then π is referred to as a first order matching prior, while if $u=1$, π is referred to as a second order matching prior.

Consider that $X_1,\ldots,X_{n_1}$ are independent and identically distributed random variables according to the exponential distribution with mean μ_1 and $Y_1,\ldots,$ Y_{n_2} are independent and identically distributed random variables according to the exponential distribution with mean μ_2. Then the likelihood function of μ_1 and μ_2 given $\mathbf{x}=(x_1,\ldots,x_{n_1})$ and $\mathbf{y}=(y_1,\ldots,y_{n_2})$ is

$$L(\mu_1,\mu_2|\mathbf{x},\mathbf{y}) = \left(\frac{1}{\mu_1}\right)^{n_1}\left(\frac{1}{\mu_2}\right)^{n_2}\exp\left(-\sum_{i=1}^{n_1}\frac{x_i}{\mu_1}-\sum_{i=1}^{n_2}\frac{y_i}{\mu_2}\right), \tag{2.3}$$

where $\mu_1>0$ and $\mu_2>0$.

In order to find matching priors π, let

$$\theta_1 = c_1\mu_1 - c_2\mu_2 \text{ and } \theta_2 = \frac{c_2n_1\mu_2 + c_1n_2\mu_1}{\mu_1\mu_2},$$

where $c_i = -\log(1-p_i), i=1,2$ and $c_i\mu_i$ is the p_i^{th} quantile of the exponential distribution with mean $\mu_i, i=1,2$. With this parameterization, the likelihood function of (θ_1, θ_2) given $(\mathbf{x}, \mathbf{y})$ is given by

$$\begin{aligned} L(\theta_1, \theta_2|\mathbf{x}, \mathbf{y}) \propto\ & \theta_2^{n_1+n_2}(c_1c_2(n_1+n_2)+\theta_1\theta_2+g(\theta_1,\theta_2)^{1/2})^{-n_1} \\ & \times(c_1c_2(n_1+n_2)-\theta_1\theta_2+g(\theta_1,\theta_2)^{1/2})^{-n_2} \\ & \times\exp\left(-\sum_{i=1}^{n_1}\frac{2c_1\theta_2x_i}{c_1c_2(n_1+n_2)+\theta_1\theta_2+g(\theta_1,\theta_2)^{1/2}}\right. \\ & \left.-\sum_{i=1}^{n_2}\frac{2c_2\theta_2y_i}{c_1c_2(n_1+n_2)-\theta_1\theta_2+g(\theta_1,\theta_2)^{1/2}}\right), \end{aligned} \tag{2.4}$$

where $g(\theta_1,\theta_2) = [c_1c_2(n_2-n_1)+\theta_1\theta_2]^2+4c_1^2c_2^2n_1n_2$. Based on the above likelihood function (2.4), the Fisher information matrix is given by

$$\mathbf{I} = \begin{pmatrix} I_{11} & 0 \\ 0 & I_{22} \end{pmatrix},$$

where

$$I_{11} = \frac{2n_1n_2\theta_2^2}{g(\theta_1,\theta_2)^{1/2}[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g(\theta_1,\theta_2)^{1/2}]}$$

and

$$I_{22} = \frac{[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g(\theta_1,\theta_2)^{1/2}]}{2\theta_2^2g(\theta_1,\theta_2)^{1/2}}.$$

From the above Fisher information matrix $\mathbf{I}$, θ_1 is orthogonal to θ_2 in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of the first order probability matching prior is characterized by

$$\pi_m^{(1)}(\theta_1,\theta_2) \propto \frac{\theta_2 g(\theta_1,\theta_2)^{-1/4}d(\theta_2)}{[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g(\theta_1,\theta_2)^{1/2}]^{1/2}}, \tag{2.5}$$

where $d(\theta_2) > 0$ is an arbitrary function differentiable in its arguments.

The class of first order probability matching prior given in (2.5) is so broad, so one can narrow down this prior to the second order probability matching prior as given in Mukerjee and Ghosh (1997).

The second order probability matching prior is of the form (2.5), and also the function $d(\cdot)$ must satisfy an additional differential equation (*cf* (2.10) of Mukerjee and Ghosh (1997)), namely,

$$\frac{1}{6}d(\theta_2)\frac{\partial}{\partial\theta_1}I_{11}^{-\frac{3}{2}}L_{1,1,1}+\frac{\partial}{\partial\theta_2}I_{11}^{-\frac{1}{2}}L_{112}I^{22}d(\theta_2)=0, \tag{2.6}$$

where

$$L_{1,1,1} = E\left[\left(\frac{\partial \log L}{\partial \theta_1}\right)^3\right]$$
$$= \frac{8n_1n_2\theta_2^3 h_1(\theta_1,\theta_2)}{g(\theta_1,\theta_2)^{3/2}[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g(\theta_1,\theta_2)^{1/2}]^3}$$
$$L_{112} = E\left[\frac{\partial^3 \log L}{\partial \theta_1^2 \partial \theta_2}\right] = -\frac{2c_1c_2n_1n_2\theta_2}{g(\theta_1,\theta_2)^{3/2}}.$$

Here

$$\begin{aligned} h_1(\theta_1,\theta_2) &= c_1^3c_2^3(n_2-n_1)(n_1+n_2)^4+3c_1^2c_2^2(n_1+n_2)^2(n_1^2-n_1n_2+n_2^2)\theta_1\theta_2 \\ &+3c_1c_2(n_2^3-n_1^3)\theta_1^2\theta_2^2+(n_1^2+n_2^2)\theta_1^3\theta_2^3+[c_1^2c_2^2(n_2^2-n_1^2)(n_1+n_2)^2 \\ &+c_1c_2(n_2+n_1)(2n_1^2-n_1n_2+2n_2^2)\theta_1\theta_2+(n_2^2-n_1^2)\theta_1^2\theta_2^2]g(\theta_1,\theta_2)^{1/2}. \end{aligned}$$

Then (2.6) simplifies to

$$\frac{1}{6}d(\theta_2)\frac{\partial}{\partial \theta_1}\{w_1(\theta_1,\theta_2)\}+\frac{\partial}{\partial \theta_2}\{w_2(\theta_1,\theta_2)d(\theta_2)\}=0, \tag{2.7}$$

where

$$w_1(\theta_1,\theta_2)=\frac{2^{3/2}(n_1n_2)^{-1/2}h_1(\theta_1,\theta_2)}{g(\theta_1,\theta_2)^{3/4}[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g(\theta_1,\theta_2)^{1/2}]^{3/2}}$$

and

$$w_2(\theta_1,\theta_2) = -\frac{2^{3/2}(n_1n_2)^{1/2}c_1c_2\theta_2^2}{g(\theta_1,\theta_2)^{3/4}[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g^{1/2}]^{1/2}}.$$

However there can be no solution to (2.7) unless the function d is the function of θ_1 and θ_2. Thus the second order probability matching prior does not exist.

From the Fisher information matrix $\mathbf{I}$, Jeffreys' prior is given by

$$\pi_J(\theta_1,\theta_2) \propto \frac{1}{[(c_1c_2(n_2-n_1)+\theta_1\theta_2)^2+4c_1^2c_2^2n_1n_2]^{1/2}}. \tag{2.8}$$

Remark 2.1 In the original parameterization (μ_1,μ_2), the first order probability matching prior is given by

$$\pi_m^{(1)}(\mu_1,\mu_2) \propto \mu_1^{-1}\mu_2^{-1}\left(\frac{c_2^2n_1}{\mu_1^2}+\frac{c_1^2n_2}{\mu_2^2}\right)^{1/2} d(\theta_2(\mu_1,\mu_2)).$$

And Jeffreys' prior is given by

$$\pi_J(\mu_1,\mu_2) \propto \mu_1^{-1}\mu_2^{-1}. \tag{2.9}$$

Notice that the matching prior (2.5) includes many different matching priors because of the arbitrary selection of the function d. However all functions are not permissible in the construction of priors. For instance, we consider any function of the form θ_2^{-a}. If a is negative integer, then the posterior distribution under function of the form θ_2^{-a} is proper. But the condition of propriety in this form strongly depend on a . Moreover the posterior under this form is complex. Also there does not seem to be any improvement in the coverage probabilities with this posterior distribution. So we have chosen d to be a constant function. The resulting prior is given by

$$\pi_m^{(1)}(\theta_1,\theta_2) \propto \frac{\theta_2 g(\theta_1,\theta_2)^{-\frac{1}{4}}}{[c_1c_2(n_1+n_2)^2+(n_2-n_1)\theta_1\theta_2+(n_1+n_2)g(\theta_1,\theta_2)^{\frac{1}{2}}]^{\frac{1}{2}}}. \quad (2.10)$$

Thus in the original parameterization (μ_1,μ_2), the first order matching prior is given by

$$\pi_m^{(1)}(\mu_1,\mu_2) \propto \mu_1^{-1}\mu_2^{-1}\left(\frac{c_2^2 n_1}{\mu_1^2}+\frac{c_1^2 n_2}{\mu_2^2}\right)^{1/2}. \quad (2.11)$$

3. Implementation of the Bayesian Procedure

We investigate the propriety of posteriors for general priors which include Jeffreys' prior (2.9) and the first order probability matching prior (2.11). We consider the class of priors

$$\pi_g(\mu_1,\mu_2) \propto \mu_1^{-a}\mu_2^{-b}\left(\frac{c_2^2 n_1}{\mu_1^2}+\frac{c_1^2 n_2}{\mu_2^2}\right)^{c}, \quad (3.1)$$

where $a>0$, $b>0$ and $c\geq 0$. The following theorem can be proved.

Theorem 3.1 *The posterior distribution of (μ_1,μ_2) under the prior (3.1) is proper if $n_1+a-1>0$ and $n_2+b-1>0$.*

Proof: Under the prior (3.1), the joint posterior for μ_1 and μ_2 given $\mathbf{x}$ and $\mathbf{y}$ is

$$\pi(\mu_1,\mu_2|\mathbf{x},\mathbf{y}) \propto \left(\frac{1}{\mu_1}\right)^{n_1+a}\left(\frac{1}{\mu_2}\right)^{n_2+b}\left(\frac{c_2^2 n_1}{\mu_1^2}+\frac{c_1^2 n_2}{\mu_2^2}\right)^{c}\exp\left(-\sum_{i=1}^{n_1}\frac{x_i}{\mu_1}-\sum_{i=1}^{n_2}\frac{y_i}{\mu_2}\right).$$

For $1 \le \mu_1 < \infty$ and $1 \le \mu_2 < \infty$,

$$\int_1^\infty \int_1^\infty \pi(\mu_1, \mu_2|\mathbf{x}, \mathbf{y}) d\mu_1 d\mu_2 \le \int_1^\infty \int_1^\infty \left(\frac{1}{\mu_1}\right)^{n_1+a} \left(\frac{1}{\mu_2}\right)^{n_2+b} (c_2^2 n_1 + c_1^2 n_2)^c \times \exp\left(-\sum_{i=1}^{n_1} \frac{x_i}{\mu_1} - \sum_{i=1}^{n_2} \frac{y_i}{\mu_2}\right) d\mu_1 d\mu_2 < \infty,$$

if $n_1 + a - 1 > 0$ and $n_2 + b - 1 > 0$. For $0 < \mu_1 < 1$ and $1 \le \mu_2 < \infty$,

$$\int_1^\infty \int_0^1 \pi(\mu_1, \mu_2|\mathbf{x}, \mathbf{y}) d\mu_1 d\mu_2 \le \int_1^\infty \int_0^1 \left(\frac{1}{\mu_1}\right)^{n_1+a+2c} \left(\frac{1}{\mu_2}\right)^{n_2+b} (c_2^2 n_1 + c_1^2 n_2)^c \times \exp\left(-\sum_{i=1}^{n_1} \frac{x_i}{\mu_1} - \sum_{i=1}^{n_2} \frac{y_i}{\mu_2}\right) d\mu_1 d\mu_2 < \infty,$$

if $n_1 + a + 2c - 1 > 0$ and $n_2 + b - 1 > 0$. For $0 < \mu_1 < 1$ and $0 < \mu_2 < 1$,

$$\int_0^1 \int_0^1 \pi(\mu_1, \mu_2|\mathbf{x}, \mathbf{y}) d\mu_1 d\mu_2 \le \int_0^1 \int_0^1 \left(\frac{1}{\mu_1}\right)^{n_1+a+2c} \left(\frac{1}{\mu_2}\right)^{n_2+b+2c} (c_2^2 n_1 + c_1^2 n_2)^c \times \exp\left(-\sum_{i=1}^{n_1} \frac{x_i}{\mu_1} - \sum_{i=1}^{n_2} \frac{y_i}{\mu_2}\right) d\mu_1 d\mu_2 < \infty,$$

if $n_1 + a + 2c - 1 > 0$ and $n_2 + b + 2c - 1 > 0$. This completes the proof. □

Theorem 3.2 *The marginal posterior density of θ_1 under the matching prior (2.11) is given by*

$$\begin{aligned}
&\pi_m(\theta_1|\mathbf{x}, \mathbf{y}) \\
&\propto \int_0^\infty \theta_2^{n_1+n_2+1} [c_1 c_2 (n_1 + n_2) + \theta_1\theta_2 + g^{\frac{1}{2}}]^{-n_1} [c_1 c_2 (n_1 + n_2) - \theta_1\theta_2 + g^{\frac{1}{2}}]^{-n_2} \\
&\times g^{-\frac{1}{4}} [c_1 c_2 (n_1 + n_2)^2 + (n_2 - n_1)\theta_1\theta_2 + (n_1 + n_2) g^{\frac{1}{2}}]^{-\frac{1}{2}} \\
&\times \exp\left(-\sum_{i=1}^{n_1} \frac{2c_1\theta_2 x_i}{c_1 c_2 (n_1 + n_2) + \theta_1\theta_2 + g^{\frac{1}{2}}} - \sum_{i=1}^{n_2} \frac{2c_2\theta_2 y_i}{c_1 c_2 (n_1 + n_2) - \theta_1\theta_2 + g^{\frac{1}{2}}}\right) d\theta_2,
\end{aligned}$$

where $g \equiv g(\theta_1, \theta_2) = (c_1 c_2 (n_2 - n_1) + \theta_1\theta_2)^2 + 4c_1^2 c_2^2 n_1 n_2$. And the marginal posterior density of θ_1 under Jeffreys' prior (2.9) is given by

$$\begin{aligned}
&\pi_J(\theta_1|\mathbf{x}, \mathbf{y}) \\
&\propto \int_0^\infty \theta_2^{n_1+n_2} [c_1 c_2 (n_1 + n_2) + \theta_1\theta_2 + g^{\frac{1}{2}}]^{-n_1} [c_1 c_2 (n_1 + n_2) - \theta_1\theta_2 + g^{\frac{1}{2}}]^{-n_2} g^{-\frac{1}{2}} \\
&\times \exp\left(-\sum_{i=1}^{n_1} \frac{2c_1\theta_2 x_i}{c_1 c_2 (n_1 + n_2) + \theta_1\theta_2 + g^{\frac{1}{2}}} - \sum_{i=1}^{n_2} \frac{2c_2\theta_2 y_i}{c_1 c_2 (n_1 + n_2) - \theta_1\theta_2 + g^{\frac{1}{2}}}\right) d\theta_2.
\end{aligned}$$

Actually the normalizing constant for the marginal density of θ_1 requires two dimensional integration. Therefore we have the marginal posterior density of θ_1, and so it is to compute the marginal moment of θ_1.

4. Numerical Studies and Discussion

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posteriors density of θ_1 under the noninformative prior π given in section 3 for several configurations (p_1, p_2), (μ_1, μ_2) and (n_1, n_2). That is to say, the frequentist coverage of a $(1-\alpha)$ posterior quantile should be close to $(1-\alpha)$. This is done numerically. Table 1 gives numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the our priors. The computation of these numerical values is based on the following algorithm for any fixed true (μ_1, μ_2) and any prespecified probability α. Here α is 0.05 (0.95). Let $\theta_1^\pi(\alpha|\mathbf{X}, \mathbf{Y})$ be the posterior α-quantile of θ_1 given $\mathbf{X}$ and $\mathbf{Y}$. That is to say, $F(\theta_1^\pi(\alpha|\mathbf{X}, \mathbf{Y})|\mathbf{X}, \mathbf{Y}) = \alpha$, where $F(\cdot|\mathbf{X}, \mathbf{Y})$ is the marginal posterior distribution of θ_1. Then the frequentist coverage probability of this one sided credible interval of θ_1 is

$$P_{(\mu_1,\mu_2)}(\alpha; \theta_1) = P_{(\mu_1,\mu_2)}(0 < \theta_1 \le \theta_1^\pi(\alpha|\mathbf{X}, \mathbf{Y})).$$

The estimated $P_{(\mu_1,\mu_2)}(\alpha; \theta_1)$ when $\alpha = 0.05(0.95)$ is shown in Table 4.1. In particular, for fixed , we take 10,000 independent random samples of $\mathbf{X}$ and $\mathbf{Y}$ from the model (2.3). For the cases presented in Table 4.1, we see that the first order matching prior meets very well the target coverage probabilities for small and moderate values of n_1 and n_2. Also the results of tables are not much sensitive to the change of the values of (μ_1, μ_2) and (p_1, p_2). Thus we can recommend to use the first order matching prior when using the matching criterion. Note that Jeffreys' prior does not satisfy the first order matching criterion but it meets the target coverage probabilities well.

Example 4.1 The following data, given by Proschan (1963), are time intervals of successive failures of the air conditioning equipment in Boeing 720 aircraft.

Aircraft 1	102 209 14 57 54 32 67 59 134 152 27 14 230 66 61 34
Aircraft 2	50 44 102 72 22 39 3 15 197 188 79 88 46 5 5 36 22 139 210 97 30 23 13 14

For aircraft 1, the Kolmogorov-Smirnov test statistic is 0.1143 and its p-value is 0.88. For aircraft 2, the Kolmogorov-Smirnov test statistic is 0.1791 and its

Table 4.1: Frequentist Coverage Probabilities of 0.05 (0.95) Posterior Quantiles for θ_1

p_1	p_2	μ_1	μ_2	n_1	n_2	π_J		π_m	
0.1	0.9	0.1	0.1	5	5	0.061	(0.954)	0.059	(0.952)
				5	10	0.043	(0.955)	0.040	(0.950)
				10	10	0.044	(0.952)	0.044	(0.951)
				10	15	0.052	(0.954)	0.051	(0.952)
		0.1	1.0	5	5	0.053	(0.947)	0.052	(0.946)
				5	10	0.051	(0.953)	0.051	(0.952)
				10	10	0.049	(0.951)	0.049	(0.951)
				10	15	0.050	(0.953)	0.049	(0.952)
		0.1	10	5	5	0.050	(0.952)	0.050	(0.952)
				5	10	0.050	(0.952)	0.050	(0.952)
				10	10	0.051	(0.946)	0.051	(0.946)
				10	15	0.049	(0.950)	0.049	(0.950)
		1.0	0.1	5	5	0.056	(0.963)	0.067	(0.946)
				5	10	0.061	(0.964)	0.065	(0.945)
				10	10	0.055	(0.957)	0.057	(0.946)
				10	15	0.056	(0.957)	0.057	(0.946)
		1.0	1.0	5	5	0.055	(0.949)	0.053	(0.946)
				5	10	0.050	(0.957)	0.048	(0.953)
				10	10	0.051	(0.956)	0.050	(0.953)
				10	15	0.051	(0.952)	0.050	(0.950)
		1.0	10	5	5	0.050	(0.950)	0.049	(0.949)
				5	10	0.048	(0.949)	0.048	(0.949)
				10	10	0.051	(0.950)	0.051	(0.950)
				10	15	0.051	(0.953)	0.051	(0.953)
		10	0.1	5	5	0.042	(0.942)	0.053	(0.940)
				5	10	0.047	(0.951)	0.053	(0.950)
				10	10	0.043	(0.950)	0.050	(0.951)
				10	15	0.045	(0.946)	0.049	(0.948)
		10	1.0	5	5	0.057	(0.961)	0.066	(0.942)
				5	10	0.056	(0.963)	0.060	(0.944)
				10	10	0.056	(0.958)	0.059	(0.948)
				10	15	0.054	(0.960)	0.056	(0.947)
		10	10	5	5	0.049	(0.952)	0.048	(0.950)
				5	10	0.057	(0.954)	0.055	(0.950)
				10	10	0.050	(0.950)	0.049	(0.947)
				10	15	0.053	(0.951)	0.051	(0.948)

p-value is 0.62. So we can assume that the time between successive failures for each plane is exponentially distributed.

Under Jeffreys' prior and the matching prior, Table 4.3 shows the Bayes esti-

Table 4.2: (Continued)

p_1	p_2	μ_1	μ_2	n_1	n_2	π_J		π_m	
0.9	0.1	0.1	0.1	5	5	0.046	(0.942)	0.049	(0.944)
				5	10	0.048	(0.942)	0.050	(0.944)
				10	10	0.049	(0.959)	0.051	(0.960)
				10	15	0.048	(0.959)	0.050	(0.959)
		0.1	1.0	5	5	0.039	(0.944)	0.057	(0.934)
				5	10	0.043	(0.948)	0.053	(0.942)
				10	10	0.047	(0.947)	0.058	(0.944)
				10	15	0.045	(0.951)	0.053	(0.950)
		0.1	10	5	5	0.057	(0.960)	0.058	(0.948)
				5	10	0.055	(0.964)	0.053	(0.951)
				10	10	0.055	(0.962)	0.054	(0.954)
				10	15	0.053	(0.956)	0.050	(0.948)
		1.0	0.1	5	5	0.050	(0.950)	0.050	(0.950)
				5	10	0.052	(0.953)	0.053	(0.953)
				10	10	0.052	(0.950)	0.052	(0.950)
				10	15	0.051	(0.951)	0.051	(0.951)
		1.0	1.0	5	5	0.046	(0.947)	0.050	(0.949)
				5	10	0.047	(0.954)	0.048	(0.955)
				10	10	0.048	(0.945)	0.050	(0.946)
				10	15	0.048	(0.950)	0.049	(0.951)
		1.0	10	5	5	0.037	(0.942)	0.056	(0.934)
				5	10	0.042	(0.948)	0.052	(0.941)
				10	10	0.043	(0.945)	0.055	(0.941)
				10	15	0.045	(0.947)	0.053	(0.945)
		10	0.1	5	5	0.050	(0.950)	0.050	(0.950)
				5	10	0.049	(0.950)	0.049	(0.950)
				10	10	0.053	(0.952)	0.053	(0.952)
				10	15	0.051	(0.950)	0.051	(0.950)
		10	1.0	5	5	0.049	(0.950)	0.049	(0.950)
				5	10	0.050	(0.951)	0.050	(0.951)
				10	10	0.047	(0.949)	0.047	(0.949)
				10	15	0.050	(0.950)	0.050	(0.950)
		10	10	5	5	0.046	(0.950)	0.050	(0.951)
				5	10	0.047	(0.948)	0.049	(0.949)
				10	10	0.050	(0.949)	0.052	(0.951)
				10	15	0.046	(0.948)	0.048	(0.949)

mates and the 95% Bayesian credible intervals of $\theta_1 = -\log(1-p_1)\mu_1 + \log(1-p_2)\mu_2$ with several values of p_1 and p_2.

From Table 4.3, the credible intervals under two priors have similar values and the length of the credible interval under the matching prior is shorter than

Table 4.3: Bayes Estimates and 95% Bayesian Credible Intervals of θ_1

p_1	p_2	π_J		π_m	
0.9	0.1	195.73	(115.82,325.57)	196.01	(116.12,325.85)
0.7	0.3	82.16	(38.80,150.59)	82.89	(39.71,151.19)
0.5	0.5	14.66	(-18.93,56.86)	14.93	(-18.27,56.76)
0.3	0.7	-49.15	(-91.34,-15.82)	-50.11	(-91.99,-17.40)
0.1	0.9	-144.79	(-221.29,-93.11)	-145.34	(-221.81,-93.70)

Jeffreys' prior.

5. Conclusion

In the exponential distributions, we have found the first order matching prior and Jeffreys' prior for the difference of two quantiles. We have proved that the second order matching prior does not exist. And these first order matching priors possess good frequentist properties in the sense that the coverage probabilities of credible intervals for the difference of two quantiles based on this prior match their frequentist counterparts very closely even for small and moderate sample sizes. Also Jeffreys' prior does not satisfy the first order matching criterion. From our simulation results and example, we recommend to use the first order matching prior for the Bayesian inference of the difference of two quantiles.

References

Albers, W. and Löhnberg, P. (1984). An approximate confidence interval for the difference between quantiles in a bio-medical problem. *Statistica Neerlandica*, **38**, 20–22.

Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Reinhart and Winston, New York.

Berger, J. O. and Bernardo, J. M. (1989). Estimating a product of means : Bayesian analysis with reference priors. *Journal of the American Statistical Association*, **84**, 200–207.

Berger, J. O. and Bernardo, J. M. (1992a). Reference priors in a variance components problem. *Bayesian Analysis in Statistics and Econometrics* (P. Goel and N. S. Iyengar eds.), 177–194, Springer-Verlag, New York.

Berger, J. O. and Bernardo, J. M. (1992b). On the development of reference priors (with discussion). *Bayesian Statistics IV* (J. M. Bernardo, *et. al.* eds.), 35–60, Oxford University Press, Oxford.

Bernardo, J. M. (1979). Reference posterior distributions for Bayesian inference (with discussion). *Journal of the Royal Statistical Society*, Ser. B, **41**, 113–147.

Bristol, D. R. (1990). Distribution-free confidence intervals for the difference between quantiles. *Statistica Neerlandica*, **44**, 87–90.

Cox, D. R. and Reid, N. (1987). Orthogonal parameters and approximate conditional inference (with discussion). *Journal of Royal Statistical Society*, Ser. B, **49**, 1–39.

Datta, G. S. and Ghosh, J. K. (1995a). On priors providing frequentist validity for Bayesian inference. *Biometrika*, **82**, 37–45.

Datta, G. S. and Ghosh, M. (1995b). Some remarks on noninformative priors. *Journal of the American Statistical Association*, **90**, 1357–1363.

Datta, G. S. and Ghosh, M. (1996). On the invariance of noninformative priors. *The Annal of Statistics*, **24**, 141–159.

Davis, D. J. (1952). An analysis of some failure data. *Journal of the American Statistical Association*, **47**, 113–150.

DiCiccio, T. J. and Stern, S. E. (1994). Frequentist and Bayesian Bartlett correction of test statistics based on adjusted profile likelihood. *Journal of the Royal Statistical Society*, Ser. B, **56**, 397–408.

Epstein, B. and Sobel, M. (1953). Life testing. *Journal of the American Statistical Association*, **48**, 486–502.

Ghosh, J. K. and Mukerjee, R. (1992). Noninformative priors (with discussion). *Bayesian Statistics IV* (J. M. Bernardo, *et. al.* eds.), 195–210, Oxford University Press, Oxford.

Guo, H. and Krishnamoorthy, K. (2005). Comparison between two quantiles: the normal and exponential cases. *Communications in Statistics: Simulation and Computation*, **34**, 243–252.

Huang, L. F. and Johnson, R. A. (2006). Confidence regions for the ratio of percentiles. *Statistics & Probability Letters*, **76**, 384–392.

Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data.* John Wiley & Sons, Hoboken, New Jersey.

Mukerjee, R. and Dey, D. K. (1993). Frequentist validity of posterior quantiles in the presence of a nuisance parameter : higher order asymptotics. *Biometrika*, **80**, 499–505.

Mukerjee, R. and Ghosh, M. (1997). Second-order probability matching priors. *Biometrika*, **84**, 970–975.

Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, **5**, 375–383.

Stein, C. M. (1985). On the coverage probability of confidence sets based on a prior distribution. *Sequential Methods in Statistics*, Banach Center Publications, **16**, 485–514.

Tibshirani, R. (1989). Noninformative priors for one parameter of many. *Biometrika*, **76**, 604–608.

Welch, B. L. and Peers, H. W. (1963). On formulae for confidence points based on integrals of weighted likelihoods. *Journal of the Royal Statistical Society*, Ser. B, **25**, 318–329.

[Received June 2007, Accepted July 2007]